Universal Arithmetick:

OR, A

TREATISE

OF

ARITHMETICAL

COMPOSITION and RESOLUTION.

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COMPOSITION and RESOLUTION.

To which is added,

Dr. HALLEY's Method of finding the Roots of EQUATIONS Arithmetically.

Written in LATIN by Sir ISAAC NEWTON, *and Translated by the late Mr.* RALPHSON, *and Revised and Corrected by Mr.* CUNN.

The Second Edition, very much Corrected.

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TO THE

READER.

TO say any thing in Praise of the ensuing Treatise, were an Attempt as needless and impertinent, as to write a Panegyrick on its Author. It is enough that the Subject is ALGEBRA; *and that it was written by Sir* Isaac Newton: *Those who know any thing of the Sciences, need not be told the Value of the former; nor those who have heard any thing of Philosophy and Mathematicks, to be instructed in the Praises of the latter. If any thing could add to the Esteem every Body has for the* Analytick *Art, it must be, that Sir* Isaac *has condescended to handle it; nor could any thing add to the Opinion the World has of that illustrious Author's Merit, but that he has written with so much Success on that wonderful Subject.*

It is true, we have already a great many Books of Algebra, *and one might even furnish a mo-*

a moderate Library purely with Authors on that Subject: But as no Body will imagine that Sir Isaac *would have taken the Pains to compose a new one, had he not found all the old ones defective; so, it will be easily allowed, that none was more able than he, either to discover the Errors and Defects in other Books, or to supply and rectify them in his own.*

The Book was originally writ for the private Use of the Gentlemen of Cambridge, *and was delivered in Lectures, at the publick Schools, by the Author, then* Lucasian *Professor in that University. Thus, not being immediately intended for the Press, the Author had not prosecuted his Subject so far as might otherwise have been expected; nor indeed did he ever find Leisure to bring his Work to a Conclusion: So that it must be observed, that all the Constructions, both Geometrical and Mechanical, which occur towards the End of the Book, do only serve for finding the first two or three Figures of Roots; the Author having here only given us the Construction of Cubick Equations, though he had a Design to have added, a general Method of constructing Biquadratick, and other higher Powers, and to have particularly shewn in what Manner the other Figures of Roots were to be extracted. In this unfinished State it continued till the Year* 1707, *when Mr.* Whiston, *the Author's Successor in the* Lucasian *Chair, considering*

ſidering that it was but ſmall in Bulk, and yet ample in Matter, not too much crowded with Rules and Precepts, and yet well furniſhed with choice Examples, (ſerving not only as Praxes *on the Rules, but as Inſtances of the great Uſefulneſs of the Art it ſelf; and, in ſhort, every Way qualified to conduct the young Student from his firſt ſetting out on this Study) thought it Pity ſo noble and uſeful a Work ſhould be doomed to a College-Confinement, and obtained Leave to make it Publick. And in order to ſupply what the Author had left undone, ſubjoyned the General and truly Noble Method of extracting the Roots of Equations, publiſhed by Dr.* Halley *in the* Philoſophical Tranſactions, *having firſt procured both thoſe Gentlemen's Leave for his ſo doing.*

As to the publiſhing a Tranſlation of this Book, the Editor is of Opinion, that it is enough to excuſe his Undertaking, that ſuch Great Men were concerned in the Original; and is perſwaded, that the ſame Reaſon which engaged Sir Iſaac *to write, and Mr.* Whiſton *to publiſh the* Latin *Edition, will bear him out in publiſhing this* Engliſh *one: Nor will the Reader require any farther Evidence, that the Tranſlator has done Juſtice to the Original, after I have aſſured him, that Mr.* Ralphſon *and Mr.* Cunn *were both concerned in this Tranſlation.*

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ADVERTISEMENT.

THIS New Edition, in *English*, of Sir ISAAC NEWTON's ALGEBRA, has been very carefully compared with the correct Edition of the Original, that was published in 1722; and suitable Emendations have been every where made accordingly. What was there wanting, is a general Method of finding, in Numbers, the Roots of Equations; and the Doctrine of the *Loci* and by their Means the Geometrical Construction of Equations. The first is supplied by Dr. HALLEY's Method, which is here annexed. The other the Reader may find delivered with great Perspicuity and Elegance, by the *Marquiss de l'Hospital*, in his *Analytical Treatise* of the *Conick Sections*: Where, besides, he will meet with great Variety of very difficult Problems, which are solved after so excellent a Manner, as not to be given over, until they are at length brought to the most elegant Construction they are capable of. As this Part of *Algebra* is the most difficult, so it is the most necessary; and has never, I believe, been handled to any good Purpose, by any Writer whatever, besides that illustrious Author. What *Schoten* has pretended to do on this Head, in general, at the End of *Cartes*'s *Geometry*, is the most clumsy Performance imaginable.

Uni-

Univerſal Arithmetick:

OR, A

TREATISE

OF

Arithmetical COMPOSITION and RESOLUTION.

COMPUTATION is either perform'd by *Numbers*, as in Vulgar Arithmetick, or by *Species*, as uſual among Algebraiſts. They are both built on the ſame Foundations, and aim at the ſame End, *viz.* *Arithmetick* Definitely and Particularly, *Algebra* Indefinitely and Univerſally; ſo that almoſt all Expreſſions that are found out by this Computation, and particularly Concluſions, may be called *Theorems*. But Algebra is particularly excellent in this, that whereas in Arithmetick Queſtions are only reſolv'd by proceeding from given Quantities to the Quantities ſought, Algebra proceeds in a retrograde Order, from the Quantities ſought, as if they were given, to the Quantities given, as if they were ſought, to the end that we may ſome way or other come to a Concluſion or Equation, from which one may bring out the Quantity ſought. And after this Way the moſt difficult Problems are reſolv'd, the Reſolutions whereof would be ſought in vain from only common Arithmetick. Yet Arithmetick in all its Operations is so subservient to Algebra, as that they seem both but to make one perfect *Science of Computing*; and therefore I will explain them both together.

Whoever goes upon this Science, must first understand the Signification of the Terms and Notes, and learn the fundamental Operations, *viz.* Addition, Substraction, Multiplication, and Division; Extraction of Roots, Reduction of Fractions, and of Radical Quantities, and the Methods of ordering the Terms of Equations, and *exterminating the unknown Quantities* (where they are more than one). Then let [the Learner] proceed to exercise himself in these Operations, by bringing Problems to Equations; and, lastly, let him consider the Nature and Resolution of Equations.

Of the Signification of some Words and Notes.

By *Number* we understand not so much a Multitude of Unities, as the abstracted Ratio of any Quantity, to another Quantity of the same Kind, which we take for Unity. And this is threefold; integer, fracted, and surd: An *Integer* is what is measured by Unity, a *Fraction*, that which a submultiple Part of Unity measures, and a *Surd*, to which Unity is incommensurable.

Every one understands the Notes of *whole Numbers*, (0, 1, 2, 3, 4, 5, 6, 7, 8, 9) and the Values of those Notes when more than one are set together. But as Numbers plac'd on the left Hand, next before Unity, denote Tens of Units, in the second Place Hundreds, in the third Place Thousands, &c. so Numbers set in the first Place after Unity, denote tenth Parts of an Unit, in the second Place hundredth Parts, in the third Place thousandth Parts, &c. and these are call'd *Decimal Fractions*, because they always decrease in a Decimal Ratio; and to distinguish the Integers from the Decimals, we place a Comma, or a Point, or a separating Line. Thus the Number 732 ∠ 569 denotes seven hundred thirty-two Units, together with five tenth Parts, six centesimal, or hundredth Parts, and nine millesimal, or thousandth Parts of Unity; which are also written thus, 732, ∠ 569; or thus, 732.569; or also thus, 732 ∠ 569, and so the Number 57104,2083 fifty-seven thousand one hundred and four Units, together with two tenth Parts, eight thousandth Parts, and three ten thousandth Parts of Unity; and the Number 0,064 denotes six centesimals and four millesimal Parts. The Notes of Surds and fracted Numbers are set down in the following Pages.

When the Quantity of any Thing is unknown, or look'd upon as indeterminate, ſo that we cannot expreſs it in Numbers, we denote it by ſome Species, *or by ſome Letter.* And if we conſider known Quantities as indeterminate, we denote them, for diſtinction ſake, with the initial Letters of the Alphabet, as *a*, *b*, *c*, *d*, and the unknown ones by the final ones, *z*, *y*, *x*, &c. Some ſubſtitute Conſonants or great Letters for known Quantities, and Vowels or little Letters for the unknown ones.

Quantities are either Affirmative, *or greater than nothing*; *or* Negative, *or leſs than nothing*. Thus in humane Affairs, Poſſeſſions or Stock may be call'd affirmative Goods, and Debts negative ones. And ſo in local Motion, Progreſſion may be call'd affirmative Motion, and Regreſſion negative Motion; becauſe the firſt augments, and the other diminiſhes the Length of the Way made. And after the ſame manner in Geometry, if a Line drawn any certain Way be reckon'd for Affirmative, then a Line drawn the contrary Way may be taken for Negative. As if AB [*See Fig.* 1.] be drawn to the right, and BC to the left; and AB be reckon'd Affirmative, then BC will be Negative; becauſe in the drawing it diminiſhes AB, and reduces it either to a ſhorter, as AC, or to none, if C chances to fall upon the Point A, or to leſs than none, if BC be longer than AB from which it is taken. A *negative* Quantity is denoted by the Sign —; the Sign + is prefix'd to an *affirmative* one; and ∓ denotes an uncertain Sign, and ± a contrary uncertain one.

In an Aggregate of Quantities the Note + ſignifies, that the Quantity it is prefix'd to, is to be added, *and the Note* —, *that it is* to be ſubtracted. And we uſually expreſs theſe Notes by the Words *Plus* (or *more*) and *Minus* (or *leſs*). Thus $2+3$, or 2 more 3, denotes the Sum of the Numbers 2 and 3, that is 5. And $5-3$, or 5 leſs 3, denotes the Difference which ariſes by ſubducting 3 from 5, that is 2: And $-5+3$ ſignifies the Difference which ariſes from ſubducting 5 from 3, that is -2; and $6-1+3$ makes 8. Alſo $a+b$ denotes the Sum of the Quantities a and b, and $a-b$ the Difference which ariſes by ſubducting b from a; and $a-b+c$ ſignifies the Sum of that Difference, and of the Quantity c. Suppoſe if a be 5, b 2, and c 8, then $a+b$ will be 7, and $a-b$ 3, and $a-b+c$ will be 11. Alſo $2a+3a$ is $5a$, and $3b-2a-b+3a$ is $2b+a$; for $3b-b$ makes $2b$, and $-2a+3a$ makes a, whoſe Aggregate, or Sum, is $2b+a$, and ſo in others. Theſe Notes + and — are called *Signs*. And when neither is prefix'd, the Sign + is always to be underſtood.

Multiplication, properly ſo call'd, is that which is made by Integers, as ſeeking a new Quantity, ſo many times greater than the Multiplicand, as the Multiplier is greater than Unity. But for want of a better Word, that is alſo called *Multiplication*, which is made uſe of in Fractions and Surds, to find a new Quantity in the ſame *Ratio* (whatever it be) to the Multiplicand, as the Multiplier has to Unity. Nor is Multiplication made only by abſtract Numbers, but alſo by concrete Quantities, as by Lines, Surfaces, Local Motion, Weights, &c. as far as theſe being related to ſome known Quantity of their kind, as to Unity, may expreſs the Ratios of Numbers, and ſupply their Place. As is if the Quantity A be to be multiply'd by a Line of 12 Foot, ſuppoſing a Line of 2 Foot to be Unity, there will be produc'd by that Multiplication 6 A, or ſix times A, in the ſame manner as if A were to be multiply'd by the abſtract Number 6; for 6 A is in the ſame Ratio to A, as a Line of 12 Foot has to a Line of 2 Foot. And ſo if you were to multiply any two Lines, A C [*See Fig.* 2.] and A D by one another, take A B for Unity, and draw B C, and parallel to it D E, and A E will be the Product of this Multiplication; becauſe it is to A D as A C to the Unity A B. Moreover, Cuſtom has obtain'd, that the Geneſis or Deſcription of a Surface, by a Line moving at right Angles upon another Line, ſhould be called the Multiplication of thoſe two Lines. For tho' a Line, however multiply'd, cannot become a Surface, and conſequently this Generation of a Surface by Lines is very different from Multiplication, yet they agree in this, that the Number of Unities in either Line, multiply'd by the Number of Unities in the other, produces an abſtracted Number of Unities in the Surface comprehended under thoſe Lines, if the ſuperficial Unity be defin'd as it is uſed to be, *viz.* a Square whoſe Sides are linear Unities. As if the right Line [*Fig.* 3.] A B conſiſt of four Unities, and A C of three, then the Rectangle A D will conſiſt of four times three, or 12 ſquare Unities, as from the Scheme will appear. And there is the like Analogy of a Solid and a Product made by the continual Multiplication of three Quantities. And hence it is, that the Words *to multiply into*, the *Content*, a *Rectangle*, a *Square*, a *Cube*, a *Dimenſion*, a *Side*, and the like, which are Geometrical Terms, are applied to Arithmetical Operations. For by a *Square*, or *Rectangle*, or a *Quantity of two Dimenſions*, we do not always underſtand a Surface, but moſt commonly a Quantity of ſome other kind, which is produc'd by the Multiplication of two other Quantities, and very often

a Line

a Line which is produc'd by the Multiplication of two other Lines. And so we call a *Cube*, or a *Parallelopiped*, or a *Quantity of three Dimensions*, that which is produc'd by two Multiplications. We say likewise the *Side* for a *Root*, and use *Draw* into instead of *Multiply*; and so in others.

A Number prefix'd immediately before any Species, denotes that Species to be so often to be taken. Thus $2a$ denotes two a's, $3b$ three b's, $15x$, fifteen x's.

Two or more Species immediately connected together denote a Product *or Quantity made by the Multiplication of all the Species together.* Thus ab denotes a Quantity made by multiplying a by b, and abx denotes a Quantity made by multiplying a by b, and the Product again by x. As suppose, if a were 2, and b 3 and x 5, then ab would be 6, and abx 30.

Among Quantities multiplying one another, the Sign $\times$, or the Word *by* or *into*, is made use of to denote the Product sometimes. Thus 3×5, or 3 by or into 5 denotes 15; but the chief Use of these Notes is, when compound Quantities are multiply'd together. As if $y-2b$ were to multiply $y+b$, the way is to draw a Line over each Quantity, and then write them thus, $\overline{y-2b}$ into $\overline{y+b}$, or $\overline{y-2b}\times\overline{y+b}$.

Division is properly that which is made use of for integer or whole Numbers, in finding a new Quantity so much less than the Dividend, as Unity is than the Divisor. But by Analogy, the Word may also be used when a new Quantity is sought, that shall be in any such *Ratio* to the Dividend, as Unity has to the Divisor; whether that Divisor be a Fraction or surd Number, or other Quantity of any other kind. Thus to divide the Line [*See Fig.* 4.] AE by the Line AC, AB being Unity, you are to draw ED parallel to CB, and AD will be the Quotient. Moreover, it is call'd *Division*, by reason of a certain Similitude, when a Rectangle is applied to a given Line as a Base, in order thereby to know the Heighth.

One Quantity below another, with a Line interposed, denotes a Quotient, *or a Quantity arising by the Division of the upper Quantity by the lower.* Thus $\frac{6}{2}$ denotes a Quantity arising by dividing 6 by 2, that is 3; and $\frac{5}{8}$ a Quantity arising by the Division of 5 by 8, that is one eighth Part of the Number 5. And $\frac{a}{b}$ denotes a Quantity which arises by dividing a by b; as suppose a was 15 and b 3, then $\frac{a}{b}$ would

denote

denote 5. Likewise thus $\frac{ab-bb}{a+x}$ denotes a Quantity arising by dividing $ab-bb$ by $a+x$. And so in others. These sorts of Quantities are called *Fractions*, and the upper Part is call'd by the Name of the *Numerator*, and the lower is call'd the *Denominator*.

Sometimes the Divisor is set before the divided Quantity, and separated from it by a Mark resembling an Arch of a Circle. Thus to denote the Quantity which arises by the Division of $\frac{axx}{a+b}$ by $a-b$, it may be wrote thus, $\overline{a-b)}\,\frac{axx}{a+b}$.

Although we commonly denote Multiplication by the immediate Conjunction of the Quantities, yet an Integer before a Fraction, denotes the Sum of both. Thus $3\frac{1}{2}$ denotes three and a half.

If a Quantity be multiply'd by itself, the Number of Facts *or Products is, for Shortness sake, set at the Top of the Letter.* Thus for aaa we write a^3, for $aaaa$ a^4, for $aaaaa$ a^5, and for $aaabb$ we write a^3bb, or a^3b^2; as, suppose if a were 5 and b be 2, then a^3 will be $5\times5\times5$ or 125, and a^4 will be $5\times5\times5\times5$ or 625, and a^3b^2 will be $5\times5\times5\times2\times2$ or 500. Where Note, that if a Number be written immediately between two Species, it always belongs to the former; thus the Number 3 in the Quantity a^3bb, does not denote that bb is to be taken thrice, but that a is to be thrice multiply'd by itself. Note, moreover, that these Quantities are said to be of so many *Dimensions*, or of so high a *Power* or *Dignity*, as they consist of Factors or Quantities multiplying one another; and the Number set on forwards at the Top of the Letter is called the Index of those Powers or Dimensions; thus aa is of two Dimensions, or of the 2d Power, and a^3 of three, as the Number 3 at the Top denotes. aa is also called a *Square*, a^3 a *Cube*, a^4 a *Biquadrate* or *squared Square*, a^5 a *Quadrato-Cube*, a^6 a *Cubo-Cube*, a^7 a *Quadrato-Quadrato-Cube* or *Squared-Squared Cube*, and so on: And the Quantity a, by whose Multiplication by itself these Powers are generated, is called their *Root*, *viz.* it is the Square Root of the Square aa, the Cube Root of the Cube aaa, &c.

But when a Root, multiply'd by itself, produces a Square, and that Square, multiply'd again by the Root, produces a Cube, &c. it will be (by the Definition of Multiplication) as Unity to the Root, so that Root to the Square, and that Square to the Cube, &c. And consequently *the square Root*

of

of any Quantity will be a mean Proportional between Unity and that Quantity, and *the Cube Root* the first of two mean Proportionals, and *the Biquadratick Root* the first of three, and so on. Wherefore Roots are known by these two Properties or Affections, first, that by multiplying themselves they produce the superiour Powers; 2dly, that they are mean Proportionals between those Powers and Unity. Thus, 8 is the Square Root of the Number 64, and 4 the Cube Root of it, is hence evident, because 8×8, and $4 \times 4 \times 4$ make 64, or because as 1 to 8, so is 8 to 64, and 1 is to 4 as 4 to 16, and as 16 to 64. And hence, if the Square Root of any Line, as AB [*See Fig.* 5.] is to be extracted, produce it to C, and let BC be Unity; then upon AC describe a Semicircle, and at B erect a Perpendicular, meeting the Circle in D; then will BD be the Root, because it is a mean Proportional between AB and Unity BC.

To denote the Root of any Quantity, we use to prefix this Note $\surd$ *for a Square Root, and this* $\surd\ 3:$ *if it be a Cube Root, and this* $\surd\ 4:$ *for a Biquadratick Root*, &c. Thus $\surd\ 64$ denotes 8; and $\surd\ 3:64$ denotes 4; and $\surd\ aa$ denotes a; and $\surd\ ax$ denotes the Square Root of ax; and $\surd\ 3:4axx$ the Cube Root of $4axx$. As if a be 3, and x 12; then $\surd\ ax$ will be $\surd\ 36$, or 6; and $\surd\ 3:4axx$ will be $\surd\ 3:1728$, or 12. And when these Roots cannot be extracted, the Quantities are called *Surds*, as $\surd\ ax$; or *Surd Numbers*, as $\surd\ 12$.

There are some, that to denote the Square or first Power, make use of q, and of c for the Cube, qq for the Biquadrate, and qc for the Quadrato-Cube, *&c.* After this Manner for the Square, Cube, and Biquadrate of A, they write Aq, Ac, Aqq, &c. and for the Cube Root of $abb - x^3$, they write $\surd\ c:abb - x^3$. Others make use of other sorts of Notes, but they are now almost out of Fashion.

The Mark $=$ *signifies, that the Quantities on each Side of it are* equal. Thus $x = b$ denotes x to be equal to b.

The Note $::$ *signifies that the Quantities on both Sides of it are* proportional. Thus $a.b :: c.d$ signifies, that a is to b as c to d; and $a.b.e :: c.d.f$ signifies that a, b, and e, are to one another respectively, as c, d, and f, are among themselves; or that a to c, b to d, and e to f, are in the same *Ratio*.

Lastly, the Interpretation of any Marks or Signs that may be compounded out of these, will easily be known by Analogy. Thus $\frac{3}{4}a^3bb$ denotes three quarters of a^3bb, and $3\frac{a}{c}$ signifies thrice

thrice $\frac{a}{c}$, and $7\sqrt{ax}$ seven times $\sqrt{ax}$. Also $\frac{a}{b}x$ denotes the Product of x by $\frac{a}{b}$; and $\frac{5ee}{4a+9e}z^2$ denotes the Product made by multiplying z^2 by $\frac{5ee}{4a+9e}$, that is the Quotient arising by the Division of $5ee$ by $4a+9e$; and $\frac{2a^3}{9c}\sqrt{ax}$, that which is made by multiplying $\sqrt{ax}$ by $\frac{2a^3}{9c}$; and $\frac{7\sqrt{ax}}{c}$ the Quotient arising by the Division of $7\sqrt{ax}$ by c; and $\frac{8a\sqrt{cx}}{2a+\sqrt{cx}}$ the Quotient arising by the Division of $8a\sqrt{cx}$ by the Sum of the Quantities $2a+\sqrt{cx}$. And thus $\frac{3axx-x^3}{a+x}$ denotes the Quotient arising by the Division of the Difference $3axx-x^3$ by the Sum $a+x$, and $\sqrt{\frac{3axx-x^3}{a+x}}$ denotes the Root of that Quotient, and $\overline{2a+3c}\sqrt{\frac{3axx-x^3}{a+x}}$ denotes the Product of the Multiplication of that Root by the Sum $2a+3c$. Thus also $\sqrt{\frac{1}{4}aa+bb}$ denotes the Root of the Sum of the Quantities $\frac{1}{4}aa$ and bb, and $\sqrt{\frac{1}{2}a+\sqrt{\frac{1}{4}aa+bb}}$ denotes the Root of the Sum of the Quantities $\frac{1}{2}a$ and $\sqrt{\frac{1}{4}aa+bb}$, and $\frac{2a^3}{aa-zz}\sqrt{\frac{1}{2}a+\sqrt{\frac{1}{4}aa+bb}}$ denotes the Root multiply'd by $\frac{2a^3}{aa-zz}$. And so in other Cases.

But Note, that in complex Quantities of this nature, there is no necessity of giving a particular Attention to, or bearing in your Mind the Signification of each Letter; it will suffice in general to understand, *e. g.* that $\sqrt{\frac{1}{2}a+\sqrt{\frac{1}{4}aa+bb}}$ signifies the Root of the Aggregate or Sum of $\frac{1}{2}a+\sqrt{\frac{1}{4}aa+bb}$; what-

whatever that Aggregate may chance to be, when Numbers or Lines are substituted in the room of Letters. And thus it is as sufficient to understand, that $\frac{\sqrt{\frac{1}{2}a+\sqrt{\frac{1}{4}aa+bb}}}{a-\sqrt{ab}}$ signifies the Quotient arising by the Division of the Quantity $\sqrt{\frac{1}{2}a+\sqrt{\frac{1}{4}aa+bb}}$ by the Quantity $a-\sqrt{ab}$, as much as if those Quantities were simple and known, though at present one may be ignorant what they are, and not give any particular Attention to the Constitution or Signification of each of their Parts; which I thought I ought here to admonish, lest young Beginners should be deterr'd in the very Beginning, by the Complexness of the Terms.

Of ADDITION.

THE Addition of Numbers, where *they are not very compounded*, is manifest of itself. Thus it is at first Sight evident, that 7 and 9 or 7 + 9 make 16, and that 11 + 15 make 26. But in *more compounded Numbers*, the Business is perform'd *by writing the Numbers in a Row downwards, or one under another, and singly collecting the Sums of the Columns.* As if the Numbers 1357 and 172 are to be added, write either of them (suppose 172) under the other 1357, so that the Units of the one, *viz.* 2, may exactly stand under the Units of the other, *viz.* 7, and the other Numbers of the one exactly under the correspondent ones of the other, *viz.* the Place of Tens under Tens, *viz.* 7 under 5, and that of Hundreds, *viz.* 1, under the Place of Hundreds of the other, *viz.* 3. Then beginning at the right hand, say 2 and 7 make 9, which write underneath. Also 7 and 5 make 12, the last of which two Numbers, *viz.* 2, write underneath, and reserve in your mind the other, *viz.* 1, to be added to the two next Numbers, *viz.* 1 and 3. Then say 1 and 1 make 2, which being added to 3 they make 5, which write underneath, and there will remain only 1, the first Figure of the upper Row of Numbers, which also must be writ underneath; and then you have the whole Sum, *viz.* 1529.

$$\begin{array}{r}1357\\172\\\hline1529\end{array}$$

Thus, to add the Numbers 87899 + 13403 + 885 + 1920 into one Sum, write them one under another, so that all the Units may make one Column, the Tens another, the Hundreds a third, and the Places of Thousands a fourth, and so on.

on. Then ſay, 5 and 3 make 8, and 8 + 9 make 17; then write 7 underneath, and the 1 add to the next Rank, ſaying 1 and 8 make 9, 9 + 2 make 11, and 11 + 9 make 20; and having writ the 0 underneath, ſay again as before, 2 and 8 make 10, and 10 + 9 make 19, and 19 + 4 make 23, and 23 + 8 make 31; then reſerving 3 in your Memory, write down 1 as before, and ſay again, 3 + 1 make 4, 4 + 3 make 7, and 7 + 7 make 14, wherefore write underneath 4, and laſtly ſay 1 + 1 make 2, and 2 + 8 make 10, which in the laſt Place write down, and you will have the Sum of them all.

$$\begin{array}{r} 87899 \\ 13403 \\ 1920 \\ 885 \\ \hline 104107 \end{array}$$

After the ſame manner we alſo add Decimals, as in the following Example may be ſeen:

$$\begin{array}{r} 630,953 \\ 51,0807 \\ 305,25 \\ \hline 987,3037 \end{array}$$

Addition *is perform'd in Algebraick Terms or Species, by connecting the Quantities to be added with their proper Signs, and moreover by uniting into one Sum thoſe that can be ſo united.* Thus a and b make $a+b$; a and $-b$ make $a-b$; $-a$ and $-b$ make $-a-b$; $7a$ and $9a$ make $7a+9a$; $-a\sqrt{ac}$ and $b\sqrt{ac}$ make $-a\sqrt{ac}+b\sqrt{ac}$, or $b\sqrt{ac}-a\sqrt{ac}$; for it is all one, in what Order ſoever they are written.

Affirmative Quantities, which agree in Species, are united together, by adding the prefix'd Numbers that are multiply'd into thoſe Species. Thus $7a+9a$ make $16a$. And $11bc+15bc$ make $26bc$. Alſo $3\frac{a}{c}+5\frac{a}{c}$ make $8\frac{a}{c}$, and $2\sqrt{ac}+7\sqrt{ac}$ make $9\sqrt{ac}$, and $6\sqrt{ab-xx}+7\sqrt{ab-xx}$ make $13\sqrt{ab-xx}$. And in like manner, $6\sqrt{3}+7\sqrt{3}$ make $13\sqrt{3}$. Moreover $a\sqrt{ac}+b\sqrt{ac}$ make $\overline{a+b}\sqrt{ac}$, by adding together a and b as Numbers multiplying $\sqrt{ac}$. And ſo $\overline{2a+3c}\sqrt{\frac{3axx-x^3}{a+x}}+3a\sqrt{\frac{3axx-x^3}{a+x}}$ make $\overline{5a+3c}\sqrt{\frac{3axx-x^3}{a+x}}$ becauſe $2a+3c$ and $3a$ make $5a+3c$.

Affir-

Affirmative Fractions, that have the same Denominator, are united by adding their Numerators. Thus $\frac{1}{5}+\frac{2}{5}$ make $\frac{3}{5}$, and $\frac{2ax}{b}+\frac{3ax}{b}$ make $\frac{5ax}{b}$; and thus $\frac{8a\sqrt{cx}}{2a+\sqrt{cx}}+\frac{17a\sqrt{cx}}{2a+\sqrt{cx}}$ make $\frac{25a\sqrt{cx}}{2a+\sqrt{cx}}$, and $\frac{aa}{c}+\frac{bx}{c}$ make $\frac{aa+bx}{c}$.

Negative Quantities are added after the same way as Affirmative. Thus -2 and -3 make -5; $-\frac{4ax}{b}$ and $-\frac{11ax}{b}$ make $-\frac{15ax}{b}$; $-a\sqrt{ax}$ and $-b\sqrt{ax}$ make $-\overline{a-b}\sqrt{ax}$. But when a *Negative Quantity is to be added to an Affirmative one*, the Affirmative must be diminish'd by the Negative one. Thus 3 and -2 make 1; $\frac{11ax}{b}$ and $-\frac{4ax}{b}$ make $\frac{7ax}{b}$; $-a\sqrt{ac}$ and $b\sqrt{ac}$ make $\overline{b-a}\sqrt{ac}$. And Note, that when the Negative Quantity is greater than the Affirmative, the Aggregate or Sum will be Negative. Thus 2 and -3 make -1; $-\frac{11ax}{b}$ and $\frac{4ax}{b}$ make $-\frac{7ax}{b}$, and $2\sqrt{ac}$ and $-7\sqrt{ac}$ make $-5\sqrt{ac}$.

In the Addition of a greater Number of Quantities, or more compound ones, it will be convenient to observe the Method or Form of Operation we have laid down above in the Addition of Numbers. As if $17ax-14a+3$, and $4a+2-8ax$, and $7a-9ax$, were to be added together, dispose them so in Columns, that the Terms that contain the same Species may stand in a Row one under another, *viz.* the Numbers 3 and 2 in one Column, the Species $-14a$, and $4a$, and $7a$, in another Column, and the Species $17ax$ and $-8ax$ and $-9ax$ in a third. Then I add the Terms of each Column by themselves, saying 2 and 3 make 5, which I write underneath, then $7a$ and

$$\begin{array}{rrr} 17ax & -14a & +3 \\ -8ax & +4a & +2 \\ -9ax & +7a & \\ \hline * & -3a & +5 \end{array}$$

and 4 a make 11 a and moreover — 14 a make — 3 a, which I alſo write underneath, laſtly, — 9 ax and — 8 ax make — 17 ax, to which 17 ax added make 0. And ſo the Sum comes out — 3 a + 5.

After the ſame manner the Buſineſs is done in the following Examples.

$$\begin{array}{r} 12x + 7a \\ 7x + 9a \\ \hline 19x + 16a \end{array} \qquad \begin{array}{r} 11bc - 7\sqrt{ac} \\ 15bc + 2\sqrt{ac} \\ \hline 26bc - 5\sqrt{ac} \end{array} \qquad \begin{array}{r} -\frac{4ax}{b} + 6\sqrt{3} + \frac{1}{5} \\ +\frac{11ax}{b} - 7\sqrt{3} + \frac{2}{5} \\ \hline \frac{7ax}{b} - \sqrt{3} + \frac{3}{5} \end{array}$$

$$\begin{array}{r} -6xx + \frac{3}{7}x \\ 5x^3 + \frac{5}{7}x \\ \hline 5x^3 - 6xx + \frac{8}{7}x \end{array}$$

$$\begin{array}{rrrrr} & & aay & + 2a^3 & - \frac{a^4}{2y} \\ & -2ayy & -4aay & - a^3 & \\ y^3 & +2ayy & -\frac{1}{2}aay & & \\ \hline y^3 & * & -3\frac{1}{2}aay & + 3a^3 & - \frac{a^4}{2y} \end{array}$$

$$\begin{array}{rrrl} 5x^4 & + 2ax^3 & & \\ -3x^4 & - 2ax^3 & + 8\frac{1}{4}a^3\sqrt{aa+xx} & \\ -2x^4 & + 5bx^3 & - 20a^3\sqrt{aa-xx} & \\ & - 4bx^3 & - 7\frac{1}{4}a^3\sqrt{aa+xx} & \\ \hline * & bx^3 & + a^3\sqrt{aa+xx} & - 20a^3\sqrt{aa-xx}. \end{array}$$

Of SUBTRACTION.

THE Invention of the Difference of Numbers that are not *too much compounded*, is of itſelf evident; as if you take 9 from 17, there will remain 8. But in *more compounded Numbers*, Subtraction is perform'd *by ſubſcribing* or ſetting underneath *the Subtrahend*, *and ſubtracting each of the lower Figures from each of the upper ones*. Thus to ſubtract 63543 from 782579, having ſubſcrib'd 63543, ſay, 3 from 9 and there remains 6, which write underneath; and 4 from 7 and there remains 3, which write likewiſe underneath; then

5 from

5 from 5 and there remains nothing, which in like manner ſet underneath; then 3 comes to be taken from 2; but becauſe 3 is greater than 2, you muſt borrow 1 from the next Figure 8, which, together with 2, make 12, from which 3 may be taken and there will remain 9, which write likewiſe underneath; and then when beſides 6 there is alſo 1 to be taken from 8, add the 1 to the 6, and the Sum 7 being taken from 8, there will be left 1, which in like manner write underneath. Laſtly, when in the lower Rank of Numbers there remains nothing to be taken from 7, write underneath the 7, and ſo you have the Difference 719036.

782579
63543
719036

But eſpecial Care is to be taken, that the Figures of the Subtrahend be placed or ſubſcribed in their proper or homogeneous Places; viz. the Units of the one under the Units of the other, and the Tens under the Tens, and likewiſe the Decimals under the Decimals, *&c.* as we have ſhewn in Addition. Thus to take the Decimal, 0,63 from the Integer 547, they are not to be diſpoſed thus $\begin{matrix}547\\0{,}63\end{matrix}$, but thus $\begin{matrix}547\\0{,}63\end{matrix}$; *viz.* ſo that the 0, which ſupplies the Place of Units in the Decimal, muſt be placed under the Units of the other Number. Then 0 being underſtood to ſtand in the empty Places of the upper Number, ſay, 3 from 0, which ſince it cannot be, 1 ought to be borrow'd from the foregoing Place, which will make 10, from which 3 is to be taken, and there remains 7, which write underneath. Then that 1 which was borrow'd added to 6 make 7, and this is to be taken from 0 above it; but ſince that cannot be, you muſt again borrow 1 from the foregoing Place to make 10, then 7 from 10 leaves 3, which in like manner is to be writ underneath; then that 1 being added to 0, makes 1, which 1 being taken from 7 leaves 6, which again write underneath. Laſtly, write the two Figures 54 (ſince nothing remains to be taken from them) underneath, and you'll have the Remainder 546,37.

547
0,63
546,37

For Exerciſe ſake, we here ſet down ſome more Examples, both in Integers and Decimals.

1673	1673	458074	35,72	46,5003	308,7
1541	1580	9205	14,32	3,078	25,74
132	93	448869	21,4	43,4223	282,96

If a greater Number is to be taken from a less, you must first subtract the less from the greater, and then prefix a negative Sign to the Remainder. As if from 1541 you are to subtract 1673, on the contrary I subtract 1541 from 1673, and to the Remainder 132 I prefix the Sign —.

In Algebraick Terms, Subtraction *is perform'd by connecting the Quantities, after having changed all the Signs of the Subtrahend, and by uniting those together which can be united,* as we have done in Addition. Thus $+7a$ from $+9a$ leaves $9a-7a$ or $2a$; $-7a$ from $+9a$ leaves $+9a+7a$, or $16a$; $+7a$ from $-9a$ leaves $-9a-7a$, or $-16a$; and $-7a$ from $-9a$ leaves $-9a+7a$, or $-2a$; so $3\frac{a}{c}$ from $5\frac{a}{c}$ leaves $2\frac{a}{c}$; $7\sqrt{ac}$ from $2\sqrt{ac}$ leaves $-5\sqrt{ac}$; $\frac{2}{9}$ from $\frac{5}{9}$ leaves $\frac{3}{9}$; $-\frac{4}{7}$ from $\frac{3}{7}$ leaves $\frac{7}{7}$; $-\frac{2ax}{b}$ from $\frac{3ax}{b}$ leaves $\frac{5ax}{b}$; $\frac{8a\sqrt{cx}}{2a+\sqrt{cx}}$ from $\frac{-17a\sqrt{cx}}{2a+\sqrt{cx}}$ leaves $\frac{-25a\sqrt{cx}}{2a+\sqrt{cx}}$; $\frac{aa}{c}$ from $\frac{bx}{c}$ leaves $\frac{bx-aa}{c}$; $a-b$ from $2a+b$ leaves $2a+b-a+b$, or $a+2b$; $3az-zz+ac$ from $3az$ leaves $3az-3az+zz-ac$ or $zz-ac$; $\frac{2aa-ab}{c}$ from $\frac{aa+ab}{c}$ leaves $\frac{aa+ab-2aa+ab}{c}$, or $\frac{-aa+2ab}{c}$; and $\overline{a-x}\sqrt{ax}$ from $\overline{a+x}\sqrt{ax}$ leaves $\overline{a+x-a+x}\sqrt{ax}$, or $2x\sqrt{ax}$, and so in others. But where Quantities consist of more Terms, the Operation may be managed as in Numbers, as in the following Examples:

12 x

$$\begin{array}{r} 12x + 7a \\ 7x + 9a \\ \hline 5x - 2a \end{array} \qquad \begin{array}{r} 15bc + 2\sqrt{ac} \\ -11bc + 7\sqrt{ac} \\ \hline 26bc - 5\sqrt{ac} \end{array} \qquad \begin{array}{l} 5x^3 + \frac{5}{7}x \\ 6x^2 - \frac{3}{7}x \\ \hline 5x^3 - 6xx + \frac{8}{7}x \end{array}$$

$$\begin{array}{l} \frac{11ax}{b} - 7\sqrt{3} + \frac{2}{5} \\ \frac{4ax}{b} - 6\sqrt{3} - \frac{1}{5} \\ \hline \frac{7ax}{b} - \sqrt{3} + \frac{3}{5} \end{array}$$

Of MULTIPLICATION.

NUMBERS which arise or are produced by the Multiplication of any two Numbers, not greater than 9, are to be learnt and retain'd in the Memory: As that 5 into 7 makes 35, and that 8 by 9 make 72, &c. and then the Multiplication of greater Numbers is to be perform'd after the Rule of these Examples.

If 795 is to be multiply'd by 4, write 4 underneath, as you see here. Then say, 4 into 5 makes 20, whose last Figure, *viz.* 0, set under the 4, and reserve the former 2 for the next Operation. Say moreover, 4 into 9 makes 36, to which add the former 2, and there is made 38, whose latter Figure 8 write underneath as before, and reserve the former 3. Lastly, say, 4 into 7 makes 28, to which add the former 3 and there is made 31, which being also set underneath, you'll have the Number 3180, which comes out by multiplying the whole 795 by 4.

$$\begin{array}{r} 795 \\ 4 \\ \hline 3180 \end{array}$$

Moreover, if 9043 be to be multiply'd by 2305, write either of them, *viz.* 2305 under the other 9043 as before, and multiply the upper 9043 first by 5, after the Manner shewn, and there will come out 45215; then by 0, and there will come out 0000; thirdly, by 3, and there will come out 27129; lastly, by 2, and there will come out 18086. Then dispose these Numbers so coming out in a descending Series, or under one another, so that the last Figure of every lower Row shall stand one Place nearer to the left Hand than the last of the next superiour Row. Then add all these

$$\begin{array}{r} 9043 \\ 2305 \\ \hline 45215 \\ 0000 \\ 27129 \\ 18086 \\ \hline 20844115 \end{array}$$

together,

together, and there will arise 20844115, the Number that is made by multiplying the whole 9043 by the whole 2305.

In the same manner *Decimals* are multiply'd by Integers or other Decimals, or both, as you may see in the following Examples:

72,4	50,18	3,0925
29	2,75	0,0132
6516	25090	78050
1448	35126	117075
2099,6	10036	39025
	137,9950	0,05151300

But Note, *in the Number coming out, or the Product, so many Figures must be cut off to the right Hand for Decimals, as there are Decimal Figures both in the Multiplyer and the Multiplicand.* And if by Chance there are not so many Figures in the Product, the deficient Places must be fill'd up to the left Hand with o's, as here in the third Example.

Simple Algebraick Terms are multiply'd by multiplying the Numbers into the Numbers, and the Species into the Species, and by making the Product Affirmative, *if both the Factors are Affirmative. or both Negative; and* Negative *if otherwise.*

Thus $2a$ into $3b$, or $-2a$ into $-3b$ make $6ab$, or $6ba$: for it is no matter in what order they are placed. Thus also $2a$ by $-3b$, or $-2a$ by $3b$ make $-6ab$. And thus, $2ac$ into $8bcc$ make $16abccc$, or $16abc^3$; and $7axx$ into $-12aaxx$ make $-84a^3x^4$; and $-16cy$ into $31ay^3$ make $-496acy^4$; and $-4z$ into $-3\sqrt{az}$ make $12z$ $\sqrt{az}$. And so 3 into -4 make -12, and -3 into -4 make 12.

Fractions are multiply'd, by multiplying their Numerators by their Numerators, and their Denominators by their Denominators. Thus $\frac{2}{5}$ into $\frac{3}{7}$ make $\frac{6}{35}$; and $\frac{a}{b}$ into $\frac{c}{d}$ make $\frac{ac}{bd}$; and $2\frac{a}{b}$ into $3\frac{c}{d}$ make $6\times\frac{a}{b}\times\frac{c}{d}$, or $6\frac{ac}{bd}$; and $\frac{3acy}{2bb}$ into $\frac{-7cyy}{4b^3}$ make $\frac{-21accy^3}{8b^5}$; and $\frac{-4z}{c}$

into

into $\frac{-3\sqrt{az}}{c}$ make $\frac{12z\sqrt{az}}{cc}$; and $\frac{a}{b}x$ into $\frac{c}{d}xx$ make $\frac{ac}{bd}x^3$. Alſo 3 into $\frac{2}{5}$ make $\frac{6}{5}$ as may appear, if 3 be reduced to the Form of a Fraction, *viz.* $\frac{3}{1}$ by making uſe of Unity for the Denominator. And thus $\frac{15aaz}{cc}$ into $2a$ make $\frac{30a^3z}{cc}$. Whence note by the way, that $\frac{ab}{c}$ and $\frac{a}{c}b$ are the ſame; as alſo $\frac{abx}{c}$, $\frac{ab}{c}x$, and $\frac{a}{c}bx$, alſo $\frac{\overline{a+b}\sqrt{cx}}{c}$ and $\frac{a+b}{a}\sqrt{cx}$; and ſo in others.

Radical Quantities of the ſame Denomination (that is, if they are both Square Roots, or both Cube Roots, or both Biquadratick Roots, &c.) are multiply'd by multiplying the Terms together under the ſame Radical Sign. Thus $\sqrt{3}$ into $\sqrt{5}$ make $\sqrt{15}$; and $\sqrt{ab}$ into $\sqrt{cd}$ make $\sqrt{abcd}$; and $\sqrt{:3}\,5ayy$ into $\sqrt{:3}\,7ayz$ make $\sqrt{:3}\,35aay^3z$; and $\sqrt{\frac{a^3}{c}}$ into $\sqrt{\frac{abb}{c}}$ make $\sqrt{\frac{a^4bb}{cc}}$ that is * $\frac{aab}{c}$. And $2a\sqrt{az}$ into $3b\sqrt{az}$ make $6ab\sqrt{aazz}$, that is $6aabz$; and $\frac{3xx}{\sqrt{ac}}$ into $\frac{-2x}{\sqrt{ac}}$ make $\frac{-6x^3}{\sqrt{aacc}}$, that is $\frac{-6x^3}{ac}$; and $\frac{-4x\sqrt{ab}}{7a}$ into $\frac{-3dd\sqrt{5cx}}{10ee}$ make $\frac{12ddx\sqrt{5abcx}}{70aee}$.

Quantities that conſiſt of ſeveral Parts, are multiply'd by multiplying all the Parts of the one into all the Parts of the other, as is ſhewn in the Multiplication of Numbers. Thus, $c-x$ into a make $ac-ax$, and $aa+2ac-bc$ into $a-b$ make $a^3+2aac-aab-3bac+bbc$. For $aa+2ac-bc$ into $-b$ make $-aab-2acb+bbc$, and into a make $a^3+2aac-abc$, the Sum whereof is

* See the Chapter of Notation.

$a^3 + 2aac - aab - 3abc + bbc$. A Specimen of this Sort of Multiplication, together with other like Examples, you have underneath:

$$\begin{array}{r}
aa + 2ac - bc \\
a - b \\
\hline
- aab - 2abc + bbc \\
a^3 + 2aac - abc \qquad\qquad \\
\hline
a^3 + 2aac - aab - 3abc + bbc
\end{array}
\qquad
\begin{array}{r}
a + b \\
a + b \\
\hline
ab + bb \\
aa + ab \qquad \\
\hline
aa + 2ab + bb
\end{array}$$

$$\begin{array}{r}
a + b \\
a - b \\
\hline
- ab - bb \\
aa + ab \qquad \\
\hline
aa \quad * - bb
\end{array}
\qquad
\begin{array}{r}
yy + 2ay - \tfrac{1}{2}aa \\
yy - 2ay + aa \\
\hline
aayy + 2a^3y - \tfrac{1}{2}a^4 \\
- 2ay^3 - 4aayy + a^3y \qquad \\
y^4 + 2ay^3 - \tfrac{1}{2}aayy \qquad\qquad \\
\hline
y^4 \quad * \quad - 3\tfrac{1}{2}aayy + 3a^3y - \tfrac{1}{2}a^4
\end{array}$$

$$\begin{array}{l}
\dfrac{2ax}{c} - \sqrt{\dfrac{a^3}{c}} \\
3a + \sqrt{\dfrac{abb}{c}} \\
\hline
\dfrac{2ax}{c}\sqrt{\dfrac{abb}{c}} - \dfrac{aab}{c} \\
\dfrac{6aax}{c} - 3a\sqrt{\dfrac{a^3}{c}} \\
\hline
\dfrac{6aax}{c} - 3a\sqrt{\dfrac{a^3}{c}} + \dfrac{2ax}{c}\sqrt{\dfrac{aab}{c}} - \dfrac{aab}{c}.
\end{array}$$

Of

Of DIVISION.

DIVISION *is performed in Numbers, by seeking how many times the Divisor is contained in the Dividend, and as often subtracting, and writing so many Units in the Quotient; and by repeating that Operation upon Occasion, as often as the Divisor can be subtracted.*

Thus, to divide 63 by 7, seek how many times 7 is contained in 63, and there will come out precisely 9 for the Quotient; and consequently $\frac{63}{7}$ is equal to 9. Moreover, to divide 371 by 7, prefix the Divisor 7, and beginning at the first Figures of the Dividend, coming as near them as possible, say, how many times 7 is contained in 37, and you will find 5; then writing 5 in the Quotient, subtract 5 × 7, or 35, from 37, and there will remain 2, to which set the last Figure of the Dividend, *viz.* 1; and then 21 will be the remaining Part of the Dividend for the next Operation; say therefore as before, how many times 7 is contained in 21? and the Answer will be 3; wherefore writing 3 in the Quotient, take 3 × 7, or 21, from 21 and there will remain 0. Whence it is manifest, that 53 is precisely the Number, that arises from the Division of 371 by 7.

7) 371 (53
35
21
21
0

And thus to divide 4798 by 23, first beginning with the initial Figures 47, say, how many times is 23 contained in 47? Answer 2; wherefore write 2 in the Quotient, and from 47 subtract 2 × 23, or 46, and there will remain 1, to which join the next Number of the Dividend, *viz.* 9, and you will have 19 to work upon next Say therefore, how many times is 23 contained in 19? Answer 0; wherefore write 0 in the Quotient; and from 19 subtract 0 × 23, or 0, and there remains 19, to which join the last Number 8, and you will have 198 to work upon next.

23) 4798 (208,6086, &c.
46
19
00
198
184
140
138
20
00
200
184
160

 Where-

Wherefore in the last Place say, how many times is 23 contained in 198 (which may be guess'd at from the first Figures of each, 2 and 19, by taking notice how many times 2 is contain'd in 19)? I answer 8; wherefore write 8 in the Quotient, and from 198 subtract 8 × 23, or 184, and there will remain 14 to be farther divided by 23; and so the Quotient will be $208\frac{14}{23}$. And if this Fraction is not liked, you may continue the Division in Decimal Fractions as far as you please, by adding always a Cypher to the remaining Number. Thus to the Remainder 14 add 0, and it becomes 140. Then say, how many times 23 in 140? Answer 6; write therefore 6 in the Quotient; and from 140 subtract 6 × 23, or 138, and there will remain 2; to which set a Cypher (or 0) as before. And thus the Work being continued as far as you please, there will at length come out this Quotient, *viz.* 208,6086, *&c.*

After the same Manner the Decimal Fraction 3,5218 is divided by the Decimal Fraction 46,1, and there comes out 0,07639, *&c.* *Where note, that there must be so many Figures cut off in the Quotient, for Decimals, as there are more in the last Dividend than in the Divisor:* As in this Example 5, because there are 6 in the last Dividend, *viz.* 0,004370, and 1 in the Divisor 46,1.

```
46,1) 3,5218 (0,07639
      322,7
      -----
        2948
        2766
        ----
         1820
         1383
         ----
          4370
```

We have here subjoin'd more Examples, for Clearness sake, *viz.*

```
9043) 20844115 (2305.
      18086
      -----
       27581
       27129
       -----
         45215
         45215
         -----
             0
```

```
72,4) 2099,6 (29
      1448
      ----
       6516
       6516
       ----
          0
```

```
50,18) 137,995 (2,75        0,0132) 0,051513 (3,9025
       10036                        396
       -----                        ----
       37635                        1191
       35126                        1188
       -----                        -----
        25090                         330
        25090                         264
        -----                         ---
            0                          660
                                       660
                                       ---
                                         0
```

In Algebraick Terms Division is performed by the Resolution of what is compounded by Multiplication. Thus, ab divided by a gives for the Quotient b, $6ab$ divided by $2a$ gives $3b$; and divided by $-2a$ gives $-3b$. $-6ab$ divided by $2a$ gives $-3b$; and divided by $-2a$ gives $3b$. $16abc^2$ divided by $2ac$ gives $8bcc$. $-84a^3x^4$ divided by $-12aaxx$ gives $7axx$. Likewise $\frac{6}{35}$ divided by $\frac{2}{5}$ gives $\frac{3}{7}$. $\frac{ac}{bd}$ divided by $\frac{a}{b}$ gives $\frac{c}{d}$. $\frac{-21accy^3}{8b^5}$ divided by $\frac{3acy}{2bb}$ gives $\frac{-7cyy}{4b^3}$. $\frac{6}{5}$ divided by 3 gives $\frac{2}{5}$; and reciprocally $\frac{6}{5}$ divided by $\frac{2}{5}$ gives $\frac{3}{1}$, or 3. $\frac{30a^3z}{cc}$ divided by $2a$ gives $\frac{15aaz}{cc}$; and reciprocally divided by $\frac{15aaz}{cc}$ gives $2a$. Likewise $\sqrt{15}$ divided by $\sqrt{3}$ gives $\sqrt{5}$. $\sqrt{abcd}$ divided by $\sqrt{cd}$ gives $\sqrt{ab}$. $\sqrt{a^3c}$ by $\sqrt{ac}$ gives $\sqrt{aa}$, or a. $\sqrt{3:}\,35aay^3z$ divided by $\sqrt{3:}\,5ayy$ gives $\sqrt{3:}\,7ayz$. $\sqrt{\frac{a^4bb}{cc}}$ divided by $\frac{a^3}{c}$ gives $\sqrt{\frac{abb}{c}}$. $\frac{12ddx\sqrt{5abcx}}{70aee}$ divided by $\frac{-3dd\sqrt{5cx}}{10ee}$ gives $\frac{-4x\sqrt{ab}}{7a}$. And so $\overline{a+b}\sqrt{ax}$ divided by $a+b$ gives $\sqrt{ax}$; and reciprocally divided by

$\sqrt{ax}$

$\sqrt{ax}$ gives $a+b$. And $\frac{a}{a+b}\sqrt{ax}$ divided by $\frac{1}{a+b}$ gives $a\sqrt{ax}$, or divided by a gives $\frac{1}{a+b}\sqrt{ax}$, or $\frac{\sqrt{ax}}{a+b}$; and reciprocally divided by $\frac{\sqrt{ax}}{a+b}$ gives a. But in Divisions of this sort you are to take care, that the Quantities divided by one another be of the same kind, *viz.* that Numbers be divided by Numbers, and Species by Species, Radical Quantities by Radical Quantities, Numerators of Fractions by Numerators, and Denominators by Denominators; also in Numerators, Denominators, and Radical Quantities, the Quantities of each kind must be divided by homogeneous ones, or Quantities of the same kind.

Now if the Quantity to be divided cannot be thus resolved by the Divisor proposed, it is sufficient, when both the Quantities are Integers, to write the Divisor underneath, with a Line between them. Thus to divide ab by c, write $\frac{ab}{c}$; and to divide $\overline{a+b}\sqrt{cx}$ by a, write $\frac{\overline{a+b}\sqrt{cx}}{a}$, or $\frac{a+b}{a}\sqrt{cx}$. And so $\sqrt{ax-xx}$ divided by $\sqrt{cx}$ gives $\frac{\sqrt{ax-xx}}{\sqrt{cx}}$, or $\sqrt{\frac{ax-cx}{cx}}$. And $\overline{aa+ab}\sqrt{aa-2xx}$ divided by $\overline{a-b}\sqrt{aa-xx}$ gives $\frac{aa+ab}{a-b}\sqrt{\frac{ax-2xx}{aa-xx}}$. And $12\sqrt{5}$ divided by $4\sqrt{7}$ gives $3\sqrt{\frac{5}{7}}$.

But when these Quantities are Fractions, multiply the Numerator of the Dividend into the Denominator of the Divisor, and the Denominator into the Numerator, and the *first Product* will be the Numerator, and the *latter* the Denominator of the Quotient. Thus to divide $\frac{a}{b}$ by $\frac{c}{d}$ write $\frac{ad}{bc}$, that is, multiply a by d and b by c. In like manner,

$\frac{3}{7}$

$\frac{3}{7}$ by $\frac{5}{4}$ gives $\frac{12}{35}$. And $\frac{3a}{4c}\sqrt{ax}$ divided by $\frac{2c}{5a}$ gives $\frac{15aa}{8cc}\sqrt{ax}$, and divided by $\frac{2c\sqrt{aa-xx}}{5a\sqrt{ax}}$ gives $\frac{15a^3x}{8cc\sqrt{aa-xx}}$. After the same manner, $\frac{ad}{b}$ divided by $c\left(\text{or by } \frac{c}{1}\right)$ gives $\frac{ad}{bc}$. And $c\left(\text{or } \frac{c}{1}\right)$ divided by $\frac{ad}{b}$ gives $\frac{bc}{ad}$. And $\frac{3}{7}$ divided by 5 gives $\frac{3}{35}$. And 3 divided by $\frac{5}{4}$ gives $\frac{12}{5}$. And $\frac{a+b}{c}\sqrt{cx}$ divided by a gives $\frac{a+b}{ac}\sqrt{cx}$. And $\overline{a+b}\sqrt{cx}$ divided by $\frac{a}{c}$ gives $\frac{ac+bc}{a}\sqrt{cx}$. And $2\sqrt{\frac{axx}{c}}$ divided by $3\sqrt{cd}$ gives $\frac{2}{3}\sqrt{\frac{axx}{ccd}}$; and divided by $3\sqrt{\frac{cd}{x}}$ gives $\frac{2}{3}\sqrt{\frac{ax^3}{ccd}}$. And $\frac{1}{5}\sqrt{\frac{7}{11}}$ divided by $\frac{1}{2}\sqrt{\frac{3}{7}}$ gives $\frac{2}{5}\sqrt{\frac{49}{33}}$, and so in others.

A Quantity compounded of several Terms, is divided by dividing each of its Terms by the Divisor. Thus $aa + 3ax - xx$ divided by a gives $a + 3x - \frac{xx}{a}$. But when the Divisor consists also of several Terms, the Divsion is perform'd as in Numbers. Thus to divide $a^3 + 2aac - aab - 3abc + bbc$ by $a - b$, say, how many times is a contained in a^3, *viz.* the first Term of the Divisor in the first Term of the Dividend? Answer aa. Wherefore write aa in the Quotient; and having subtracted $a - b$ multiply'd into aa, or $a^3 - aab$ from the Dividend, there will remain $2aac - 3abc + bbc$ yet to be divided. Then say again, how many times a in $2aac$? Answer $2ac$. Wherefore write also $2ac$ in the Quotient, and having subtracted $a - b$ into $2ac$, or $2aac - 2abc$ from the aforesaid Remainder, there will yet remain $-abbc + bc$. Wherefore say again, how many times a in $-abc$? Answer $-bc$, and then write

write $-bc$ in the Quotient; and having, in the laſt Place, ſubtracted $+a-b$ into $-bc$, *viz.* $-abc+bbc$ from the laſt Remainder, there will remain nothing; which ſhews that the Diviſion is at an end, and the Quotient coming out $aa+2ac-bc$.

But that theſe Operations may be duly reduced to the Form which we uſe in the Diviſion of Numbers, *the Terms both of the Dividend and the Diviſor muſt be diſpoſed in order, according to the Dimenſions of that Letter which is judged moſt proper for the Operation*; ſo that thoſe Terms may ſtand firſt, in which that Letter is of moſt Dimenſions, and thoſe in the ſecond Place whoſe Dimenſions are next higheſt; and ſo on to thoſe wherein that Letter is not at all involv'd, or into which it is not at all multiply'd, which ought to ſtand in the laſt Place. Thus in the Example we juſt now brought, if the Terms are diſpoſed according to the Dimenſions of the Letter a, the following Diagram will ſhew the Form of the Work, *viz.*

$$a-b)\; a^3 {+2aac \atop -aab} - 3abc + bbc \;(aa + 2ac - bc$$
$$\underline{a^3 - aab}$$
$$0 + 2aac - 3abc$$
$$\underline{2aac - 2abc}$$
$$0 - abc + bbc$$
$$\underline{- abc + bbc}$$
$$0 \qquad 0$$

Where may be ſeen, that the Term a^3, or a of three Dimenſions, ſtands in the firſt Place of the Dividend, and the Terms ${2aac \atop -aab}$, in which a is of two Dimenſions, ſtand in the ſecond Place, and ſo on. The Dividend might alſo have been writ thus;

$$a^3 {+2c \atop -b} aa - 3bca + bbc.$$

Where the Terms that ſtand in the ſecond Place are united, by collecting together the Factors of the Letter according to which the Order is made. And thus if the Terms were to be diſpoſed according to the Dimenſions of the Letter b, the Buſineſs muſt be performed as in the following Diagram, the Explication whereof we ſhall here ſubjoin.

$-b$

$$-b+a)\,cbb\begin{matrix}-3ac\\-aa\end{matrix}b\begin{matrix}+a^3\\+2aac\end{matrix}\left(-cb\begin{matrix}+2ac\\+aa\end{matrix}\right.$$

$$\begin{array}{l}
\underline{cbb-acb}\\
0\begin{matrix}-2ac\\-aa\end{matrix}b\begin{matrix}+a^3\\+2aac\end{matrix}\\
\underline{\begin{matrix}-2ac\\-aa\end{matrix}b\begin{matrix}+2aac\\+a^3\end{matrix}}\\
\quad 0 \qquad\quad 0
\end{array}$$

Say, How many times is $-b$ contain'd in cbb? Anſwer $-cb$. Wherefore having writ $-cb$ in the Quotient, ſubtract $\overline{-b+a}\times -cb$, or $bbc-abc$, and there will remain in the ſecond Place $\begin{matrix}-2ac\\-aa\end{matrix}b$. To this Remainder add, if you pleaſe, the Quantities that ſtand in the laſt Place, *viz.* $\begin{matrix}a^3\\2aac,\end{matrix}$ and ſay again, how many times is $-b$ contain'd in $\begin{matrix}-2ac\\-aa\end{matrix}b$? Anſwer $\begin{matrix}+2ac\\+aa\end{matrix}$. Theſe therefore being writ in the Quotient, ſubtract $-b+a$ multiply'd by $\begin{matrix}+2ac\\+aa\end{matrix}$ or $\begin{matrix}-2ac\\-aa\end{matrix}b\begin{matrix}+2aac\\+a^3\end{matrix}$, and there will remain nothing. Whence it is manifeſt, that the Diviſion is at an End, the Quotient coming out $-cb+2ac+aa$, as before.

nd thus, if you were to divide $aay^4-aac^4+yyc^4+y^6-2y^4cc-a^6-2a^4cc-a^4yy$ by $yy-aa-cc$. I order or place the Quantities according to the Dimenſions of the Letter y, thus:

$$yy \begin{matrix} -aa \\ -cc \end{matrix}\Big)\, y^6 \begin{matrix} +aa \\ -2cc \end{matrix} y^4 \begin{matrix} -a^4 \\ +c^4 \end{matrix} yy \begin{matrix} -a^6 \\ -2a^4cc. \\ -aac^4 \end{matrix}$$

Then I divide as in the following Diagram. Here are added other Examples, in which you are to take Notice, that where the Dimensions of the Letter, which this Method of ordering ranges, don't always proceed in the same Arithmetical Progression, but sometimes interrupted, in the defective Places this Mark $*$ is put.

$$yy \begin{matrix} -aa \\ -cc \end{matrix}\Big)\, y^6 \begin{matrix} +aa \\ -2cc \end{matrix} y^4 \begin{matrix} -a^4 \\ +c^4 \end{matrix} yy \begin{matrix} -a^6 \\ -2a^4cc. \\ -aac^4 \end{matrix}$$

$$y^6 \begin{matrix} -aa \\ -cc \end{matrix} y^4 \qquad \Big(y^4 \begin{matrix} +2aa \\ -cc \end{matrix} y^2 \begin{matrix} +a^4 \\ +aacc. \end{matrix}$$

$$0 \begin{matrix} +2aa \\ -cc \end{matrix} y^4 \begin{matrix} -2a^4 \\ -aacc \\ +c^4 \end{matrix} y^2$$

$$0 \qquad \begin{matrix} +a^4 \\ +aacc \end{matrix} y^2$$

$$\begin{matrix} +a^4 \\ +aacc \end{matrix} y^2 \begin{matrix} -a^6 \\ -2a^4cc \\ -aac^4 \end{matrix}$$

$$0 \qquad 0$$

$$a+b)\ aa * -bb\ (a-b$$

$$aa+ab$$

$$0-ab$$

$$-ab-bb$$

$$0 \qquad 0$$

$$
\begin{array}{l}
yy - 2ay + aa)\; y^4 * - 3\tfrac{1}{2}aayy + 3a^3y - \tfrac{1}{2}a^4 \;(yy + 2ay - \tfrac{1}{2}aa \\
\qquad\qquad \underline{y^4 - 2ay^3 + aayy} \\
\qquad\qquad 0 + 2ay^3 - 4\tfrac{1}{2}aayy \\
\qquad\qquad\quad \underline{+ 2ay^3 - 4\,aayy + 2a^3y} \\
\qquad\qquad\qquad 0 \;-\; \tfrac{1}{2}aayy + a^3y \\
\qquad\qquad\qquad\quad \underline{- \tfrac{1}{2}aayy + a^3y - \tfrac{1}{2}a^4} \\
\qquad\qquad\qquad\qquad 0 \qquad 0 \qquad 0
\end{array}
$$

$$
\begin{array}{l}
aa + ab\sqrt{2} + bb)\; a^4 \;*\; *\; *\; + b^4 \;(aa - ab\sqrt{2} + bb \\
\qquad\qquad \underline{a^4 + a^3b\sqrt{2} + aabb} \\
\qquad\qquad\quad - a^3b\sqrt{2} - aabb \\
\qquad\qquad\quad \underline{- a^3b\sqrt{2} - 2aabb - ab^3\sqrt{2}} \\
\qquad\qquad\qquad\qquad + aabb + ab^3\sqrt{2} \\
\qquad\qquad\qquad\qquad \underline{+ aabb + ab^3\sqrt{1} + b^4} \\
\qquad\qquad\qquad\qquad\quad 0 \qquad 0 \qquad 0
\end{array}
$$

Some begin Division from the last Terms, but it comes to the same Thing, if, inverting the Order of the Terms, you begin from the first. There are also other Methods of dividing, but it is sufficient to know the most easy and commodious.

Of EXTRACTION of ROOTS.

WHEN *the Square Root* of any Number is to be extracted, *it is first to be noted with Points in every other Place, beginning from Unity; then you are to write down such a Figure for the Quotient, or Root, whose Square shall be equal to, or nearest, less than the Figure or Figures to the first Point. And then subtracting that Square, the other Figures of the Root will be found one by one, by dividing the Remainder by the double of the Root as far as extracted, and each Time taking from that Remainder the*

Square of the Figure that laſt came out, and the Decuple of the aforeſaid Diviſor augmented by that Figure.

Thus to extract the Root out of 99856, firſt Point it after this Manner, 99856; then ſeek a Number whoſe Square ſhall equal the firſt Figure 9, *viz.* 3, and write it in the Quotient; and then having ſubtracted from 9, 3×3, or 9, there will remain 0; to which ſet down the Figures to the next Point, *viz.* 98 for the following Operation. Then taking no Notice of the laſt Figure 8, ſay, How many times is the Double of 3, or 6, contained in the firſt Figure 9? Anſwer 1; wherefore having writ 1 in the Quotient, ſubtract the Product of 1 × 61, or 61, from 98, and there will remain 37, to which connect the laſt Figures 56, and you will have the Number 3756, in which the Work is next to be carried on. Wherefore alſo neglecting the laſt Figure of this, *viz.* 6, ſay, How many times is the double of 31, or 62, contained in 375, (which is to be gueſſed at from the initial Figures 6 and 37, by taking Notice how many times 6 is contained in 37?) Anſwer 6; and writing 6 in the Quotient, ſubtract 6 × 626, or 3756, and there will remain 0; whence it appears that the Buſineſs is done; the Root coming out 316.

```
99856 (316
9
098
 61
 3756
 3756
    0
```

Otherwiſe with the Diviſors ſet down it will ſtand thus:

```
    99856 (316
    9
  6)98
    61
 62)3756
    3756
       0
```

And ſo in others.

And ſo if you were to extract the Root out of 22178791, firſt having pointed it, ſeek a Number, whoſe Square (if it cannot be exactly equalled) ſhall be the next leſs Square to 22, the Figures to the firſt Point, and you will find

find it to be 4. For 5 × 5, or 25, is greater than 22; and 4 × 4, or 16, less; wherefore 4 will be the first Figure of the Root. This therefore being writ in the Quotient, from 22 take the Square 4 × 4, or 16, and to the Remainder 6 adjoin moreover the next Figures 17, and you will have 617, from whose Division by the double of 4 you are to obtain the second Figure of the Root; *viz.* neglecting the last Figure 7, say, how many times is 8 contained in 61? Answer 7; wherefore write 7 in the Quotient, and from 617 take the Product of 7 into 87, or 609, and there will remain 8; to which join the two next Figures 87, and you will have 887, by the Division whereof by the double of 47, or 94, you are to obtain the third Figure; as say, How many times is 94 contain'd in 88? Answer 0; wherefore write 0 in the Quotient, and adjoin the two last Figures 91, and you will have 88791, by whose Division by the double of 470, or 940, you are to obtain the last Figure, *viz.* say, How many times 940 in 8879? Answer 9; wherefore write 9 in the Quotient, and you will have the Root 4709.

```
22178791 (4709,43637 &c.
16
617
609
  88791
  84681
   411000
   376736
    3426400
    2825649
     60075100
     56513196
      356190400
      282566169
       73624231
```

But since the Product 9 × 9409 or 84681 subtracted from 88791, leaves 4110, that is a Sign that the Number 4709 is not the Root of the Number 22178791 precisely, but that it is a little less. And in this Case, and in others like it, if you desire the Root should approach nearer, you must carry on the Operation in Decimals, by adding to the Remainder two Cyphers in each Operation. Thus the Remainder 4110 having two Cyphers added to it, becomes 411000; by the Division whereof by the double of 4709 or 9418, you will have the first Decimal Figure 4. Then having writ 4 in the Quotient, subtract 4 × 94184, or 376736 from 411000, and there will remain 34264. And so having added two more Cyphers, the Work may be carried

ried on at Pleaſure, the Root at length coming out 4709,43637, &c.

But when the Root is carried on half-way, or above, the reſt of the Figures may be obtained by Diviſion alone. As in this Example, if you had a mind to extract the Root to nine Figures, after the five former 4709,4 are extracted, the four latter may be had, by dividing the Remainder by the double of 4709,4.

And after this manner, if the Root of 32976 was to be extracted to five Places in Numbers: After the Figures are pointed, write 1 in the Quotient, as being the Figure whoſe Square 1 × 1, or 1, is the greateſt that is contained in 3 the Figure to the firſt Point; and having taken the Square of 1 from 3, there will remain 2; then having ſet the two next Figures, *viz*, 29 to it, (*viz.* to 2) ſeek how many times the double of 1, or 2, is contained in 22, and you will find indeed that it is contained more than ten times; but you are never to take your Diviſor ten times, no, nor nine times in this Caſe; becauſe the Product of 9 × 29, or 261, is greater than 229, from which it would be to be taken.

```
      . . .
      32976 (181,59
      1
      ---
   2) 229
      224
   ---------
  36) 576
      361
   ---------
 362) 215 (59, &c.
```

Wherefore ſay only 8: And then having writ 8 in the Quotient, and ſubtracted 8 × 28, or 224, there will remain 5; and having ſet down to this the Figures 76, ſeek how many times the double of 18, or 36, is contained in 57, and you will find 1, and ſo write 1 in the Quotient; and having ſubtracted 1 × 361, or 361 from 576, there will remain 215. Laſtly, to obtain the remaining Figures, divide this Number 215 by the Double of 181, or 362, and you will have the Figures 59, which being writ in the Quotient, you will have the Root 181,59.

After the ſame way Roots are alſo extracted out of Decimal Numbers. Thus the Root of 329,76 is 18,159; and the Root of 3,2976 is 1,8152; and the Root of 0,032976 is 0,18159, and ſo on. But the Root of 3297,6 is 57,4247; and the Root of 32,976 is 5,74247. And thus ths Root of 9,9856 is 3,16. But the Root of 0,99856 is 0,999279, &c. as will appear from the following Diagrams.

```
       3297,60 (57,4247 &c.
       25
  10)  797
       749
 114)  4860
       4576
1148)  284 (247
```

```
      9,9856 (3,16
      9
  6)  98
      61
 62)  3756
      3756
      0
```

```
       0,998560 (0,999279 &c.
        81
  18)   1885
        1701
 198)   18460
        17901
1998)   559 (279
```

I will comprehend the Extraction of the Cubick Root, and of all others, under one general Rule, consulting rather the Ease of understanding the Praxis than the Expeditiousness of it, lest I should too much retard the Learner in Things that are of no frequent Use, *viz. every third Figure beginning from Unity is first of all to be pointed, if the Root to be extracted be a* Cubick one; *or every fifth, if it be a* Quadrato-Cubick, *or of the fifth Power*, &c. *and then such a Figure is to be writ in the Quotient, whose greatest Power* (i. e. *whose* Cube, *if it be a* Cubick *Power, or whose* Quadrato-Cube, *if it be the* fifth Power, *&c.*) *shall either be equal to the Figure or Figures before the first Point, or the next less; and then having subtracted that Power, the next Figure will be found by dividing the Remainder augmented by the next Figure of the Resolvend, by the next greatest Power of the Quotient, multiplied by the Index of the Power to be extracted, that is, by the* triple Square *of the Quotient, if the Root be a* Cubic one; *or by the* quintuple Biquadrate, *i. e.* five times the Biquadrate *if the Root be of* the fifth Power, *&c. And having again subtracted the greatest Power of the whole Quotient from the first Resolvend, the third Figure will be found by dividing that Remainder augmented by the next Figure of the Resolvend, by the next greatest Power of the whole Quotient multiplied by the Index of the Power to be extracted*; and so on *in infinitum*.

Thus

Thus to extract the Cube Root of 13312053, the Number is firſt to be pointed after this manner, *viz.* 13312053. Then you are to write in the Quotient the Figure 2, whoſe Cube 8 is the next leſs Cube to the Figures 13, [which is not a perfect Cube Number] or to the firſt Point; and having ſubtracted that Cube, there will remain 5; which being augmented by the next Figure of the Reſolvend 3, and divided by the triple Square of the Quotient 2, by ſeeking how many times 3×4, or 12, is contained in 53, it gives 4 for the ſecond Figure of the Quotient. But ſince the Cube of the Quotient 24, *viz.* 13824 would come out too great to be ſubtracted from the Figures 13312 that preceed the ſecond Point, there muſt only 3 be writ in the Quotient. Then the Quotient 23 being in a ſeparate Paper or Place multiplied by 23 gives the Square 529, which again multiplied by 23 gives the Cube 12167, and this taken from 13312, will leave 1145; which augmented by the next Figure of the Reſolvend 0, and divided by the triple Square of the Quotient 23, *viz.* by ſeeking how many times 3×529, or 1587, is contained in 11450, it gives 7 for the third Figure of the Quotient. Then the Quotient 237, multiplied by 237, gives the Square 56169, which again multiplied by 237 gives the Cube 13312053, and this taken from the Reſolvend leaves 0. Whence it is evident that the Root ſought is 237.

```
                      13312053 (237
Subtract the Cube     8
          12) rem.    53 (4 or 3
    Subtract Cube     12167
      1587) rem.      11450 (7
    Subtract Cube     13312053
         Remains      0
```

And ſo to extract the Quadrato-Cubical Root of 36430820, it muſt be pointed over every fifth Figure, and the Figure 3, whoſe Quadrato-Cube [or fifth Power] 243 is the next leſs to 364, *viz.* to the firſt Point muſt be writ in the Quotient. Then the Quadrato-Cube 243 being ſubtracted from 364, there remains 121, which augmented by the next Figure of the Reſolvend, *viz.* 3, and divided by five times the Biquadrate of the Quotient, *viz.* by ſeeking how many times 5×81, or 405, is contained in

```
            36430820 (32,5
            243
      405)  1213 (2
            33554432
  5242880)  2876388,0 (5
```

in 1213, it gives 2 for the second Figure. That Quotient 32 being thrice multiplied by itself, makes the Biquadrate 1048576; and this again multiplied by 32, makes the Quadrato-Cube 33554432, which being subtracted from the Resolvend leaves 2876388. Therefore 32 is the Integer Part of the Root, but not the exact Root; wherefore, if you have a mind to prosecute the Work in Decimals, the Remainder, augmented by a Cypher, must be divided by five times the aforesaid Biquadrate of the Quotient, by seeking how many times 5×1048576, or 5242880, is contained in 2876388,0, and there will come out the third Figure, or the first Decimal 5. And so by subtracting the Quadrato-Cube of the Quotient 32,5 from the Resolvend, and dividing the Remainder by five times its Biquadrate, the fourth Figure may be obtained. And so on *in infinitum.*

When the Biquadratick Root is to be extracted, you may extract twice the Square Root, because $\sqrt{4}$ is as much as $\sqrt{2 \times 2}$. *And when the Cubo-Cubick Root is to be extracted, you may first extract the Cube-Root, and then the Square-Root of that Cube-Root, because* $\sqrt{6}$ is the same as $\sqrt{2 \times 3}$; *whence some have called these Roots not Cubo-Cubick ones, but Quadrato-Cubes.* And the same is to be observed in other Roots, whose Indexes are not prime Numbers.

The Extraction of Roots out of simple Algebraick Quantities, is evident, even from the Notation itself; as that $\sqrt{aa}$ is a, and that $\sqrt{aacc}$ is ac, and that $\sqrt{9aacc}$ is $3ac$, and that $\sqrt{49a^4xx}$ is $7aax$. And also that $\sqrt{\frac{a^4}{cc}}$, or $\frac{\sqrt{a^4}}{\sqrt{cc}}$ is $\frac{aa}{c}$, and that $\sqrt{\frac{a^4bb}{cc}}$ is $\frac{aab}{c}$, and that $\sqrt{\frac{9aazz}{25bb}}$ is $\frac{3az}{5b}$, and that $\sqrt{\frac{4}{9}}$ is $\frac{2}{3}$, and that $\sqrt{3}\,\frac{8b^6}{27a^3}$ is $\frac{2bb}{3a}$, and that $\sqrt{4}\,aabb$ is $\sqrt{ab}$. Moreover, that $b\sqrt{aacc}$, or b into $\sqrt{aacc}$, is b into ac or abc. And that $3c\sqrt{\frac{9aazz}{25bb}}$ is $3c \times \frac{3az}{5b}$, or $\frac{9acz}{5b}$. And that $\frac{a+3x}{c}\sqrt{\frac{4bbx^4}{81aa}}$ is $\frac{a+3x}{c} \times \frac{2bxx}{9a}$, or $\frac{2abxx+6bx^3}{9ac}$.

I say, these are all evident, because it will appear, at first Sight, that the proposed Quantities are produced by multiplying the Roots into themselves (as aa from $a \times a$, $aacc$ from ac into ac, $9aacc$ from $3ac$ into $3ac$, &c.) But when Quantities consist of several Terms, the Business is performed as in Numbers. Thus, to extract the Square Root out of $aa + 2ab + bb$, in the first Place, write the Root of the first Term aa, viz. a in the Quotient, and having subtracted its Square $a \times a$, there will remain $2ab + bb$ to find the Remainder of the Root by. Say therefore, How many times is the double of the Quotient, or $2a$, contained in the first Term of the Remainder $2ab$? I answer b; therefore write b in the Quotient, and having subtracted the Product of b into $\overline{2a + b}$, or $2ab + bb$, there will remain nothing. Which shews that the Work is finished, the Root coming out $a + b$.

$$\begin{array}{l} aa + 2ab + bb \;(a + b \\ aa \\ \overline{} \\ 0 \\ \quad + 2ab + bb \\ \quad \overline{\quad\; 0 \qquad 0 \quad} \end{array}$$

And thus, to extract the Root out of $a^4 + 6a^3b + 5a^2bb - 12ab^3 + 4b^4$; first, set in the Quotient the Root of the first Term a^4, viz. aa, and having subtracted its Square $aa \times aa$, or a^4, there will remain $6a^3b + 5aabb - 12ab^3 + 4b^4$ to find the Remainder of the Root. Say therefore, How many times is $2aa$ contained in $6a^3b$? Answer $3ab$; wherefore write $3ab$ in the Quotient, and having subtracted the Product of $3ab$ into $2aa + 3ab$, or $6a^3b + 9aabb$, there will yet remain $-4aabb - 12ab^3 + 4b^4$ to carry on the Work. Therefore say again, How many times is the Double of the Quotient, *viz.* $2aa + 6ab$ contained in $-4aabb - 12ab^3$, or, which is the same Thing, say, How many times is the Double of the first Term of the Quotient, or $2aa$, contained in the first Term of the Remainder $-4aabb$? Answer $-2bb$. Then having writ $-2bb$ in the Quotient, and subtracted the Product $-2bb$ into $2aa + 6ab - 2bb$, or $-4aabb - 12ab^3 + 4b^4$, there will remain nothing. Whence it follows, That the Root is $aa + 3ab - 2bb$.

$a^4 +$

$$\begin{array}{l} a^4 + 6a^3b + 5aabb - 12ab^3 + 4b^4 \; (aa + 3ab - 2bb \\ \underline{a^4} \\ 0 \\ \quad \underline{+6a^3b + 9aabb} \\ \quad\quad 0 \quad -4aabb \\ \quad\quad\quad \underline{-4aabb - 12ab^3 + 4b^4} \\ \quad\quad\quad 0 \quad\quad 0 \quad\quad 0 \end{array}$$

And thus the Root of the Quantity $xx - ax + \frac{1}{4}aa$ is $x - \frac{1}{2}a$; and the Root of the Quantity $y^4 + 4y^3 - 8y + 4$ is $yy + 2y - 2$; and the Root of the Quantity $16a^4 - 24aaxx + 9x^4 + 12bbxx - 16aabb + 4b^4$ is $3xx - 4aa + 2bb$, as may appear by the Diagrams underneath:

$$\begin{array}{l} xx - ax + \frac{1}{4}aa \; (x - \frac{1}{2}a \\ \underline{xx} \\ 0 \\ \quad \underline{-ax + \frac{1}{4}aa} \\ \quad\quad 0 \quad\quad 0 \end{array}$$

$$\begin{array}{l} 9x^4 \begin{matrix} -24aa \\ +12bb \end{matrix} x^2 \begin{matrix} +16a^4 \\ -16aab^2 \\ +4b^4 \end{matrix} \left(3x^2 \begin{matrix} -4aa \\ +2bb \end{matrix}\right. \\ \underline{9x^4} \\ 0 \\ \quad \underline{\begin{matrix} -24aa \\ +12bb \end{matrix} x^2 \begin{matrix} +16a^4 \\ -16a^2b^2 \\ +4b^4 \end{matrix}} \\ \quad\quad 0 \quad\quad\quad 0 \end{array}$$

$$\begin{array}{l} y^4 + 4y^3 * - 8y + 4 \; (yy + 2y - 2 \\ \underline{y^4} \\ 0 \\ \quad \underline{4y^3 + 4yy} \\ \quad 0 \quad -4yy \\ \quad\quad \underline{-yy - 8y + 4} \\ \quad\quad 0 \quad 0 \quad 0 \end{array}$$

If you would extract the Cube Root of $a^3 + 3aab + 3abb + b^3$, the Operation is performed thus:

$$\begin{array}{l} a^3 + 3aab + 3abb + b^3 \ (a+b \\ a^3 \\ 3aa)\ \overline{0 + 3aab}\ (b \\ \overline{a^3 + 3aab + 3abb + b^3} \\ 0 \quad 0 \quad 0 \quad 0 \end{array}$$

Extract first the Cube Root of the first Term a^3, viz: a, and set it down in the Quotient: Then, subtracting its Cube a^3, say, How many times is its triple Square, or $3aa$, contained in the next Term of the Remainder $3aab$? and there comes out b; wherefore write b in the Quotient, and subtracting the Cube of the Quotient, there will remain 0. Therefore $a + b$ is the Root.

After the same manner, if the Cube Root is to be extracted out of $z^6 + 6z^5 - 40z^3 + 96z - 64$, it will come out $zz + 2z - 4$. And so in higher Roots.

Of the REDUCTION *of* FRACTIONS *and* RADICAL *Quantities.*

THE Reduction of Fractions and Radical Quantities is of Use in the preceding Operations, and that either to the least Terms, or to the same Denomination.

Of the REDUCTION *of* FRACTIONS *to the least Terms.*

FRACTIONS *are reduced to the least Terms, by dividing the Numerators and Denominators by the greatest Divisor.* Thus the Fraction $\frac{aac}{bc}$ is reduced to a more Simple one $\frac{aa}{b}$ by dividing both aac and bc by c; and $\frac{203}{667}$ is reduced to a more Simple one $\frac{7}{23}$ by dividing both 203 and 667 by 29; and $\frac{203aac}{667bc}$ is reduced to $\frac{7aa}{23b}$ by dividing

viding by $29c$. And so $\frac{6a^3-9acc}{6aa+3ac}$ becomes $\frac{2aa-3cc}{2a+c}$ by dividing by $3a$. And $\frac{a^3-aab+abb-b^3}{aa-ab}$ becomes $\frac{aa+bb}{a}$ by dividing by $a-b$.

And after this Method, the Terms after Multiplication or Division, may be for the most part abridged. As if you were to multiply $\frac{2ab^3}{3ccd}$ by $\frac{9acc}{bdd}$, or divide it by $\frac{bdd}{9acc}$, there will come out $\frac{18aab^3cc}{3bccd^3}$, and by Reduction $\frac{6aabb}{d^3}$. But in these Cases, it is better to abbreviate the Terms before the Operation, by dividing those Terms first by the greatest common Divisor, which you would be obliged to do afterwards. Thus, in the Example before us, if I divide $2ab^3$ and bdd by the common Divisor b, and $3ccd$ and $9acc$ by the common Divisor $3cc$, there will come out the Fraction $\frac{2abb}{d}$ to be multiplied by $\frac{3a}{dd}$, or to be divided by $\frac{dd}{3a}$, there coming out $\frac{6aabb}{d^3}$ as above. And so $\frac{aa}{c}$ into $\frac{c}{b}$ becomes $\frac{aa}{1}$ into $\frac{1}{b}$, or $\frac{aa}{b}$. And $\frac{aa}{c}$ divided by $\frac{b}{c}$ becomes aa divided by b, or $\frac{aa}{b}$. And $\frac{a^3-axx}{xx}$ into $\frac{cx}{aa+ax}$ becomes $\frac{a-x}{x}$, into $\frac{c}{1}$, or $\frac{ac}{x}-c$. And 28 divided by $\frac{7}{3}$ becomes 4 divided by $\frac{1}{3}$, or 12.

Of

Of the Invention of Divisors.

TO this Head may be referred the Invention of Divisors, by which any Quantity may be divided. *If it be a simple Quantity, divide it by its least Divisor, and the Quotient by its least Divisor, till there remain an indivisible Quotient, and you will have all the prime Divisors of that Quantity. Then multiply together each Pair of these Divisors, each Ternary or three of them, each Quaternary,* &c. *and you will also have all the compounded Divisors.* As, if all the Divisors of the Number 60 are required, divide it by 2, and the Quotient 30 by 2, and the Quotient 15 by 3, and there will remain the indivisible Quotient 5. Therefore the prime Divisors are 1, 2, 2, 3, 5; those composed of the Pairs 4, 6, 10, 15; of the Ternaries 12, 20, 30; and of all of them 60. Again, If all the Divisors of the Quantity $21abb$ are desired, divide it by 3, and the Quotient $7abb$ by 7, and the Quotient abb by a, and the Quotient bb by b, and there will remain the prime Quotient b. Therefore the prime Divisors are 1, 3, 7, a, b, b; and those composed of the Pairs 21, $3a$, $3b$, $7a$, $7b$, ab, bb; those composed of the Ternaries $21a$, $21b$, $3ab$, $3bb$, $7ab$, $7bb$, abb; and those of the Quaternaries $21ab$, $21bb$, $3abb$, $7abb$; that of the Quinaries $21abb$. After the same Way all the Divisors of $2abb - 6aac$ are 1, 2, a, $bb - 3ac$, $2a$, $2bb - 6ac$, $abb - 3aac$, $2abb - 6aac$.

If after a Quantity is divided by all its simple Divisors, it remains still compounded, and you suspect it has some compounded Divisor, dispose it according to the Dimensions of any of the Letters in it, and in the Room of that Letter substitute successively three or more Terms of this Arithmetical Progression, viz. 3, 2, 1, 0, −1, −2, *and set the resulting Terms together with all their Divisors, by the corresponding Terms of the Progression, setting down also the Signs of the Divisors, both Affirmative and Negative. Then set also down the Arithmetical Progressions which run through the Divisors of all the Numbers proceeding from the greater Terms to the less, in the Order that the Terms of the Progression* 3, 2, 1, 0, −1, −2 *proceed, and whose Terms differ either by Unity, or by some Number which divides the highest Term of the Quantity proposed. If any Progression of this kind occurs, that Term of it which stands in the same Line with the Term* 0 *of the first Progression, divided by the Difference of the Terms, and join-*

ed

ed with its Sign to the aforesaid Letter, will compose the Quantity by which you are to attempt the Division.

As if the Quantity be $x^3 - xx - 10x + 6$, by substituting, one by one, the Terms of this Progression 1. 0. — 1, for x, there will arise the Numbers — 4, 6, + 14, which, together with all their Divisors, I place right against the Terms of the Progression 1. 0. — 1. after this Manner.

1	4	1. 2. 4.	+ 4.
0	6	1. 2. 3. 6	+ 3.
— 1	14	1. 2. 7. 14	+ 2.

Then, because the highest Term x^3 is divisible by no Number but Unity, I seek among the Divisors a Progression whose Terms differ by Unity, and (proceeding from the highest to the lowest) decrease as the Terms of the lateral Progression 1. 0. — 1. And I find only one Progression of this Sort, *viz.* 4. 3. 2. whose Term therefore + 3 I chuse, which stands in the same Line with the Term 0 of the first Progression 1. 0. — 1. and I attemp the Division by $x + 3$, and find it succeeds, there coming out $xx - 4x + 2$.

Again, if the Quantity be $6y^4 - y^3 - 21yy + 3y + 20$, for y I substitute successively 2. 1. 0. — 1. — 2. and the resulting Numbers 30. 7. 20. 3. 34. with all their Divisors, I place by them as follows.

2	30	1. 2. 3. 5. 6. 10. 15. 30	+ 10.
1	7	1. 7.	+ 7.
0	20	1. 2. 4. 5. 10. 20	+ 4.
— 1	3	1. 3.	+ 1.
— 2	34	1. 2. 17. 34	— 2.

And among the Divisors I perceive there is this decreasing Arithmetical Progression + 10. + 7. + 4. + 1. — 2. The Difference of the Terms of this Progression, *viz.* 3, divides the highest Term of the Quantity $6y^4$. Wherefore I adjoin to the Letter y the Term + 4, which stands in the Row opposite to the Term 0, divided by the Difference of the Terms, *viz.* 3, and I attempt the Division by $y + \frac{4}{3}$, or, which is the same Thing, by $3y + 4$, and the Business succeeds, there coming out $2y^3 - 3yy - 3y + 5$.

And so, if the Quantity be $24a^5 - 50a^4 + 49a^3 - 140a^2 + 64a + 30$; the Operation will be as follows.

2 | 24

2	42	1. 2. 3. 6. 7. 14. 21. 42	+ 3. + 3. + 7.
1	23	1. 23.	+ 1. — 1. + 1.
0	30	1. 2. 3. 5. 6. 10. 15. 30.	— 1. — 5. — 5.
— 1	297	1. 3. 9. 11. 27. 33. 99. 297	— 3. — 9. — 11.

Here are three Progressions, whose Terms — 1. — 5. — 5, divided by the Differences of the Terms 2, 4, 6, give three Divisors to be tried $a - \frac{1}{2}$, $a - \frac{5}{4}$; and $a - \frac{5}{6}$. And the Division by the last Divisor $a - \frac{5}{6}$, or $6a - 5$, succeeds, there coming out $4a^4 - 5a^3 + 4aa - 20a - 6$.

If no Divisor occur by this Method, or none that divides the Quantity proposed, we are to conclude, that that Quantity does not admit a Divisor of one Dimension. But perhaps it may, if it be a Quantity of more than three Dimensions, admit a Divisor of two Dimensions. And if so, that Divisor will be found by this Method. *Substitute in that Quantity for the Letter or Species as before, four or more Terms of this Progression* 3, 2, 1, 0. — 1. — 2. — 3. *Let all the Divisors of the Numbers that result be singly added to and subtracted from the Squares of the correspondent Terms of that Progression, multiplied into some Numeral Divisor of the highest Term of the Quantity proposed, and right against the Progression let be placed the Sums and Differences. Then note all the collateral Progressions which run through those Sums and Differences. Then suppose* ∓ C *to be a Term of such like Progressions that stands against the Term* 0 *of the first Progression, and* ∓ B *the Difference which arises by subducting* ∓ C *from the next superior Term which stands against the Term* 1 *of the first Progression, and* A *to be the aforesaid Numeral Divisor of the highest Term, and* l *to be the Letter which is the proposed Quantity, then* $All \pm Bl \pm C$ *will be the Divisor to be tried.*

Thus suppose the proposed Quantity to be $x^4 - x^3 - 5xx + 12x - 6$, for x I write successively 3, 2, 1, 0. — 1, — 2, and the Numbers that come out 39. 6. 1. — 6. — 21. — 26. I dispose or place together with their Divisors in the same Line with them, and I add and subtract the Divisors to and from the Squares of the Terms of the first Progression, multiplied by the Numeral Divisor of the Term x^4, which is Unity, *viz.* to and from the Terms 9. 4. 1. 0. 1. 4, and I dispose likewise the Sums and Differences on the Side. Then I write, as follows, the Progressions which occur among the same. Then I make use of the Terms of these Progressions

2 and — 3, which ſtands oppoſite to the Term o in that Progreſſion which is in the firſt Column, ſucceſſively for ∓ C,

3	39	1. 3. 13. 39	9	—30. —4. 6. 8. 10. 12. 22. 48	—4.	6
2	6	1. 2. 3. 6	4	—2. 1. 2. 3. 5. 6. 7. 10.	—2.	3
1	1	1.	1	0. 2.	0.	0
0	6	1. 2. 3. 6	0	—6. —3. —2. —1. 1. 2. 3. 6	2.	—3
—1	21	1. 3. 7. 21	1	—20. —6. —2. 0. 2. 4. 8. 22	4.	—6
—2	26	1. 2. 13. 26	4	—22. 9. 2. 3. 5. 6. 17. 30	6.	—9

and I make uſe of the Differences that ariſe by ſubtracting theſe Terms from the ſuperior Terms o and o, *viz.* — 2 and + 3 reſpectively for ∓ B. Alſo Unity for A; and x for l. And ſo in the room of $All \pm Bl \pm C$, I have theſe two Diviſors to try, viz. $xx + 2x - 2$, and $xx - 3x + 3$, by both of which the Buſineſs ſucceeds.

Again, if the Quantity $3y^5 - 6y^4 + y^3 - 8yy - 14y + 14$ be propoſed, the Operation will be as follows. Firſt, I attempt the Buſineſs by adding and ſubtracting the Diviſors to and from the Squares of the Terms of the Progreſſion 2. 1. 0. — 1, making uſe of 1 for A, but the Buſineſs does not

3	170		27		—7.	17
2	38	1. 2. 19. 38	12	—16. —7. 10. 11. 13. 14. 31. 50.	—7.	—11
1	10	1. 2. 5. 10	3	—7. 2. 1. 2. 4. 5. 8. 13.	—7.	5
0	14	1. 2. 7. 14	0	—14. —7. —2. —1. 1. 2. 7. 14.	—7.	—1
—1	10	1. 2. 5. 10	3	—7. —2. 1. 2. 4. 5. 8. 13.	—7.	—7
—2	190		12		—7.	—13

ſucceed. Wherefore in the room of A, I make uſe of 3, the other numeral Diviſor of the higheſt Term $3y^5$; and theſe Squares being multiplied by 3, I add and ſubtract the Diviſors to and from the Products, *viz.* 12. 3. 0. 3, and I find theſe two Progreſſions in the reſulting Terms, — 7. — 7. — 7. — 7, and 11. 5. — 1. — 7. For Expedition ſake, I neglected the Diviſors of the outermoſt Terms 170 and 190. Wherefore, the Progreſſions being continued upwards and downwards, I take the next Terms, *viz.* — 7 and 17 at the Top, and — 7 and — 13 at Bottom, and I try if theſe being ſubducted from the Numbers 27 and 12, which ſtand againſt them in the 4th Column, their Differences divide thoſe Numbers 170 and 190, which ſtand againſt them in the ſecond Column. And the Difference between 27 and — 7, that is, 34, divides 170; and the Difference of 12 and — 7, that is, 19, divides 190.

 Alſo

Also the Difference between 27 and 17, that is, 10, divides 170; but the Difference between 12 and — 13, that is, 25, does not divide 190. Wherefore I reject the latter Progression. According to the former, $\mp$ C is — 7, and $\mp$ B is nothing; the Terms of the Progression having no Difference. Wherefore the Divisor to be tried $All \pm Bl \pm C$ will be $3yy + 7$. And the Division succeeds, there coming out $y^3 - 2yy - 2y + 2$.

If after this way, there can be found no Divisor which succeeds, we are to conclude, that the proposed Quantity will not admit of a Divisor of two Dimensions. The same Method may be extended to the Invention of Divisors of more Dimensions, by seeking in the aforesaid Sums and Differences not Arithmetical Progressions, but some others, the first, second, and third, &c. Differences of whose Terms are in Arithmetical Progression: But the Learner ought not to be detained about them.

Where there are two Letters in the proposed Quantity, and all its Terms ascend to equally high Dimensions, put Unity for one of those Letters; then, by the preceding Rules, seek a Divisor, and compleat the deficient Dimensions of this Divisor, by restoring that Letter for Unity.

As if the Quantity be $6y^4 - cy^3 - 21ccyy + 3c^3y + 20c^4$, where all the Terms are of four Dimensions; for c I put 1, and the Quantity becomes $6y^4 - y^3 - 21yy + 3y + 20$, whose Divisor, as above, is $3y + 4$; and having compleated the deficient Dimension of the last Term by a correspondent Dimension of c, you have $3y + 4c$ for the Divisor sought. So, if the Quantity be $x^4 - bx^3 - 5bbxx + 12b^3x - 6b^4$; putting 1 for b, and having found $xx + 2x - 2$ the Divisor of the resulting Quantity $x^4 - x^3 - 5xx + 12x - 6$, I compleat its deficient Dimensions by respective Dimensions of b, and so I have $xx + 2bx + 2bb$ the Divisor sought.

When there are three or more Letters in the Quantity proposed, and all its Terms ascend to the same Dimensions, the Divisor may be found by the precedent Rules; but more expeditiously after this way: *Seek all the Divisors of all the Terms in which some one of the Letters is not, and also of all the Terms in which some other of the Letters is not; as also of all the Terms in which a third, fourth, and fifth Letter is not, if there are so many Letters; and so run over all the Letters: And in the same Line with those Letters place the Divisors respectively. Then see, if in any Series of Divisors going*

going through all the Letters, all the Parts involving only one Letter can be as often found as there are Letters (excepting only one) in the Quantity proposed; and likewise if the Parts involving two Letters *may be found as often as there are Letters (excepting two) in the Quantity proposed. If so; all those Parts taken together under their proper Signs will be the Divisor sought.*

As if there were proposed the Quantity $12x^3 - 14bxx + 9cxx - 12bbx - 6bcx + 8ccx + 8b^3 - 12bbc - 4bcc + 6c^3$; the Divisors of one Dimension of the Terms $8b^3 - 12bbc - 4bcc + 6c^3$, in which x is not, will be found by the preceding Rules to be $2b - 3c$ and $4b - 6c$; and of the Terms $12x^3 + 9cxx + 8ccx + 6c^3$, in which b is not, there will be only one Divisor $4x + 3c$; and of the Terms $12x^3 - 14bxx - 12bbx + 8b^3$, in which there is not c, there will be the Divisors $2x - b$ and $4x - 2b$. I dispose these Divisors in the same Lines with the Letters x, b, c, as you here see;

x	$2b - 3c.\ 4b - 6c.$
b	$4x + 3c.$
c	$2x - b.\ 4x - 2b.$

Since there are three Letters, and each of the Parts of the Divisors only involve one of the Letters, those Parts ought to be found twice in the Series of Divisors. But the Parts $4b$, $6c$, $2x$, b of the Divisors $4b - 6c$ and $2x - b$, only occur once, and are not found any where out of those Divisors whereof they are Parts. Wherefore I neglect those Divisors. There remain only three Divisors $2b - 3c$, $4x + 3c$ and $4x - 2b$. These are in the Series going through all the Letters x, b, c, and each of the Parts $2b$, $3c$, $4x$, are found in them twice, as ought to be, and that with the same Signs, provided the Signs of the Divisor $2b - 3c$ be changed, and in its Place you write $-2b + 3c$. For you may change the Signs of any Divisor. I take therefore all the Parts of these, viz. $2b$, $3c$, $4x$, once apiece under their proper Signs, and the Aggregate $-2b + 3c + 4x$ will be the Divisor which was to be found. For if by this you divide the proposed Quantity, there will come out $3xx - 2bx + 2cc - 4bb$.

 Again

Again, if the Quantity be $12x^5 - 10ax^4 - 9bx^4 - 26a^2x^3 + 12abx^3 + 6bbx^3 + 24a^3xx - 8aabxx - 8abbxx - 24b^3xx - 4a^3bx + 6aabbx - 12ab^3x + 18b^4x + 12a^4b + 32aab^3 - 12b^5$; I place the Divisors of the Terms in which x is not, by x; and those Terms in which a is not, by a; and those in which b is not, by b, as you here see. Then I perceive that all those that

x	$b.\ 2b.\ 4b.\ aa+3bb.\ 2aa+6bb.\ 4aa+12bb.$ $bb-3aa.\ 2bb-6aa.\ 4bb-12aa.$
a	$4xx-3bx+2bb.\ 12xx-9bx+6bb.$
b	$x.\ 2x.\ 3x-4a.\ 6x-8a.\ 3xx-4ax.\ 6xx-8ax.$ $2xx+ax-3aa.\ 4xx+2ax-6aa.$

are but of one Dimension are to be rejected, because the Simple ones, $b.\ 2b.\ 4b.\ x.\ 2x$, and the Parts of the compounded ones, $3x-4a.\ 6x.-8a$, are found but once in all the Divisors; but there are three Letters in the proposed Quantity, and those Parts involve but one, and so ought to be found twice. In like manner, the Divisors of two Dimensions, $aa+3bb.\ 2aa+6bb.\ 4aa+12bb.\ bb-3aa.$ and $4bb-12aa$ I reject, because their Parts $aa.\ 2aa.\ 4aa.\ bb.$ and $4bb.$ involving only one Letter a or b, are not found more than once. But the Parts $2bb$ and $6aa$ of the Divisor $2bb-6aa$, which is the only remaining one in the Line with x, and which likewise involve only one Letter, are found again or twice, *viz.* the Part $2bb$ in the Divisor $4xx-3bx+2bb$, and the Part $6aa$ in the Divisor $4xx+2ax-6aa$. Moreover, these three Divisors are in a Series standing in the same Lines with the three Letters x, a, b; and all their Parts $2bb$, $6aa$, $4xx$, which involve only one Letter, are found twice in them, and that under their proper Signs; but the Parts $3bx$, $2ax$, which involve two Letters, occur but once in them. Wherefore, all the divers Parts of these three Divisors, $2bb$, $6aa$, $4xx$, $3bx$, $2ax$, connected under their proper Signs, will make the Divisors sought, *viz.* $2bb-6aa+4xx-3bx+2ax$. I therefore divide the Quantity proposed by this Divisor, and there arises $3x^3-4axx-2aab-6b^3$.

If all the Terms of any Quantity are not equally high, the deficient Dimensions must be filled up by the Dimensions of any assumed Letter; then having found a Divisor by the precedent Rules, the assumed Letter is to be blotted out.

As

As if the Quantity be $12x^3 - 14bxx + 9xx - 12bbx - 6bx + 8x + 8b^3 - 12b^2 - 4b + 6$; assume any Letter, as c, and fill up the Dimensions of the Quantity proposed by its Dimensions, after this Manner, $12x^3 - 14bxx + 9cxx - 12bbx - 6bcx + 8ccx + 8b^3 - 12bbc - 4bcc + 6c^3$. Then having found out its Divisor $4x - 2b + 3c$, blot out c; and you will have the Divisor required, *viz.* $4x - 2b + 3$.

Sometimes Divisors may be found more easily than by these Rules. As if some Letter in the proposed Quantity be of only one Dimension; you may seek for the greatest common Divisor of the Terms in which that Letter is found, and of the remaining Terms in which it is not found; for that Divisor will divide the whole. And if there is no such common Divisor, there will be no Divisor of the whole. For Example, if there be proposed the Quantity $x^4 - 3ax^3 - 8aaxx + 18a^3x + cx^3 - acxx - 8aacx + 6a^3c - 8a^4$; let there be sought the common Divisor of the Terms $+ cx^3 - acxx - 8aacx + 6a^3c$, in which c is only of one Dimension, and of the remaining Terms $x^4 - 3ax^3 - 8aaxx + 18a^3x - 8a^4$, and that Divisor, *viz.* $xx + 2ax - 2aa$, will divide the whole Quantity.

But the greatest common Divisor of two Numbers, if it is not known, or does not appear at first Sight, it is found by a perpetual Subtraction of the less from the greater, and of the Remainder from the last Quantity subtracted. For that will be the sought Divisor, which at length leaves nothing. Thus, to find the greatest common Divisor of the Numbers 203 and 667, subtract thrice 203 from 667, and the Remainder 58 thrice from 203, and the Remainder 29 twice from 58, and there will remain nothing; which shews, that 29 is the Divisor sought.

After the same Manner the common Divisor in Species, when it is compounded, is found, by subtracting either Quantity, or its Multiple, from the other; provided both those Quantities and the Remainder be ranged according to the Dimensions of any Letter, as is shewn in Division, and be each Time managed by dividing them by all their Divisors, which are either Simple, or divide each of its Terms as if it were a Simple one. Thus, to find the greatest common Divisor of the Numerator and Denominator of this Fraction

$\frac{x^4 - 3ax^3 - 8aaxx + 18a^3x - 8a^4}{x^3 - axx - 8aax + 6a^3}$, multiply the Denominator by x, that its first Term may become the same with the first Term of the Numerator. Then subtract it, and there will remain $-2ax^3 + 12a^3x - 8a^4$, which being rightly ordered by dividing by $-2a$, it becomes $x^3 - 6a^2x + 4a^3$. Subtract this from the Denominator, and there will remain $-axx - 2aax + 2a^3$; which again divided by $-a$, becomes $xx + 2ax - 2aa$. Multiply this by x, that its first Term may become the same with the first Term of the last subtracted Quantity $x^3 - 6aax + 4a^3$, from which it is to be likewise subtracted, and there will remain $-2axx - 4aax + 4a^3$, which divided by $-2a$, becomes also $xx + 2ax - 2aa$. And since this is the same with the former Remainder, and consequently being subtracted from it, will leave nothing, it will be the Divisor sought; by which the proposed Fraction, by dividing both the Numerator and Denominator by it, may be reduced to a more Simple one, *viz.* to $\frac{xx - 5ax + 4aa}{x - 3a}$.

And so, if you have the Fraction

$$\frac{6a^5 + 15a^4b - 4a^3cc - 10aabcc}{9a^3b - 27aabc - 6abcc + 18bc^3},$$

its Terms must be first abbreviated, by dividing the Numerator by aa, and the Denominator by $3b$: Then subtracting twice $3a^3 - 9aac - 2acc + 6c^3$ from $6a^3 + 15aab - 4acc - 10bcc$, there will remain $\begin{matrix} +15b \\ +18c \end{matrix} aa \begin{matrix} -10bcc \\ -12c^3 \end{matrix}$. Which being ordered, by dividing each Term by $5b + 6c$ after the same Way as if $5b + 6c$ was a simple Quantity, it becomes $3aa - 2cc$. This being multiplied by a, subtract it from $3a^3 - 9aac - 2acc + 6c^3$, and there will remain $-9aac + 6c^3$, which being again ordered by a Division by $-3c$, becomes also $3aa - 2cc$, as before. Wherefore $3aa - 2cc$ is the Divisor sought. Which being found, divide by it the Parts of the proposed Fraction, and you will have $\frac{2a^3 + 5aab}{3ab - 9bc}$.

Now,

Now, if a common Divisor cannot be found after this Way, it is certain there is none at all; unless, perhaps, it may arise out of the Terms that abbreviate the Numerator and Denominator of the Fraction. As, if you have the Fraction $\frac{aadd - ccdd - aacc + c^4}{4aad - 4acd - 2acc + 2c^3}$, and so dispose its Terms, according to the Dimensions of the Letter d, that the Numerator may become $\begin{matrix} aa \\ -cc \end{matrix} dd \begin{matrix} -aacc \\ +c^4 \end{matrix}$, and the Denominator $\begin{matrix} 4aa \\ -4ac \end{matrix} d \begin{matrix} -2acc \\ +2c^3 \end{matrix}$. These must first be abbreviated by dividing each Term of the Numerator by $aa - cc$, and each of the Denominator by $2a - 2c$, just as if $aa - cc$ and $2a - 2c$ were simple Quantities. And so, in Room of the Numerator there will come out $dd - cc$, and in Room of the Denominator $2ad - cc$, from which, thus prepared, no common Divisor can be obtained. But, out of the Terms $aa - cc$ and $2a - 2c$, by which both the Numerator and Denominator are abbreviated, there comes out a Divisor, *viz.* $a - c$, by which the Fraction may be reduced to this, *viz.* $\frac{add + cdd - acc - c^3}{4ad - 2cc}$. Now, if neither the Terms $aa - cc$ and $2a - 2c$ had not had a common Divisor, the proposed Fraction would have been irreducible.

And this is a general Method of finding common Divisors: *But most commonly they are more expeditiously found by seeking all the prime Divisors of either of the Quantities, that is, such as cannot be divided by others, and then by trying if any of them will divide the other without a Remainder.* Thus, to reduce $\frac{a^3 - aab + abb - b^3}{aa - ab}$ to the least Terms, you must find the Divisors of the Quantity $aa - ab$, viz. a and $a - b$; then you must try whether either a, or $a - b$, will also divide $a^3 - aab + abb - b^3$ without any Remainder.

Of

Of the Reduction *of* Fractions *to a common Denominator.*

FRACTIONS *are reduced to a common Denominator by multiplying the Terms of each by the Denominator of the other.*

Thus, having $\frac{a}{b}$ and $\frac{c}{d}$ multiply the Terms of one $\frac{a}{b}$ by d, and alſo the Terms of the other $\frac{c}{d}$ by b, and they will become $\frac{ad}{bd}$ and $\frac{bc}{bd}$, whereof the common Denominator is bd. And thus a and $\frac{ab}{c}$, or $\frac{a}{1}$ and $\frac{ab}{c}$ become $\frac{ac}{c}$ and $\frac{ab}{c}$. But where the Denominators have a common Diviſor, it is ſufficient to multiply them alternately by the Quotients. Thus the Fraction $\frac{a^3}{bc}$ and $\frac{a^3}{bd}$ are reduced to theſe $\frac{a^3d}{bcd}$, and $\frac{a^3c}{bcd}$, by multiplying alternately by the Quotients c and d, ariſing by the Diviſion of the Denominators by the common Diviſor b.

This Reduction is moſtly of Uſe in the Addition and Subtraction of Fractions, which, if they have different Denominators, muſt be firſt reduced to the ſame Denominator before they can be added. Thus $\frac{a}{b}+\frac{c}{d}$ by Reduction becomes $\frac{ad}{bd}+\frac{bc}{bd}$, or $\frac{ad+bc}{bd}$, and $a+\frac{ab}{c}$ becomes $\frac{ac+ab}{c}$. And $\frac{a^3}{bc}-\frac{a^3}{bd}$ becomes $\frac{a^3d-a^3c}{bcd}$, or $\frac{d-c}{bcd}a^3$. And $\frac{c^4+x^4}{cc-xx}-cc-xx$ becomes $\frac{2x^4}{cc-xx}$. And ſo $\frac{2}{3}+\frac{5}{7}$ becomes $\frac{14}{21}+\frac{15}{21}$, or $\frac{14+15}{21}$, that is, $\frac{29}{21}$. And $\frac{11}{6}-\frac{3}{4}$ becomes $\frac{22}{12}-\frac{9}{12}$, or $\frac{13}{12}$. And $\frac{3}{4}-\frac{5}{12}$ becomes

becomes $\frac{9}{12} - \frac{5}{12}$, or $\frac{4}{12}$, that is $\frac{1}{3}$. And $3\frac{4}{7}$, or $\frac{3}{1} + \frac{4}{7}$ becomes $\frac{21}{7} + \frac{4}{7}$, or $\frac{25}{7}$. And $25\frac{1}{2}$ becomes $\frac{51}{2}$.

Where there are more Fractions than two, they are to be added gradually. Thus, having $\frac{aa}{x} - a + \frac{2xx}{3a} - \frac{ax}{a-x}$, from $\frac{aa}{x}$, take a, and there will remain $\frac{aa-ax}{x}$, to this add $\frac{2xx}{3a}$, and there will come out $\frac{3a^3 - 3aax + 2x^3}{3ax}$, from whence, lastly, take away $\frac{ax}{a-x}$, and there will remain $\frac{3a^4 - 6a^3x + 2ax^3 - 2x^4}{3aax - 3axx}$. And so if you have $3\frac{4}{7} - \frac{2}{3}$, first, you are to find the Aggregate of $3\frac{4}{7}$, *viz.* $\frac{25}{7}$, and then to take from it $\frac{2}{3}$ and there will remain $\frac{61}{21}$.

Of the Reduction *of* Radical *Quantities to their least Terms.*

A Radical Quantity, where the Root of the whole cannot be extracted, is reduced by extracting the Root of some Divisor of it.

Thus $\sqrt{aabc}$, by extracting the Root of the Divisor aa, becomes $a\sqrt{bc}$. And $\sqrt{48}$, by extracting the Root of the Divisor 16, becomes $4\sqrt{3}$. And $\sqrt{48aabc}$, by extracting the Root of the Divisor $16aa$, becomes $4a\sqrt{3bc}$. And $\sqrt{\frac{a^3b - 4aabb + 4ab^3}{cc}}$, by extracting the Root of its Divisor $\frac{aa - 4ab + 4bb}{cc}$, becomes $\frac{a-2b}{c}\sqrt{ab}$. And $\sqrt{\frac{aaoomm}{ppzz} + \frac{4aammmm}{pzz}}$, by extracting the Root of the

Divisor $\frac{aamm}{ppzz}$, becomes $\frac{am}{pz}\sqrt{oo+4mp}$. And $6\sqrt{\frac{75}{98}}$, by extracting the Root of the Divisor $\frac{25}{49}$, becomes $\frac{30}{7}\sqrt{\frac{3}{2}}$, or $\frac{30}{7}\sqrt{\frac{6}{4}}$, and by farther extracting the Root of the Denominator, it becomes $\frac{15}{7}\sqrt{6}$. And so $a\sqrt{\frac{b}{a}}$, or $a\sqrt{\frac{ab}{aa}}$, by extracting the Root of the Denominator, becomes $\sqrt{ab}$. And $\sqrt[3]{8a^3b+16a^4}$, by extracting the Cube Root of its Divisor $8a^3$, becomes $2a\sqrt[3]{b+2a}$. After the same manner $\sqrt[4]{a^3x}$, by extracting the Square Root of its Divisor aa, becomes $\sqrt{a}$ into $\sqrt[4]{ax}$, or by extracting the Biquadratick Root of the Divisor a^4, it becomes $a\sqrt[4]{\frac{x}{a}}$. And so $\sqrt[6]{a^7x^5}$ is changed into $a\sqrt[6]{ax^5}$, or into $ax\sqrt[6]{\frac{a}{x}}$, or into $\sqrt{ax}\times\sqrt[3]{aax}$.

Moreover, this Reduction is not only of use for abbreviating of Radical Quantities, but also for their Addition and Subtraction, if they agree in their Roots when they are reduced to the most simple Form; for then they may be added, which otherwise they cannot. Thus, $\sqrt{48}+\sqrt{75}$ by Reduction becomes $4\sqrt{3}+5\sqrt{3}$, that is $9\sqrt{3}$. And $\sqrt{48}-\sqrt{\frac{16}{27}}$ by Reduction becomes $4\sqrt{3}-\frac{4}{9}\sqrt{3}$, that is, $\frac{32}{9}\sqrt{3}$. And thus, $\sqrt{\frac{4ab^3}{cc}}+\sqrt{\frac{a^3b-4aabb+4ab^3}{cc}}$, by extracting what is Rational in it, becomes $\frac{2b}{c}\sqrt{ab}+\frac{a-2b}{c}\sqrt{ab}$, that is, $\frac{a}{c}\sqrt{ab}$. And $\sqrt[3]{8a^3b+16a^4}-\sqrt[3]{b^4+2ab^3}$ becomes $2a\sqrt[3]{b+2a}-b\sqrt[3]{b+2a}$, that is, $\overline{2a-b}\sqrt[3]{b+2a}$.

Of the REDUCTION *of* RADICAL *Quantities to the same Denomination.*

WHEN you are to multiply or divide Radicals of a different Denomination, you must first reduce them to the same Denomination, by prefixing that radical Sign whose Index is the least Number, which their Indices divide without a Remainder, and by multiplying the Quantities under the Signs so many times, excepting one, as that Index is become greater.

For so $\sqrt{ax}$ into $\sqrt[3]{aax}$ becomes $\sqrt[6]{a^3x^3}$ into $\sqrt[6]{a^4xx}$, that is $\sqrt[6]{a^7x^5}$. And $\sqrt{a}$ into $\sqrt[4]{ax}$ becomes $\sqrt[4]{aa}$ into $\sqrt[4]{ax}$, that is, $\sqrt[4]{a^3x}$. And $\sqrt{6}$ into $\sqrt[4]{\frac{5}{6}}$ becomes $\sqrt[4]{36}$ into $\sqrt[4]{\frac{5}{6}}$, that is, $\sqrt[4]{30}$. By the same Reason, $a\sqrt{bc}$ becomes $\sqrt{aa}$ into $\sqrt{bc}$, that is $\sqrt{aabc}$. And $4a\sqrt{3bc}$ becomes $\sqrt{16aa}$ into $\sqrt{3bc}$, that is, $\sqrt{48aabc}$. And $2a\sqrt[3]{b+2a}$ becomes $\sqrt[3]{8a^3}$ into $\sqrt[3]{b+2a}$, that is, $\sqrt[3]{8a^3b+16a^4}$. And so $\frac{\sqrt{ac}}{b}$ becomes $\frac{\sqrt{ac}}{\sqrt{bb}}$, or $\sqrt{\frac{ac}{bb}}$. And $\frac{6ab}{\sqrt{18ab^3}}$ becomes $\frac{\sqrt{36aab^2}}{\sqrt{18ab^3}}$, or $\sqrt{2ab}$. And so in others.

Of the REDUCTION *of* RADICALS *to more simple Radicals, by the Extraction of Roots.*

THE Roots of Quantities, which are composed of Integers and Radical Quadraticks, extract thus:

Let A *denote the greater Part of any Quantity, and* B *the lesser Part*; and $\frac{A+\sqrt{AA-BB}}{2}$ *will be the Square of the greater Part of the Root*; *and* $\frac{A-\sqrt{AA-BB}}{2}$ *will be the Square of the lesser Part, which is to be joined to the greater Part with the Sign of* B.

As

As if the Quantity be $3 + \sqrt{8}$, by writing 3 for A, and $\sqrt{8}$ for B, $\sqrt{AA - BB} = 1$, and thence the Square of the greater Part of the Root $\frac{3+1}{2}$, that is, 2, and the Square of the less $\frac{3-1}{2}$, that is, 1. Therefore the Root is, $1 + \sqrt{2}$. Again if you are to extract the Root of $\sqrt{32} - \sqrt{24}$, by putting $\sqrt{32}$ for A, and $\sqrt{24}$ for B, $\sqrt{AA - BB}$ will be $= \sqrt{8}$, and thence $\frac{\sqrt{32} + \sqrt{8}}{2}$, and $\frac{\sqrt{32} - \sqrt{8}}{2}$, that is, $3\sqrt{2}$ and $\sqrt{2}$ will be the Squares of the Parts of the Root. The Root therefore is $\sqrt[4]{18} - \sqrt[4]{2}$. After the same manner, if out of $aa + 2x\sqrt{aa - xx}$ you are to extract the Root, for A write aa, and for B write $2x\sqrt{aa - xx}$, and $AA - BB$ will be $= a^4 - 4aaxx + 4x^4$, the Root whereof is $aa - 2xx$. Whence the Square of one Part of the Root will be $aa - xx$, and that of the other xx; and so the Root will be $x + \sqrt{aa - xx}$. Again, if you have $aa + 5ax - 2a\sqrt{ax + 4xx}$, by writing $aa + 5ax$ for A, and $2a\sqrt{ax + 4xx}$ for B, $AA - BB$ will be $= a^4 + 6a^3x + 9aaxx$, whose Root is $aa + 3ax$. Whence the Square of the greater Part of the Root will be $aa + 4ax$, and that of the lesser Part ax, and the Root $\sqrt{aa + 4ax} - \sqrt{ax}$. Lastly, If you have $6 + \sqrt{8} - \sqrt{12} - \sqrt{24}$, putting $6 + \sqrt{8} = A$, and $-\sqrt{12} - \sqrt{24} = B$, $AA - BB$ will be $= 8$; whence the greater Part of the Root is $\sqrt{3 + \sqrt{8}}$, that is as above $1 + \sqrt{2}$, and the lesser Part $\sqrt{3}$, and consequently the Root itself $1 + \sqrt{2} - \sqrt{3}$. But where there are more of this sort of Radical Terms, the Parts of the Root may be sooner found, by dividing the Product of any two of the Radicals by some third Radical, which shall produce a Rational and Integer Quotient. For the Root of twice that Quotient will be double of the Part of the Root sought. As in the last Example, $\frac{\sqrt{8} \times \sqrt{12}}{\sqrt{24}} = 2$, $\frac{\sqrt{8} \times \sqrt{24}}{\sqrt{12}} = 4$, And $\frac{\sqrt{12} \times \sqrt{24}}{\sqrt{8}} = 6$. Therefore the Parts of the Root are 1, $\sqrt{2}$, $\sqrt{3}$ as above.

There

There is also a Rule of extracting higher Roots out of Numeral Quantities consisting of two Parts, whose Squares are commensurable.

Let there be the Quantity A + B. *And its greater Part* A. *And the Index of the Root to be extracted* c. Seek the least Number *n, whose Power* n^c *may be divided by* AA − BB *without any Remainder*, and let the Quotient be Q. *Compute* $\sqrt[c]{\overline{A+B}\times\sqrt{Q}}$ *in the nearest Integer Numbers. Let it be* r. *Divide* A$\sqrt{Q}$ *by the greatest rational Divisor. Let the Quotient be* s, *and let* $\frac{r+\frac{n}{r}}{2s}$ *in the next greatest Integers be called* t. And $\frac{ts+\sqrt{ttss-n}}{\sqrt[2c]{Q}}$ *will be the Root sought, if the Root can be extracted.*

As if the Cube Root be to be extracted out of $\sqrt{968}+25$; AA − BB will be $=343$; and 7, 7, 7, will be its Divisors; therefore $n=7$, and $Q=1$. Moreover, $\overline{A+B}\times\sqrt{Q}$, or $\sqrt{968}+25$, having extracted the former Part of the Root, is a little greater than 56; and its Cube Root in the nearest Numbers is 4; therefore $r=4$. Moreover, A$\sqrt{Q}$ or $\sqrt{968}$, by taking out whatever is Rational, becomes $22\sqrt{2}$. Therefore $\sqrt{2}$ its Radical Part is *s*, and $\frac{r+\frac{n}{r}}{2s}$, or $\frac{5\frac{3}{4}}{2\sqrt{2}}$ in the nearest Integer Numbers is 2. Therefore $t=2$. Lastly, *ts* is $2\sqrt{2}$, $\sqrt{ttss-n}$ is 1, and $\sqrt[2c]{Q}$, or $\sqrt[6]{1}$, is 1. Therefore $2\sqrt{2}+1$ is the Root sought, if it can be extracted. I try therefore by Multiplication if the Cube of $2\sqrt{2}+1$ be $\sqrt{968}+25$, and it succeeds.

Again, if the Cube Root is to be extracted out of $68-\sqrt{4374}$, AA − BB will be $=250$, whose Divisors are 5, 5, 5, 2. Therefore $n=5\times2=10$, and $Q=4$. And $\sqrt[c]{\overline{A+B}\times\sqrt{Q}}$, or $\sqrt[3]{\overline{68+\sqrt{4374}}\times2}$ in the nearest Integer Numbers is $7=r$. Moreover, A$\sqrt{Q}$, or $68\sqrt{4}$, by ex-

extracting or taking out what is Rational, becomes $136\sqrt{1}$.

Therefore $s=1$, and $\frac{r+\frac{n}{r}}{2s}$, or $\frac{7+\frac{10}{7}}{2}$ in the nearest Integer Numbers is $4=t$. Therefore $ts=4$, $\sqrt{ttss-n}=\sqrt{6}$, and $\sqrt[c]{Q}=\sqrt[6]{4}$, or $\sqrt[3]{2}$; and so the Root to be try'd is $\frac{4-\sqrt{6}}{\sqrt[3]{2}}$.

Again, if the fifth Root be to be extracted out of $29\sqrt{6}+41\sqrt{3}$; $AA-BB$ will be $=3$, and consequently $n=3$, $Q=81$, $r=5$, $s=\sqrt{6}$, $t=1$, $ts=\sqrt{6}$, $\sqrt{ttss-n}=\sqrt{3}$, and $\sqrt[2c]{Q}=\sqrt[10]{81}$, or $\sqrt[5]{9}$; and so the Root to be tried is $\frac{\sqrt{6}+\sqrt{3}}{\sqrt[5]{9}}$.

But if in these Sorts of Operations, the Quantity be a Fraction, or its Parts have a common Divisor, extract separately the Roots of the Terms, and of the Factors. As if the Cube Root be to be extracted out of $\sqrt{242}-12\frac{1}{2}$; this, having reduced its Parts to a common Denominator, will become $\frac{\sqrt{968}-25}{2}$. Then having extracted separately the Cube Root of the Numerator and the Denominator, there will come out $\frac{2\sqrt{2}-1}{\sqrt[3]{2}}$. Again, if you are to extract any Root out of $\sqrt[3]{3993}+\sqrt[6]{17578125}$; divide the Parts by the common Divisor $\sqrt[3]{3}$, and there will come out $11+\sqrt{125}$. Whence the proposed Quantity is $\sqrt[3]{3}$ into $11+\sqrt{125}$, whose Root will be found by extracting separately the Root of each Factor $\sqrt[3]{3}$, and $11+\sqrt{125}$.

Of

Of the Form of an Equation.

EQUATIONS are Ranks of Quantities either equal to one another, or, taken together, equal to nothing. These are to be considered chiefly after two Ways; either as the last Conclusions to which you come in the Resolution of Problems; or as Means, by the Help whereof you are to obtain final Equations. An Equation of the former Kind is composed only out of one unknown Quantity involved with known ones, if the Problem be determined, and proposes something certain to be found out. But those of the latter Kind involve several unknown Quantities, which, for that Reason, must be compared among one another, and so connected, that out of all there may emerge a new Equation, in which there is only one unknown Quantity which we seek mixed with known Quantities. Which Quantity, that it may be the more easily discovered, that Equation must be transformed most commonly various Ways, until it becomes the most Simple that it can, and also like some of the following Degrees of them, in which x denotes the Quantity sought, according to whose Dimensions the Terms, as you see, are ordered, and p, q, r, s denote any other Quantities from which, being known and determined, x is also determined, and may be investigated by Methods hereafter to be explained.

$x = p$ Or, $x - p = 0$.
$xx = px + q$. $xx - px - q = 0$.
$x^3 = px^2 + qx + r$. $x^3 - pxx - qx - r = 0$.
$x^4 = px^3 + qx^2 + rx + s$. $x^4 - px^3 - qx^2 - rx - s = 0$.
&c. &c.

After this Manner therefore the Terms of Equations are to be ordered according to the Dimensions of the unknown Quantity, so that those may be in the first Place, in which the unknown Quantity is of the most Dimensions, as x, xx, x^3, x^4, &c. and those in the second Place, in which x is of the next greatest Dimension, as p, px, px^2, px^3, and so on. As to what regards the Signs, they may stand any how; and one or more of the intermediate Terms may be sometimes wanting. Thus, $x^3 * - bbx + b^3 = 0$, or $x^3 = bbx - b^3$, is an Equation of the third Degree, and

$z^4 \begin{matrix}+a\\-b\end{matrix} z^3 * \begin{matrix}+ab^3\\-b^4\end{matrix} = 0$, is an Equation of the fourth Degree: For the Degree of an Equation is always estimated by the greatest Dimension of the unknown Quantity, without any regard to the known ones, or to the intermediate Terms. But by the Defect of the intermediate Terms, the Equation is most commonly rendered much more simple, and may be sometimes depressed to a lower Degree. For thus, $x^4 = qxx + s$ is to be reckoned an Equation of the second Degree, because it may be resolved into two Equations of the second Degree. For, supposing $xx = y$, and y being accordingly writ for xx in that Equation, there will come out in its stead $yy = qy + s$, an Equation of the second Degree; by the Help whereof when y is found, the Equation $xx = y$ also of the second Degree, will give x.

And these are the Conclusions to which Problems are to be brought. But before I go upon their Resolution, it will be necessary to shew the Methods of transforming and reducing Equations into Order, and the Methods of finding the final Equations. I shall comprize the Reduction of a Single Equation in the following Rules.

Of ordering a Single EQUATION.

RULE I. *If there are any Quantities that may destroy one another, or may be joined into one by Addition or Subtraction, the Terms are that Way to be diminished.*

As if you have $5b - 3a + 2x = 5a + 3x$, take from each Side $2x$, and add $3a$, and there will come out $5b = 8a + x$. And thus, $\frac{2ab + bx}{a} - b = a + b$, by striking out the equivalent Quantities $\frac{2ab}{a} - b = b$, becomes $\frac{bx}{a} = a$.

To this Rule may also be referred the Ordering of the Terms of an Equation, which is usually performed by the Transposition of the Members to the contrary Sides under the contrary Sign. As if you had the Equation $5b = 8a + x$, you are to find x; take from each Side $8a$, or, which is the same Thing, transpose $8a$ to the contrary Side with

its

its Sign changed, and there will come out $5b - 8a = x$. After the same way, if you have $aa - 3ay = ab - bb + by$, and you are to find y; transpose $-3ay$ and $ab - bb$, so that there may be the Terms multiplied by y on the one Side, and the other Terms on the other Side, and there will come out $aa - ab + bb = 3ay + by$, whence you will have y by the fifth Rule following, *viz.* by dividing each Part by $3a + b$, for there will come out $\frac{aa - ab + bb}{3a + b} = y$. And thus the Equation $abx + a^3 - aax = abb - 2abx - x^3$, by due ordering and transposition becomes $x^3 = \begin{matrix} aa \\ -3ab \end{matrix} x \begin{matrix} -a^3 \\ +abb \end{matrix}$ or $x^3 \begin{matrix} -aa \\ +3ab \end{matrix} x \begin{matrix} +a^3 \\ -abb \end{matrix} = 0$.

RULE II. *If there is any Quantity by which all the Terms of the Equation are multiplied, all of them must be divided by that Quantity; or, if all are divided by the same Quantity, all must be multiplied by it too.*

Thus, having $15bb = 24ab + 3bx$, divide all the Terms by b, and you will have $15b = 24a + 3x$; then by 3, and you will have $5b = 8a + x$. Or having $\frac{b^3}{ac} - \frac{bbx}{cc} = \frac{xx}{c}$, multiply all by c, and there comes out $\frac{b^3}{a} - \frac{bbx}{c} = xx$.

RULE III. *If there be any irreducible Fraction, in whose Denominator there is found the Letter, according to whose Dimensions the Equation is to be ordered, all the Terms of the Equation must be multiplied by that Denominator, or by some Divisor of it.*

As if the Equation $\frac{ax}{a - x} + b = x$ be to be ordered according to x, multiply all its Terms by $a - x$ the Denominator of the Fraction $\frac{ax}{a - x}$ seeing x is contained therein, and there comes out $ax + ab - bx = ax - xx$, or $ab - bx = -xx$, and transposing each Part you will have $xx = bx - ab$. And so if you have $\frac{a^3 - aab}{2cy - cc} = y - c$, and the Terms are to be ranged according to the Dimensions of y, multiply them by the Denominator $2cy - cc$, or, at

 least,

least, by its Divisor $2y - c$, that y may vanish in the Denominator, and there will come out $\frac{a^3 - abb}{c} = 2yy - 3cy + cc$, and by farther ordering $\frac{a^3 - abb}{c} - cc + 3cy = 2yy$. After the same manner $\frac{aa}{x} - a = x$, by being multiplied by x, becomes $aa - ax = xx$, and $\frac{aabb}{cxx} = \frac{xx}{a + b - x}$, by multiplying first by xx, and then by $a + b - x$, becomes $\frac{a^3bb + aab^3 - aabbx}{c} = x^4$.

RULE IV. *If that particular Letter, according to whose Dimensions the Equation is to be ordered, be involved with an irreducible Surd, all the other Terms are to be transposed to the other Side, their Signs being changed, and each Part of the Equation must be once multiplied by itself, if the Root be a* Square one, *or* twice, *if it be a* Cubick one, *&c.*

Thus, to order the Equation $\sqrt{aa - ax} + a = x$ according to the Letter x, transpose a to the other Side, and you have $\sqrt{aa - ax} = x - a$; and having squared the Parts, $aa - ax = xx - 2ax + aa$, or $0 = xx - ax$, that is, $x = a$. So also $\sqrt[3]{aax + 2axx - x^3} - a + x = 0$, by transposing $-a + x$, becomes $\sqrt[3]{aax + 2axx - x^3} = a - x$, and multiplying the Parts cubically, $aax + 2axx - x^3 = a^3 - 3aax + 3axx - x^3$, or $xx = 4ax - aa$. And so $y = \sqrt{ay + yy - a\sqrt{ay - yy}}$, having squared the Parts, becomes $yy = ay + yy - a\sqrt{ay - yy}$, and the Terms being rightly transposed, it becomes $ay = a\sqrt{ay - yy}$, or $y = \sqrt{ay - yy}$, and the Parts being again squared $yy = ay - yy$, and lastly by transposing $2yy = ay$, or $2y = a$.

RULE V. *The Terms, by help of the preceding Rules, being disposed according to the Dimensions of some one of the Letters, if the highest Dimension of that Letter be multiplied by any known Quantity, the whole Equation must be divided by that Quantity.*

Thus,

Thus, $2y = a$, by dividing by 2, becomes $y = \frac{1}{2}a$. And $\frac{bx}{a} = a$, by dividing by $\frac{b}{a}$, becomes $x = \frac{aa}{b}$. And $\begin{array}{l} 2ac \\ -cc \end{array} x^3 \begin{array}{l} +a^3 \\ +aac \end{array} xx \begin{array}{l} -2a^3c \\ +aacc \end{array} x - a^4cc = 0$, by dividing by $2ac - cc$, becomes $x^3 + \frac{\begin{array}{l} a^3 \\ aac \end{array} x^2 \begin{array}{l} -2a^3c \\ +aacc \end{array} x - a^4cc}{2ac - cc} = 6$, or $x^3 + \frac{a^3 + aac}{2ac - cc} xx - aax - \frac{a^3c}{2a - c} = 0$.

RULE VI. *Sometimes the Reduction may be performed by dividing the Equation by some compounded Quantity.*

For thus $y^3 = \begin{array}{l} -2c \\ +b \end{array} yy + 3bcy - bbc$, is reduced to this, *viz.* $yy = -2cy + bc$, by transferring all the Terms to the same Side thus, $y^3 \begin{array}{l} +2c \\ -b \end{array} yy - 3bcy + bbc = 0$, and dividing by $y - b$, as is shewn in the Chapter of *Division*, for there will come out $yy + 2cy - bc = 0$. But the Invention of this sort of Divisors is difficult, and we have taught it already.

RULE VII. *Sometimes also the Reduction is performed by Extraction of the Root out of each Part of the Equation.*

As if you have $xx = \frac{1}{4}aa - bb$, having extracted the Root on both Sides, there comes out $x = \sqrt{\frac{1}{4}aa - bb}$. If you have $xx + aa = 2ax + bb$, transpose $2ax$, and there will arise $xx - 2ax + aa = bb$, and extracting the Roots of the Parts $x - a = +$ or $-b$, or $x = a \pm b$. So also having $xx = ax - bb$, add on each Side $-ax + \frac{1}{4}aa$, and there comes out $xx - ax + \frac{1}{4}aa = \frac{1}{4}aa - bb$, and extracting the Root on each Side $x - \frac{1}{2}a = \pm\sqrt{\frac{1}{4}aa - bb}$, or $x = \frac{1}{2}a \pm \sqrt{\frac{1}{4}aa - bb}$.

And thus universally if you have $xx = .\, px.\, q$; x will be $= .\, \frac{1}{2}p \pm \sqrt{\frac{1}{4}pp.\, q}$. Where $\frac{1}{2}p$ and q are to be affected with the same Signs as p and q in the former Equation; but $\frac{1}{4}pp$ must be always made Affirmative. And this Example is a Rule according to which all Quadratick Equations may be reduced to the Form of Simple ones. For example,

ample, having proposed the Equation $yy = \frac{2xxy}{a} + xx$, to extract the Root y, compare $\frac{2xx}{a}$ with p, and xx with q, that is, write $\frac{xx}{a}$ for $\frac{1}{2}p$, and $\frac{x^4}{aa} + xx$ for $\frac{1}{4}pp.q$, and there will arise $y = \frac{xx}{a} + \sqrt{\frac{x^4}{aa} + xx}$ or $y = \frac{xx}{a} - \sqrt{\frac{x^4}{aa} + xx}$. After the same way, the Equation $yy = ay - 2cy + aa - cc$, by comparing $a - 2c$ with p, and $aa - cc$ with q, will give $y = \frac{1}{2}a - c \pm \sqrt{\frac{5}{4}aa - ac}$.

Moreover, the Biquadratick Equation $x^4 = -aaxx + ab^3$, whose odd Terms are wanting, by help of this Rule becomes $xx = -\frac{1}{2}aa \pm \sqrt{\frac{1}{4}a^4 + ab^3}$, and extracting again the Root $x = \sqrt{-\frac{1}{2}aa \pm \sqrt{\frac{1}{4}a^4 + ab^3}}$. And so in others.

And these are the Rules for ordering one only Equation, the Use whereof, when the Analyst is sufficiently acquainted with, so that he knows how to dispose any proposed Equation according to any of the Letters contained in it, and to obtain the Value of that Letter if it be of one Dimension, or of its greatest Power if it be of more: The Comparison of several Equations among one another will not be difficult to him; which I am now going to shew.

Of the Transformation of two or more EQUATIONS *into one, in order to exterminate the unknown Quantities.*

WHEN, in the Solution of any Problem, there are more Equations than one to comprehend the State of the Question, in each of which there are several unknown Quantities; those Equations (two by two, if there are more than two) are to be so connected, that one of the unknown Quantities may be made to vanish at each of the Operations, and so produce a new Equation. Thus, having the Equations $2x = y + 5$, and $x = y + 2$, by taking equal Things out of equal Things, there will come out $x = 3$. And you are to to know, that by each Equation one unknown Quantity may be taken away, and consequently, when there are as many Equations as unknown Quantities, all may at length be reduc'd into one, in which there shall be only one Quantity unknown. But if there be more unknown Quantities by one than there are Equations, then there will remain in the Equation last resulting two unknown Quantities; and if there are more unknown Quantities by two than there are Equations, then in the last resulting Equation there will remain three, and so on.

There may also, perhaps, two or more unknown Quantities be made to vanish, by only two Equations. As if you have $ax - by = ab - az$, and $bx + by = bb + az$: Then adding Equals to Equals, there will come out $ax + bx = ab + bb$, both y and z being exterminated. But such Cases either argue some Fault to lie hid in the State of the Question, or that the Calculation is erroneous, or not artificial enough. The Method by which one unknown Quantity may be exterminated or taken away by each of the Equations, will appear by what follows.

The Extermination of an unknown Quantity by an Equality of its Values.

WHEN the Quantity to be exterminated is only of one Dimension in both Equations, both its Values are to be sought by the Rules already delivered, and the one made equal to the other.

Thus, putting $a + x = b + y$ and $2x + y = 3b$, that y may be exterminated, the first Equation will give $a + x - b = y$, and the second will give $3b - 2x = y$. Therefore $a + x - b$ is $= 3b - 2x$, or by due ordering $x = \frac{4b - a}{3}$.

And thus, $2x = y$ and $5 + x = y$ give $2x = 5 + x$ or $x = 5$.

And $ax - 2by = ab$ and $xy = bb$ give $\frac{ax - ab}{2b} (= y) = \frac{bb}{x}$; or by due ordering the Terms $xx - bx - \frac{2b^3}{a} = 0$.

Also

Also $\frac{bbx - aby}{a} = ab + xy$, and $bx + \frac{ayy}{c} = 2aa$, by taking away x, give $\frac{aby + aab}{bb - ay} (= x) = \frac{2aac - ayy}{bc}$; and by Reduction $y^3 - \frac{bb}{a} yy - \frac{2aac + bbc}{a} y + bbc = 0$.

Lastly, $x + y - z = 0$ and $ay = xz$ by taking away z give $x + y (= z) = \frac{ay}{x}$ or $xx + xy = ay$.

The same is also performed by subtracting either of the Values of the unknown Quantities from the other, and making the Remainder equal to nothing. Thus, in the first of the Examples, take away $3b - 2x$ from $a + x - b$, and there will remain $a + 3x - 4b = 0$, or $x = \frac{4b - a}{3}$.

The Extermination of an unknown Quantity, by substituting its Value for it.

WHEN, at least, in one of the Equations the Quantity to be exterminated is only of one Dimension, its Value is to be sought in that Equation, and then to be substituted in its room in the other Equation. Thus, having proposed $xyy = b^3$, and $xx + yy = by - ax$; to exterminate x, the first will give $\frac{b^3}{yy} = x$; wherefore I substitute in the second $\frac{b^3}{yy}$ in the room of x, and there comes out $\frac{b^6}{y^4} + yy = by - \frac{ab^3}{yy}$, and by Reduction $y^6 - by^5 + ab^3 yy + b^6 = 0$.

But having proposed $ayy + aay = z^3$, and $yz - ay = az$, to take away y, the second will give $y = \frac{az}{z - a}$. Wherefore for y I substitute $\frac{az}{z - a}$ into the first, and there comes out $\frac{a^3 zz}{zz - 2az + aa} + \frac{a^3 z}{z - a} = z^3$. And by Reduction, $z^4 - 2az^3 + aazz - 2a^3 z + a^4 = 0$.

In

In the like manner, having proposed $\frac{xy}{c} = z$ and $cy + zx = cc$, to take away z, I substitute in its room $\frac{xy}{c}$ in the second Equation, and there comes out $cy + \frac{xxy}{c} = cc$.

But a Person used to these Sorts of Computations, will oftentimes find shorter Methods than these by which the unknown Quantity may be exterminated. Thus, having $ax = \frac{bbx - b^3}{z}$ and $x = \frac{az}{x - b}$ if equal Quantities are multiplied by Equals, there will come out equal Quantities, *viz.* $axx = abb$, or $x = b$.

But I leave particular Cases of this kind to be found out by the Students as Occasion shall offer.

The Extermination of an unknown Quantity of several Dimensions in each Equation.

WHEN the Quantity to be taken away is of more than one Dimension in both the Equations, the Value of its greatest Power must be sought in both; then if those Powers are not the same, the Equation that involves the lesser Power must be multiplied by the Quantity to be taken away, or by its Square, or Cube, &c. that it may become of the same Power with the other Equation. Then the Values of those Powers are to be made Equal, and there will come out a new Equation, where the greatest Power or Dimension of the Quantity to be taken away is diminished. And by repeating this Operation, the Quantity will at length be taken away.

As if you have $xx + 5x = 3yy$ and $2xy - 3xx = 4$; to take away x, the first Equation will give $xx = -5x + 3yy$, and the second $xx = \frac{2xy - 4}{3}$. I put therefore $3yy - 5x = \frac{2xy - 4}{3}$, and so x is reduced to only one Dimension, and so may be taken away by what I have before shewn, *viz.* by a due Reduction of the last Equation there comes out $9yy - 15x = 2xy - 4$, or $x = \frac{9yy + 4}{2y + 15}$.

I there-

I therefore substitute this Value for x in one of the Equations first proposed (as in $xx+5x=3yy$) and there arises $\frac{81y^4+72yy+16}{4yy+60y+225}+\frac{45yy+20}{2y+15}=3yy$. To reduce which into order, I multiply by $4yy+60yy+225$, and there comes out $81y^4+72yy+16+90y^3+40y+675yy+300=12y^4+180y^3+675yy$, or $69y^4-90y^3+72yy+40y+316=0$.

Moreover, if you have $y^3=xyy+3x$, and $yy=xx-xy-3$; to take away y, I multiply the latter Equation by y, and you have $y^3=xxy-xyy-3y$, of as many Dimensions as the former. Now, by making the Values of y^3 equal to one another, I have $xyy+3x=xxy-xyy-3y$, where y is depressed to two Dimensions. By this therefore, and the most Simple one of the Equations first proposed $yy=xx-xy-3$, the Quantity y may be wholly taken away by the same Method as in the former Example.

There are moreover other Methods by which this may be done, and that oftentimes more concisely. As if there be given $yy=\frac{2x^2y}{a}+xx$, and $yy=2xy+\frac{x^4}{aa}$; that y may be extirpated, extract the Root y in each, as is shewn in the 7th Rule, and there will come out $y=\frac{xx}{a}+\sqrt{\frac{x^4}{aa}+xx}$, and $y=x+\sqrt{\frac{x^4}{aa}+xx}$. Now, by making these two Values of y equal you will have $\frac{xx}{a}+\sqrt{\frac{x^4}{aa}+xx}=x+\sqrt{\frac{x^4}{aa}+xx}$, and by rejecting the equal Quantities $\sqrt{\frac{x^4}{aa}+xx}$, there will remain $\frac{xx}{a}=x$, or $xx=ax$, and $x=a$.

Moreover, to take x out of the Equations $x+y+\frac{yy}{x}=20$, and $xx+yy+\frac{y^4}{xx}=140$, take away y from the

Parts of the first Equation, and there remains $x + \frac{yy}{x} = 20 - y$, and squaring the Parts, it becomes $xx + 2yy + \frac{y^4}{xx} = 400 - 40y + yy$, and taking away yy on both Sides, there remains $xx + yy + \frac{yyyy}{xx} = 400 - 40y$. Wherefore, since $400 - 40y$ and 140 are equal to the same Quantities, $400 - 40y$ will be equal to 140, or $y = 6\frac{1}{2}$; and so you may contract the Matter in most other Equations.

But when the Quantity to be exterminated is of several Dimensions, sometimes there is required a very laborious Calculus to exterminate it out of the Equations; but then the Labour will be much diminished by the following Examples made use of as Rules.

RULE I.

From $axx + bx + c = 0$, and $fxx + gx + h = 0$, x being exterminated, there comes out

$$\overline{ah - bg - 2cf} \times ah : + \overline{bh - cg} \times bf : \times \overline{agg + cff} \times c = 0.$$

RULE II.

From $ax^3 + bxx + cx + d = 0$, and $fxx + gx + h = 0$, x being exterminated, there comes out

$$\overline{ah - bg - 2cf} \times ahh : + \overline{bh - cg - 2df} \times bfh : + \overline{ch - dg} \times \overline{agg + cff} : + \overline{3agh + bgg + dff} \times df = 0.$$

RULE III.

From $ax^4 + bx^3 + cxx + dx + e = 0$, *and* $fxx + gx + h = 0$, x being exterminated, there comes out

$$\overline{ah - bg - 2cf} \times ah^3 : + \overline{bh - cg - 2df} \times bfhh : + \overline{agg + cff} \times \overline{chh + dgh + egg - 2efh} + \overline{3agh + bgg + dff} \times dfh : + \overline{2ahh + 3bgh - dfg + eff} \times cff : - \overline{bg - 2ah} \times efgg = 0.$$

Rule IV.

From $ax^3+bxx+cx+d=0$, and $fx^3+gx^2+hx+k=0$.

x being exterminated, there comes out

$$\overline{ah-bg-2cf}\times\overline{adhh-achk}:+\overline{ak+bh-cg-2df}\times bdfh:-\overline{ak+bh+2cg+3df}\times aakk:+\overline{cdh-ddg-cck+2bdk}\times\overline{agg+cff}:+\overline{3agh+bgg+dff-3afk}\times ddf:-\overline{3ak-bh+cg+df}\times bcfk:+\overline{bk-2dg}\times bbfk-\overline{bbk-3adh-cdf}\times agk=0.$$

For Example, to exterminate x out of the Equations $xx+5x-3yy=0$, and $3xx-2xy+4=0$: I respectively substitute in the first Rule for $a, b, c; f, g,$ and h these Quantities, $1, 5, -3yy; 3, -2y$ and 4; and duly observing the Signs $+$ and $-$, there arises $\overline{4+10y+18yy}\times 4:+\overline{20-by^2}\times 15:+\overline{4yy-27yy}\times -3yy=0$, or $16+40y+72yy+300-90y^3+69y^4=0$.

By the like Reason that y may be expunged out of the Equations $y^3-xyy-3x=0$, and $yy+xy-xx+3=0$, I substitute into the second Rule for $a, b, c, d; f, g, h$, and x, these Quantities $1, -x, 0, -3x; 1, x, -xx+3$, and y respectively, and there comes out $\overline{3-xx+xx}\times 9-6xx+x^4:-\overline{3x+x^3+6x}\times\overline{-3x+x^3}:+3xx\times xx:+\overline{9x-3x^3-x^3-3x}\times -3x=0$: Then blotting out the superfluous Quantities and multiplying, you have $27-18xx+3x^4, -9xx+x^6, +3x^4-18x^2+12x^4=0$. And ordering $x^6+18x^4-45xx+27=0$.

Hitherto we have discoursed of taking away one unknown Quantity out of two Equations. Now, if several are to be taken out of several, the Business must be done by degrees: Out of the Equations $ax=yz$, $x+y=2$ and $5x=y+3z$; if the Quantity y is to be found, first, take out one of the Quantities x or z, suppose x, by substituting for it its Value $\frac{yz}{a}$ (found by the first Equation) in the second and third Equations; and then you will have $\frac{yz}{a}+y=z$, and $\frac{5yz}{a}=y+3z$, out of which take away z as above.

Of the Method of taking away any Number of Surd Quantities out of Equations.

Hitherto may be referred the Extermination of Surd Quantities, by making them equal to any Letters. As if you have $\sqrt{ay}-\sqrt{aa-ay}=2a+\sqrt[3]{ayy}$, by writing t for $\sqrt{ay}$, and v for $\sqrt{aa-ay}$, and x for $\sqrt[3]{ayy}$, you will have the Equations $t-v=2a+x$, $tt=ay$, $vv=aa-ay$, and $x^3=ayy$, out of which taking away by degrees t, v, and x, there will result an Equation entirely free from Surdity.

How a Question may be brought to an Equation.

AFTER the Learner has been some time exercised in managing and transforming Equations, Order requires that he should try his Skill in bringing Questions to an Equation. And any Question being proposed, his Skill is particularly required to denote all its Conditions by so many Equations. To do which he must first consider whether the Propositions or Sentences in which it is expressed, be all of them fit to be denoted in Algebraick Terms, just as we express our Conceptions in *Latin* or *Greek* Characters. And if so, (as will happen in Questions conversant about Numbers or abstract Quantities) then let him give Names to both known and unknown Quantities, as far as occasion requires; and express the Sense of the Question in the Analytick Language, if I may so speak. And the Conditions thus translated to Algebraick Terms will give as many Equations as are necessary to solve it.

As if there are required three Numbers in continual Proportion, whose Sum is 20, and the Sum of their Squares 140; putting x, y, and z for the Names of the three Numbers sought, the Question will be translated out of the Verbal to the Symbolical Expression, as follows:

The Question in Words.	*The same in Symbols.*
There are sought three Numbers on these Conditions :	x, y, z?
That they shall be continually proportional.	$x : y :: y : z$, or $xz = yy$.
That the Sum shall be 20.	$x + y + z = 20$.
And the Sum of their Squares 140.	$xx + yy + zz = 140$.

And so the Question is brought to these Equations, *viz.* $xz = yy$, $x + y + z = 20$, and $xx + yy + zz = 140$, by the Help whereof x, y, and z, are to be found by the Rules delivered above.

But you must note, That the Solutions of Questions are (for the most part) so much the more expedite and artificial, by how fewer unknown Quantities you have at first. Thus, in the Question proposed, putting x for the first Number, and y for the second, $\frac{yy}{x}$ will be the third Proportional; which then being put for the third Number, I bring the Question into Equations, as follows:

The Question in Words.	*Symbolically.*
There are sought three Numbers in continual Proportion.	$x, y, \frac{yy}{x}$?
Whose Sum is 20.	$x + y + \frac{yy}{x} = 20$.
And the Sum of their Squares 140.	$xx + yy + \frac{y^4}{xx} = 140$.

You have therefore the Equations $x + y + \frac{yy}{x} = 20$, and $xx + yy + \frac{y^4}{xx} = 140$, by the Reduction whereof x and y are to be determined.

Take another Example. A certain Merchant encreases his Estate yearly by a third Part, abating 100 *l.* which he spends yearly in his Family; and after three Years he finds his Estate doubled. *Query*, What was he worth?

To

To resolve this, you must know there are or lie hid several Propositions, which are all thus found out and laid down.

In English.	*Algebraically.*
A Merchant has an Estate - - - -	x.
Out of which the first Year he expends 100 *l.*	$x - 100$.
And augments the rest by one third.	$x - 100 + \frac{x-100}{3}$ or $\frac{4x-400}{3}$.
And the second Year expends 100 *l.* - -	$\frac{4x-400}{3} - 100$ or $\frac{4x-700}{3}$.
And augments the rest by a third - -	$\frac{4x-700}{3} + \frac{4x-700}{9}$ or $\frac{16x-2800}{9}$.
And so the third Year expends 100 *l.* - -	$\frac{16x-2800}{9} - 100$ or $\frac{16x-3700}{9}$.
And by the rest gains likewise one third Part - - - -	$\frac{16x-3700}{9} + \frac{16x-3700}{27}$, or $\frac{64x-14800}{27}$.
And he becomes at length twice as rich as at first - -	$\frac{64x-14800}{27} = 2x$.

Therefore the Question is brought to this Equation, $\frac{64x-14800}{27} = 2x$, by the Reduction whereof you are to find x; *viz.* multiply it by 27, and you have $64x - 14800 = 54x$; subtract $54x$, and there remains $10x - 14800 = 0$, or $10x = 14800$, and dividing by 10, you have $x = 1480$. Wherefore, 1480 *l.* was his Estate at first, as also his Profit or Gain since.

You see therefore, that to the Solution of Questions which only regard Numbers, or the abstracted Relations of Quantities, there is scarce any Thing else required, than that the Problem be translated out of the *English*, or any other Tongue it is proposed in, into the Algebraical Language, that is, into Characters fit to denote our Conceptions of the Relations of Quantities. But it may sometimes happen, that the Language or the Words wherein the State of the Question is is expressed, may seem unfit to be turned into the Algebraical Language; but making Use of a few Changes, and attending to the Sense, rather than the Sound of the Words, the Version will become easy. Thus, the Forms of Speech among different Nations have their proper Idioms; which, where they happen, the Translation out of one into another is not to be made literally, but to be determined by the Sense. But that I may illustrate these Sorts of Problems, and make familiar the Method of reducing them to Equations; and since Arts are more easily learned by Examples than Precepts, I have thought fit to adjoin the Solutions of the following Problems.

PROBLEM I.

Having given the Sum of two Numbers, a, *and the Difference of their Squares* b, *to find the Numbers?*

Let the least of them be x, the other will be $a-x$, and their Squares xx, and $aa-2ax+xx$: the Difference whereof $aa-2ax$ is supposed b. Therefore $aa-2ax=b$, and then by Reduction $aa-b=2ax$, or $\frac{aa-b}{2a}$ $\left(=\frac{1}{2}a-\frac{b}{2a}\right)=x$.

For Example, if the Sum of the Numbers or a be 8, and the Difference of the Squares or b be 16; $\frac{1}{2}a-\frac{b}{2a}$ $(=4-1)$ will be $=3=x$, and $a-x=5$. Wherefore the Numbers are 3 and 5.

PROBLEM II.

To find three Quantities, x, y, *and* z, *the Sum of any two of which shall be given.*

If the Sum of two of them, *viz.* x and y be a; of x and z, b; and of y and z, c; there will be had three Equations to determine the three Quantities sought, x, y, and z, *viz.* $x+y=a$, $x+z=b$, and $y+z=c$. Now, that two of the unknown Quantities, *viz.* y and z may be exterminated, take away x on both Sides in the first and second Equation, and you will have $y=a-x$, and $z=b-x$, which Values substitute for y and z in the third Equation, and there will come out $a-x+b-x=c$, and by Reduction $x=\frac{a+b-c}{2}$; and having found x, the Equations above $y=a-x$ and $z=b-x$ will give y and z.

EXAMPLE.

EXAMPLE. If the Sum of x and y be 9, of x and z, 10, and y and z, 13; then, in the Values of x, y, and z, write 9 for a, 10 for b, and 13 for c; and you will have $a + b - c = 6$, and consequently $x \left(= \frac{a+b-c}{2}\right) = 3$, y $(= a - x) = 6$, and z $(= b - x) = 7$.

PROBLEM III.

To divide a given Quantity into as many Parts as you please, so that the greater Parts may exceed the least by any given Differences.

Let a be a Quantity to be divided into four such Parts, and its first or least Part call x, and the Excess of the second Part above this call b, and of the third Part c, and of the fourth d; and $x + b$ will be the second Part, $x + c$ the third, and $x + d$ the fourth, the Aggregate of all which $4x + b + c + d$ is equal to the whole Line a. Take away on both Sides $b + c + d$, and there remains $4x = a - b - c - d$, or $x = \frac{a - b - c - d}{4}$.

EXAMPLE. Let there be proposed a Line of 20 Foot, so to be divided into four Parts, that the Excess of the second above the first Part shall be 2 Foot, of the third 3 Foot, and of the fourth 7 Foot; and the four Parts will be $x \left(= \frac{a - b - c - d}{4} \text{ or } \frac{20 - 2 - 3 - 7}{4}\right) = 2$, $x + b = 4$, $x + c = 5$, and $x + d = 9$.

After the same Manner a Quantity is divided into more Parts on the same Conditions.

PROBLEM IV.

A Person being willing to distribute some Money among Beggars, wanted eight Pence to give three Pence a piece to them; he therefore gave to each two Pence, and had three Pence remaining over and above. To find the Number of the Beggars.

Let the Number of the Beggars be x, and there will be wanting eight Pence to give all $3x$ Pence; he has therefore $3x - 8$ Pence. Out of these he gives $2x$ Pence, and the remaining Pence $x - 8$ are three. That is, $x - 8 = 3$, or $x = 11$.

PRO-

PROBLEM V.

If two Post-Boys, A and B, at 59 Miles Distance from one another, set out in the Morning in order to meet. And A rides 7 Miles in two Hours, and B 8 Miles in three Hours, and B begins his Journey one Hour later than A; to find what Number of Miles A will ride before he meets B.

Call that Length x, and you will have $59 - x$, the Length of *B*'s Journey. And since *A* travels 7 Miles in two Hours, he will make the Space x in $\frac{2x}{7}$ Hours, because 7 Miles : 2 Hours : : x Miles : $\frac{2x}{7}$ Hours. And so, since *B* rides 8 Miles in 3 Hours, he will describe his Space or ride his Journey $59 - x$ in $\frac{177 - 3x}{8}$ Hours. Now, since the Difference of these Times is one Hour; to the End they may become equal, add that Difference to the shorter Time $\frac{177 - 3x}{8}$, and you will have $1 + \frac{177 - 3x}{8} = \frac{2x}{7}$, and by Reduction $35 = x$. For, multiplying by 8 you have $185 - 3x = \frac{16x}{7}$. Then multiplying also by 7 you have $1295 - 21x = 16x$, or $1295 = 37x$. And, lastly, dividing by 37, there arises $35 = x$. Therefore, 35 Miles is the Distance that *A* must ride before he meets *B*.

The same more generally.

Having given the Velocities of two moveable Bodies, A and B, tending to the same Place, together with the Interval or Distance of the Places and Times from, and in which they begin to move; to determine the Place they shall meet in.

Suppose, the Velocity of the Body *A* to be such, that it shall pass over the Space c in the Time f; and of the Body *B* to be such as it shall pass over the Space d in the Time g; and that the Interval of the Places is e, and h the Interval of the Times in which they begin to move.

CASE I. Then if both tend to the same Place, [or the same Way] and *A* be the Body that, at the Beginning of the Motion, is farthest distant from the Place they tend to: Call

call that Diſtance x, and ſubtract from it the Diſtance e, and there will remain $x - e$ for the Diſtance of B from the Place it tends to. And ſince A paſſes through the Space c in the Time f, the Time in which it will paſs over the Space x will be $\frac{fx}{c}$, becauſe the Space c is to the Time f, as the Space x to the Time $\frac{fx}{c}$. And ſo, ſince B paſſes the Space d in the Time g, the Time in which it will paſs the Space $x - e$ will be $\frac{gx - ge}{d}$. Now ſince the Difference of theſe Times is ſuppoſed h, that they may become equal, add h to the ſhorter Time, *viz.* to the Time $\frac{fx}{c}$ if B begins to move firſt, and you will have $\frac{fx}{c} + h = \frac{gx - ge}{d}$, and by Reduction $\frac{cge + cdh}{cg - df}$ or $\frac{ge + dh}{g - \frac{d}{c}f} = x$. But if A begins to move firſt, add h to the Time $\frac{gx - ge}{d}$, and you will have $\frac{fx}{c} = h + \frac{gx - ge}{d}$, and by Reduction $\frac{cge - cdh}{cg - df} = x$.

CASE II. If the moveable Bodies proceed towards one another, and x, as before, be made the initial Diſtance of the moveable Body A, from the Place it is to move to, then $e - x$ will be the initial Diſtance of the Body B from the ſame Place; and $\frac{fx}{c}$ the Time in which A will deſcribe the Diſtance x, and $\frac{ge - gx}{d}$ the Time in which B will deſcribe its Diſtance $e - x$. To the leſſer of which Times, as above, add the Difference h, *viz.* to the Time $\frac{fx}{c}$ if B begin firſt to move, and ſo you will have $\frac{fx}{c} + h = \frac{ge - gx}{d}$, and by Reduction $\frac{cge - cdh}{cg + df} = x$. But if *A* begins first to move, add h to the Time $\frac{ge - gx}{d}$ and it will become $\frac{fx}{c} = h + \frac{ge - gx}{d}$, and by Reduction $\frac{cge + cdh}{cg + df} = x$.

Example I. If the Sun moves every Day one Degree, and the Moon thirteen, and at a certain Time the Sun be at the Beginning of *Cancer*, and, in three Days after, the Moon in the Beginning of *Aries*, the Place of their next following Conjunction is demanded. Answer, in $10\frac{1}{4}$ Deg. of *Cancer*: For since they both are going towards the same Parts, and the Motion of the Moon, which is farther distant from the Conjunction, hath a later *Epocha*, the Moon will be *A*, the Sun *B*, and $\frac{cge + cdh}{cg - df}$ the Length of the Moon's Way, which, if you write 13 for c, 1 for f, d, and g, 90 for e, and 3 for h, will become $\frac{13 \times 1 \times 90 + 13 \times 1 \times 3}{13 \times 1 - 1 \times 1}$, that is, $\frac{1209}{12}$, or $100\frac{3}{4}$ Degrees; and then add these Degrees to the Beginning of *Aries*, and there will come out $10\frac{3}{4}$ Deg. of *Cancer*.

Example II. If two Post-Boys, *A* and *B*, being in the Morning 59 Miles asunder, set out to meet each other, and *A* goes 7 Miles in 2 Hours, and *B* 8 Miles in 3 Hours, and *B* begins his Journey 1 Hour later than *A*, it is demanded how far *A* will have gone before he meets *B*? Answer, 35 Miles. For since they go towards each other, and *A* sets out first, $\frac{cge + cdh}{cg + df}$ will be the Length of his Journey; and writing 7 for c, 2 for f, 8 for d, 3 for g, 59 for e, and 1 for h, this will become $\frac{7 \times 3 \times 59 + 7 \times 8 \times 1}{7 \times 3 + 8 \times 2}$, that is, $\frac{1295}{37}$ or 35.

PRO-

PROBLEM VI.

Giving the Power of any Agent, to find how many such Agents will perform a given Effect a, *in a given Time* b.

Let the Power of the Agent be such that it can produce the Effect c in the Time d, and it will be as the Time d to the Time b, so the Effect c, which that Agent can produce in the Time d, to the Effect which he can produce in the Time b, which therefore will be $\frac{bc}{d}$. Again, as the Effect of one Agent $\frac{bc}{d}$ to the Effect of all a; so that single Agent to all the Agents; and thus the Number of the Agents will be $\frac{ad}{bc}$.

EXAMPLE. If a Scribe can in 8 Days write 15 Sheets, how many such Scribes must there be to write 405 Sheets in 9 Days? Answer 24. For if 8 be substituted for d, 15 for c, 405 for a, and 9 for b, the Number $\frac{ad}{bc}$ will become $\frac{405 \times 8}{9 \times 15}$, that is, $\frac{3240}{135}$, or 24.

PROBLEM VII.

The Forces of several Agents being given, to determine x *the Time, wherein they will jointly perform a given Effect* d.

Let the Forces of the Agents A, B, C, be supposed, which in the Times e, f, g can produce the Effects a, b, c respectively; and these in the Time x will produce the Effects $\frac{ax}{e}$, $\frac{bx}{f}$, $\frac{cx}{g}$; wherefore is $\frac{ax}{e} + \frac{bx}{f} + \frac{cx}{g} = d$, and by Reduction $x = \frac{d}{\frac{a}{e} + \frac{b}{f} + \frac{c}{g}}$.

EXAMPLE. Three Workmen can do a Piece of Work in certain Times, *viz. A* once in 3 Weeks, *B* thrice in 8 Weeks, and *C* five times in 12 Weeks. It is desired to know in what Time they can finish it jointly? Here then are the Forces of the Agents *A*, *B*, *C*, which in the Times 3, 8, 12 can produce the Effects 1, 3, 5, respectively, and the Time is sought wherein in they can do one Effect. Wherefore, for $a, b, c; d; e, f, g$ write 1, 3, 5, 1, 3, 8, 12, and there will arise $x = \frac{1}{\frac{1}{3}+\frac{3}{8}+\frac{5}{12}}$, or $\frac{8}{9}$ of a Week, that is, [allowing 6 working Days to a Week, and 12 Hours to each Day] 5 Days and 4 Hours, the Time wherein they will jointly finish it.

PROBLEM VIII.

So, to compound unlike Mixtures of two or more things, that the Things mixed together may have a given Ratio to one another.

Let the given Quantity of one Mixture be $dA + eB + fC$, the same Quantity of another Mixture $gA + hB + kC$, and the same of a third $lA + mB + nC$, where A, B, C denote the Things mixed, and $d, e, f, g, h,$ &c. the Proportions of the same in the Mixtures. And let $pA + qB + rC$ be the Mixture which must be composed of these three Mixtures; and suppose x, y and z to be the Numbers, by which if the three given Mixtures be respectively multiplied, their Sum will become $pA + qB + rC$.

Therefore is $\left\{\begin{array}{l} dxA + exB + fxC \\ +gyA + hyB + kyC \\ +lzA + mzB + nzC \end{array}\right\} = pA + qB + rC.$

And then comparing the Terms by making $dx + gy + lz = p$, $ex + hy + mz = q$, and $fx + ky + nz = r$, and by Reduction $x = \frac{p - gy - lz}{d} = \frac{q - hy - mz}{e} = \frac{r - ky - nz}{f}$. And again, the Equations $\frac{p - gy - lz}{d} = \frac{q - hy - mz}{e}$, and $\frac{q - hy - mz}{e} = \frac{r - ky - nz}{f}$ by Reduction give $\frac{ep - dq + dmz - elz}{eg - dh} (= y) = \frac{fq - er + enz - fmz}{fh - ek}$: Which, if abbreviated by writing α for $ep - dq$, β for $dm - el$, γ for $eg - dh$, δ for $fq - er$, ζ for $en - fm$, and θ for $fh - ek$, will become $\frac{\alpha + \beta z}{\gamma} = \frac{\delta + \zeta z}{\theta}$, and by Reduction $\frac{\theta\alpha - \gamma\delta}{\gamma\zeta - \beta\theta} = z$. Having found z, put $\frac{\alpha + \beta z}{\gamma} = y$, and $\frac{p - gy - lz}{d} = x$.

EXAM-

Example. If there were three Mixtures of Metals melted down together; of the first of which a Pound [Averdupois] contains of Silver ℥ 12, of Brass ℥ 1, and of Tin ℥ 3; of the second, a Pound contains of Silver ℥ 1, of Brass ℥ 12, and of Tin ℥ 3; and a Pound of the third contains of Brass ℥ 14, of Tin ℥ 2, and no Silver; and let these Mixtures be so to be compounded, that a Pound of the Composition may contain of Silver ℥ 4, of Brass ℥ 9, and of Tin ℥ 3: For $d, e, f;\ g, h, k;\ l, m, n;\ p, q, r,$ write 12, 1, 3; 1, 12, 3; 0, 14, 2; 4, 9, 3 respectively, and α will be $(=ep - dq = 1 \times 4 - 12 \times 9) = -104$, and β $(= dm - el = 12 \times 14 - 1 \times 0) = 168$, and so $\gamma = -143$, $\delta = 24$, $\zeta = -40$, and $\theta = 33$. And therefore $z \left(= \frac{\theta\alpha - \gamma\delta}{\gamma\zeta - \beta\theta} = \frac{-3432 + 3432}{5720 - 5544}\right) = 0$; $y \left(= \frac{\alpha + \beta z}{\gamma} = \frac{-104 + 0}{-143}\right) = \frac{8}{11}$, and $x \left(= \frac{p - gy - lz}{d} = \frac{4 - \frac{8}{11}}{12}\right) = \frac{3}{11}$. Wherefore, if there be mix'd $\frac{8}{11}$ Parts of a Pound of the second Mixture, $\frac{3}{11}$ Parts of a Pound of the first, and nothing of the third, the Aggregate will be a Pound, containing four Ounces of Silver, nine of Brass, and three of Tin.

Problem IX.

The Prices of several Mixtures of the same Things, and the Proportions of the Things mixed together being given, to determine the Price of each of the Things mixed.

Of each of the Things A, B, C, let the Price of the Mixture $dA + gB + lC$ be p, of the Mixture $eA + hB + mC$ the Price q, and of the Mixture $fA + kB + nC$ the Price r; and of those Things A, B, C let the Prices $x, y, z,$ be demanded. For the Things A, B, C substitute their Prices $x, y, z,$ and there will arise the Equations $dx + gy + lz = p$, $ex + hy + mz = q$, and $fx + ky + nz = r$; from which, by proceeding as in the foregoing Problem, there will in like manner be got $\frac{\theta\alpha - \gamma\delta}{\gamma\zeta - \beta\theta} = z$, $\frac{\alpha + \beta z}{\gamma} = y$, and $\frac{p - gy - lz}{d} = x$.

Example

Example. One bought 40 Bushels of Wheat, 24 Bushels of Barley, and 20 Bushels of Oats together, for 15 Pounds 12 Shillings. Again, he bought of the same Grain 26 Bushels of Wheat, 30 Bushels of Barley, and 50 Bushels of Oats together, for 16 Pounds. And thirdly, he bought of the like kind of Grain, 24 Bushels of Wheat, 120 Bushels of Barley, and 100 Bushels of Oats together, for 34 Pounds. It is demanded at what Rate a Bushel of each of the Grains ought to be valued. Answer, a Bushel of Wheat at 5 Shillings, of Barley at 3 Shillings, and of Oats at 2 Shillings. For instead of d, g, l; e, h, m; f, k, n; p, q, r, by writing respectively 40, 24, 20; 26, 30, 50; 24, 120, 100; $15\frac{3}{5}$, 16, and 34, there arises α $(= ep - dq = 26 \times 15\frac{3}{5} - 40 \times 16) = -234\frac{2}{5}$; and β $(= dm - el = 40 \times 50 - 26 \times 20) = 1480$; and thus $\gamma = -576$, $\delta = -500$, $\zeta = 1400$, and $\theta = -2400$. Then $z \left(= \frac{\theta\alpha - \gamma\delta}{\gamma\zeta - \beta\theta} = \frac{562560 - 288000}{-806400 + 3552000} = \frac{274560}{2745600}\right) = \frac{1}{10}$; $y \left(= \frac{\alpha + \beta z}{\gamma} = \frac{-234\frac{2}{5} + 148}{-576}\right) = \frac{3}{20}$; and $x \left(= \frac{p - gy - lz}{d} = \frac{15\frac{3}{5} - \frac{18}{5} - 2}{40}\right) = \frac{1}{4}$. Therefore a Bushel of Wheat cost $\frac{1}{4}$ lb, or 5 Shillings, a Bushel of Barley $\frac{3}{20}$ lb, or 3 Shillings, and a Bushel of Oats $\frac{1}{10}$ lb, or 2 Shillings.

Problem X.

There being given the specifick Gravity both of the Mixture and the Things mixed, to find the Proportion of the mixed Things to one another.

Let e be the specifick Gravity of the Mixture $A + B$, a the specifick Gravity of A, and b the specifick Gravity of B; and since the absolute Gravity, or the Weight, is composed of the Bulk of the Body and the specifick Gravity, aA will be the Weight of A; bB of B; and $eA + eB$ the Weight of the Mixture $A + B$; and therefore $aA + bB = eA + eB$; and from thence $aA - eA = eB - bB$ or $e - b : a - e :: A : B$.

EXAMPLE. Suppose the Gravity or specifick Weight of Gold to be as 19, and of Silver as $10\frac{1}{3}$, and King *Hiero's* Crown as 17; and it will be 10 : 3 ($e - b : a - e :: A : B$) :: Bulk of Gold in the Crown : Bulk of Silver, or 190 : 31 ($:: 19 \times 10 : 10\frac{1}{3} \times 3 :: a \times \overline{e - b} : b \times \overline{a - e}$) :: the Weight of Gold in the Crown, to the Weight of Silver, and 221 : 31 :: the Weight of the Crown, to the Weight of the Silver.

PROBLEM XI.

If the Number of Oxen a *eat up the Meadow* b *in the Time* c; *and the Number of Oxen* d *eat up as good a Piece of Pasture* e *in the Time* f, *and the Grass grows uniformly; to find how many Oxen will eat up the like Pasture* g *in the Time* h.

If the Oxen a in the Time c eat up the Pasture b; then, by Proportion, the Oxen $\frac{e}{b}a$ in the same Time c, or the Oxen $\frac{ec}{bf}a$ in the Time f, or the Oxen $\frac{ec}{bh}a$ in the Time h will eat up the Pasture e; supposing the Grass did not grow at all after the Time c. But since, by reason of the Growth of the Grass, all the Oxen d in the Time f can eat up only the Meadow e, therefore that Growth of the Grass in the Meadow e in the Time $f - c$ will be so much as alone would be sufficient to feed the Oxen $d - \frac{eca}{bf}$ the Time f, that is as much as would suffice to feed the Oxen $\frac{df}{h} - \frac{eca}{bh}$ in the Time h. And in the Time $h - c$, by Proportion so much would be the Growth of the Grass as would be sufficient to feed the Oxen $\frac{h-c}{f-c}$ into $\frac{df}{h} - \frac{eca}{bh}$ or $\frac{bdfh - ecah - bdcf + aecc}{bfh - bch}$.

Add this Increment to the Oxen $\frac{aec}{bh}$, and there will come out $\frac{bdfh - ecah - bdcf + ecfa}{bfh - bch}$, the Number of Oxen which the Pasture e will suffice to feed in the Time h. And ſo in Proportion the Meadow g will ſuffice to feed the Oxen $\frac{gbdfh - ecagh - bdcgf + ecfga}{befh - bceh}$ during the ſame Time h.

Example. If 12 Oxen eat up $3\frac{1}{3}$ Acres of Paſture in 4 Weeks, and 21 Oxen eat up ten Acres of like Paſture in 9 Weeks; to find how many Oxen will eat up 24 Acres in 18 Weeks? Anſwer 36; for that Number will be found by ſubſtituting in $\frac{bdfgh - ecagh - bdcgf + ecfga}{befh - bceh}$ the Numbers 12, $3\frac{1}{3}$, 4, 21, 10, 9, 24, and 18 for the Letters a, b, c, d, e, f, g, and h reſpectively; but the Solution, perhaps, will be no leſs expedite, if it be brought out from the firſt Principles, in Form of the precedent literal Solution. As if 12 Oxen in 4 Weeks eat up $3\frac{1}{3}$ Acres, then by Proportion 36 Oxen in 4 Weeks, or 16 Oxen in 9 Weeks, or 8 Oxen in 18 Weeks, will eat up 10 Acres, on Suppoſition that the Graſs did not grow. But ſince by reaſon of the Growth of the Graſs 21 Oxen in 9 Weeks can eat up only 10 Acres, that Growth of the Graſs in 10 Acres for the laſt 5 Weeks will be as much as would be ſufficient to feed the Exceſs of 21 Oxen above 16, that is 5 Oxen for 9 Weeks, or what is the ſame Thing, to feed $\frac{5}{2}$ Oxen for 18 Weeks. And in 14 Weeks (the Exceſs of 18 above the firſt 4) the Increaſe of the Graſs, by Analogy, will be ſuch, as to be ſufficient to feed 7 Oxen for 18 Weeks; for it is 5 Weeks : 14 Weeks :: $\frac{5}{2}$ Oxen : 7 Oxen. Wherefore add theſe 7 Oxen, which the Growth of the Graſs alone would ſuffice to feed, to the 8, which the Graſs without Growth after 4 Weeks would feed, and the Sum will be 15 Oxen. And, laſtly, if 10 Acres ſuffice to feed 15 Oxen for 18 Weeks, then, in Proportion, 24 Acres would ſuffice 36 Oxen for the ſame Time.

Problem XII.

Having given the Magnitudes and Motions of Spherical Bodies perfectly elaſtick, moving in the ſame right Line, and ſtriking againſt one another, to determine their Motions after Reflexion.

The Reſolution of this Queſtion depends on theſe Conditions, that each Body will ſuffer as much by Re-action as the

Action

Action of each is upon the other, and that they must recede from each other after Reflexion with the same Velocity or Swiftness as they met before it. These Things being supposed, let the Velocity of the Bodies A and B, be a and b respectively; and their Motions (as being composed of their Bulk and Velocity together) will be aA and bB. And if the Bodies tend the same Way, and A moving more swiftly, follows B, make x the Decrement of the Motion aA, and the Increment of the Motion bB arising by the Percussion; and the Motions after Reflexion will be $aA-x$ and $bB+x$; and the Celerities $\frac{aA-x}{A}$ and $\frac{bB+x}{B}$, whose Difference is $=a-b$ the Difference of the Celerities before Reflexion. Therefore there arises this Equation $\frac{bB+x}{B}-\frac{aA-x}{A}=a-b$, and thence by Reduction x becomes $=\frac{2aAB-2bAB}{A+B}$, which being substituted for x in the Celerities $\frac{aA-x}{A}$, and $\frac{bB+x}{B}$, there comes out $\frac{aA-aB+2bB}{A+B}$ for the Celerity of A, and $\frac{2aA-bA+bB}{A+B}$ for the Celerity of B after Reflexion.

But if the Bodies move towards one another, then changing every where the Sign of b, the Velocities after Reflexion will be $\frac{aA-aB-2bB}{A+B}$ and $\frac{2aA+bA-bB}{A+B}$; either of which, if they come out, by Chance, Negative, it argues that Motion, after Reflexion, to tend a contrary Way to that which A tended to before Reflexion. Which is also to be understood of A's Motion in the former Case.

Example. If the homogeneous Bodies [or Bodies of the same Sort] A of 3 Pounds with 8 Degrees of Velocity, and B, a Body of 9 Pounds with 2 Degrees of Velocity, tend the same Way; then for A, a, B and b, write 3, 8, 9 and 2; and $\left(\frac{aA-aB+2bB}{A+B}\right)$ beomes -1, and

$\left(\frac{2aA - bA + bB}{A + B}\right)$ becomes 5. Therefore A will return back with one Degree of Velocity after Reflexion, and B will go on with 5 Degrees.

PROBLEM XIII.

To find three Numbers in continual Proportion, whose Sum shall be 20, *and the Sum of their Squares* 140?

Make the first of the Numbers $= x$, and the second $= y$, and the third will be $\frac{yy}{x}$, and consequently $x + y + \frac{yy}{x} = 20$; and $xx + yy + \frac{y^4}{xx} = 140$. And by Reduction $xx \begin{smallmatrix} +y \\ -20 \end{smallmatrix} x + yy = 0$, and $x^4 \begin{smallmatrix} +yy \\ -140 \end{smallmatrix} xx + y^4 = 0$. Now to exterminate x, for a, b, c, d, e, f, g, h, in the third Rule, substitute respectively $1, 0, yy - 140, 0, y^4; 1, y - 20$, and yy; and there will come out $-\overline{yy + 280} \times y^6 : + \overline{2yy - 40y + 260} \times \overline{260y^3 - 40y^5} : + 3y^4 \times y^4 : - 2yy \times y^5 - 40y^5 + 400y^4 : = 0$; and by Multiplication $1600y^5 - 20800y^5 - 67600y^4 = 0$. And by Reduction $4yy - 52y + 169 = 0$. Or (the Root being extracted) $2y - 13 = 0$, or $y = 6\frac{1}{2}$. Which is found more short by another Method before, but not so obvious as this. Moreover, to find x, substitute $6\frac{1}{2}$ for y in the Equation $xx \begin{smallmatrix} +y \\ -20 \end{smallmatrix} x + yy = 0$, and there will arise $xx - 13\frac{1}{2}x + 42\frac{1}{4} = 0$, or $xx = 13\frac{1}{2}x + 42\frac{1}{4}$, and having extracted the Root $x = 6\frac{3}{4} + \text{or} - \sqrt{3\frac{5}{16}}$; *viz.* $6\frac{3}{4} + \sqrt{3\frac{5}{16}}$ is the greatest of the three Numbers sought, and $6\frac{3}{4} - \sqrt{3\frac{5}{16}}$ the least. For x denotes ambiguously either of the extreme Numbers, and thence there will come out two Values, either of which may be x, the other being $\frac{yy}{x}$.

The same otherwise. Putting the Numbers x, y and $\frac{yy}{x}$

as

as before, you will have $x+y+\frac{yy}{x}=20$, or $xx={20 \atop -y}x-yy$, and extracting the Root $x=10-\frac{1}{2}y+\sqrt{100-10y-\frac{3}{4}yy}$ for the first Number: Take away this and y from 20, and there remains $\frac{yy}{x}=10-\frac{1}{2}y-\sqrt{100-10y-\frac{3}{4}yy}$ the third Number. And the Sum of the Squares arising from these three Numbers is $400-40y$, and so $400-40y=140$, or $y=6\frac{1}{2}$. And having found the mean Number $6\frac{1}{2}$, substitute it for y in the first and third Number above found; and the first will become $6\frac{3}{4}+\sqrt{3\frac{5}{16}}$, and the third $6\frac{3}{4}-\sqrt{3\frac{5}{16}}$, as before.

PROBLEM XIV.

To find four Numbers in continual Proportion, the two Means whereof together make 12, and the two Extremes 20.

Let x be the second Number; and $12-x$ will be the third; $\frac{xx}{12-x}$ the first; and $\frac{144-24x+xx}{x}$ the fourth; and consequently $\frac{xx}{12-x}+\frac{144-24x+xx}{x}=20$. And by Reduction $xx=12x-30\frac{6}{7}$, or $x=6+\sqrt{5\frac{1}{7}}$. Which being found, the other Numbers are given from those above.

PROBLEM XV.

To find four Numbers continually proportional, whereof the Sum a *is given, and also the Sum of their Squares* b.

Although we ought for the most Part to seek the Quantities required immediately, yet if there are two that are ambiguous, that is, that involve both the same Conditions, (as here the two Means and two Extremes of the four Proportionals) the best Way is to seek other Quantities that are not ambiguous, by which these may be determined, as suppose their Sum, or Difference, or Rectangle. Let us therefore make the Sum of the two mean Numbers to be s, and the Rectangle r; and the Sum of the Extremes will be $a-s$, and the Rectangle also r, because of the Proportionality. Now that from hence these four Numbers may be found, make x the first, and y the second; and

and $s-y$ will be the third; and $a-s-x$ the fourth; and the Rectangle under the Means $sy-yy=r$, and thence one Mean $y=\frac{1}{2}s+\sqrt{\frac{1}{4}ss-r}$, the other $s-y=\frac{1}{2}s-\sqrt{\frac{1}{4}ss-r}$. Alſo, the Rectangle under the Extremes $ax-sx-xx=r$,

and thence one Extreme $x=\frac{a-s}{2}+\sqrt{\frac{ss-2as+aa}{4}-r}$,

and the other $a-s-x=\frac{a-s}{2}-\sqrt{\frac{ss-2as+aa}{4}-r}$.

The Sum of the Squares of theſe four Numbers is $2ss-2as+aa-4r$ which is $=b$. Therefore $r=\frac{1}{2}ss-\frac{1}{2}as+\frac{1}{4}aa-\frac{1}{4}b$, which being ſubſtituted for r, there come out the four Numbers as follows:

The two Means $\begin{cases}\frac{1}{2}s+\sqrt{\frac{1}{4}b-\frac{1}{4}ss+\frac{1}{2}as-\frac{1}{4}aa}\\ \frac{1}{2}s-\sqrt{\frac{1}{4}b-\frac{1}{4}ss+\frac{1}{2}as-\frac{1}{4}aa.}\end{cases}$

The two Extremes $\begin{cases}\frac{a-s}{2}+\sqrt{\frac{1}{4}b-\frac{1}{4}ss}\\ \frac{a-s}{2}-\sqrt{\frac{1}{4}b-\frac{1}{4}ss.}\end{cases}$

Yet there remains the Value of s to be found. Wherefore, to abbreviate the Terms, for theſe Quantities ſubſtitute

$$\frac{1}{2}s+p. \qquad \frac{a-s}{2}+q$$

and

$$\frac{1}{2}s-p. \qquad \frac{a-s}{2}-q$$

And make the Rectangle under the ſecond and fourth equal to the Square of the third, ſince this Condition of the Queſtion is not yet ſatisfied, and you will have $\frac{as-ss}{4}-\frac{1}{2}qs+\frac{pa-ps}{2}-pq=\frac{1}{4}ss-ps+pp$. Make alſo the Rectangle under the firſt and third equal to the Square of the ſecond, and you will have $\frac{as-ss}{4}+\frac{1}{2}qs+\frac{-pa+ps}{2}-pq=\frac{1}{4}ss+ps+pp$. Take the firſt of theſe Equations from the latter, and there will remain $qs-pa+ps=2ps$, or $qs=$

$pa+ps$. Restore now $\sqrt{\frac{1}{4}b-\frac{1}{4}ss+\frac{1}{2}as-\frac{1}{4}aa}$ in the Place of p, and $\sqrt{\frac{1}{4}b-\frac{1}{4}ss}$ in the Place of q, and you will have $s\sqrt{\frac{1}{4}b-\frac{1}{4}ss}=\overline{a+s}\sqrt{\frac{1}{4}b-\frac{1}{4}ss+\frac{1}{2}as-\frac{1}{4}aa}$, and by squaring $ss=-\frac{b}{a}s+\frac{1}{2}aa-\frac{1}{2}b$, or $s=-\frac{b}{2a}+\sqrt{\frac{bb}{4aa}+\frac{1}{2}aa-\frac{1}{2}b}$; which being found, the four Numbers sought are given from what has been shewn above.

Problem XVI.

If an annual Pension of the Number of Pounds a, *to be paid in the five next following Years, be bought for the ready Money* c, *to find what the Compound Interest of* 100 l. per Annum *will amount to?*

Make $1-x$ the Compound Interest of the Money x for a Year, that is, that the Money 1 to be paid after one Year is worth x in ready Money: and, by Proportion, the Money a to be paid after one Year will be worth ax in ready Money, and after two Years it will be worth axx, and after three Years ax^3, and after four Years ax^4, and after five Years ax^5. Add these five Terms, and you will have $ax^5+ax^4+ax^3+axx+ax=c$, or $x^5+x^4+x^3+x^2+x=\frac{c}{a}$ an Equation of five Dimensions, by Help of which when x is found by the † Rules to be taught hereafter, put $x:1::100:y$, and $y-100$ will be the Compound Interest of 100 *l. per Annum.*

It is sufficient to have given these Instances in Questions where only the Proportions of Quantities are to be considered, without the Positions of Lines: Let us now proceed to the Solutions of Geometrical Problems.

† Viz. *by finding the first Figures of the Root by any mechanical Construction, and the remaining Figures by the Method of* Vieta.

How

How Geometrical Questions may be reduced to Equations.

GEometrical Questions may be reduced sometimes to Equations with as much Ease, and by the same Laws, as those we have proposed concerning abstracted Quantities. As if the right Line [See *Fig.* 6.] A B be to be divided in mean and extreme Proportion in C, that is, so that B E the Square of the greatest Part shall be equal to the Rectangle B D contained under the whole and the least Part; having put $AB = a$, and $BC = x$, then will A C be $= a - x$, and $xx = a$ into $a - x$; an Equation which by Reduction gives $x = -\frac{1}{2}a + \sqrt{\frac{5}{4}aa}$.

But in Geometrical Affairs, which more frequently occur, they so much depend on the various Positions and complex Relations of Lines, that they require some farther Invention and Artifice to bring them into Algebraick Terms. And though it is difficult to prescribe any Thing in these Sorts of Cases, and every Person's own Genius ought to be his Guide in these Operations; yet I will endeavour to shew the Way to Learners. You are to know therefore, that Questions about the same Lines, related after any definite Manner to one another, may be variously proposed, by making different Quantities the *Quæsita* or Things sought, from different *Data* or Things given. But of what *Data* or *Quæsita* soever the Question be proposed, its Solution will follow the same Way by an Analytick Series, without any other Variation of Circumstance besides the feigned Species of Lines, or the Names by which we are used to distinguish the given Quantities from those fought.

As if the Question be of an *Isosceles* Triangle C B D [See *Fig.* 7.] inscribed in a Circle, whose Sides B C, B D, and Base C D, are to be compared with the Diameter of the Circle A B. This may either be proposed of the Investigation of the *Diameter* from the given Sides and Base, or of the Investigation of the *Basis* from the given Sides and Diameter; or lastly, of the Investigation of the *Sides* from the given Base and Diameter; but however it be proposed, it will be reduced to an Equation by the same Series of an Analysis, *viz.* If the *Diameter* be sought, I put $AB = x$, $CD = a$, and BC or $BD = b$. Then (having drawn A C) by reason of the similar Triangles A B C, and C B E, it will be A B : B C

BC :: BC : BE, or $x : b :: b :$ BE. Wherefore BE $= \frac{bb}{x}$. Moreover CE is $= \frac{1}{2}$ CD or $\frac{1}{2}a$; and by reason of the right Angle CEB, $CEq + BEq = BCq$, that is $\frac{1}{4}aa + \frac{b^4}{xx} = bb$. Which Equation, by Reduction, will give the Quantity x sought.

But if the *Base* be sought, put AB $= c$, CD $= x$, and BC or BD $= b$. Then (AC being drawn) because of the similar Triangles ABC and CBE, there is AB : BC :: BC : BE, or $c : b :: b :$ BE. Wherefore BE $= \frac{bb}{c}$; and also CE $= \frac{1}{2}$ CD, or $\frac{1}{2}x$. And because the Angle CEB is right, $CEq + BEq = BCq$, that is, $\frac{1}{4}xx + \frac{b^4}{cc} = bb$; an Equation which will give by Reduction the sought Quantity x.

But if the Side BC or BD be sought, put AB $= c$, CD $= a$, and BC or BD $= x$. And (AC being drawn as before) by reason of the similar Triangles ABC and CBE, it is AB : BC :: BC : BE; or $c : x :: x :$ BE. Wherefore BE $= \frac{xx}{c}$. Moreover CE is $= \frac{1}{2}$ CD or $\frac{1}{2}a$; and by reason of the right Angle CEB, $CEq + BEq$ is $= BCq$, that is, $\frac{1}{4}aa + \frac{x^4}{cc} = xx$; and the Equation, by Reduction, will give the Quantity sought, *viz.* x.

You see therefore that in every Case, the Calculus, by which you come to the Equation, is the same every where, and brings out the same Equation, excepting only that I have denoted the Lines by different Letters according as I made the *Data* and *Quæsita* different. And from different *Data* and *Quæsita* there arises a Diversity in the Reduction of the Equation found: For the Reduction of the Equation $\frac{1}{4}aa + \frac{b^4}{xx} = bb$, in order to obtain $x = \frac{2bb}{\sqrt{4bb - aa}}$ the Value of AB, is different from the Reduction of the Equation

Equation $\frac{1}{4}xx + \frac{b^4}{cc} = bb$, in order to obtain $x = \frac{2b}{c}\sqrt{cc - bb}$, the Value of CD; and the Reduction of the Equation $\frac{1}{4}aa + \frac{x^4}{cc} = xx$ very different to obtain $x = \sqrt{\frac{1}{4}cc \pm \frac{1}{2}c\sqrt{cc - aa}}$ the Value of BC or BD: (as well as this also, $\frac{1}{4}aa + \frac{b^4}{cc} = bb$, to bring out c, a, or b, ought to be reduced after different Methods) but there was no Difference in the Investigation of these Equations. And hence it is that Analysts order us to make no Difference between the given and sought Quantities. For since the same Computation agrees to any Case of the given and sought Quantities, it is convenient that they should be conceived and compared without any Difference, that we may the more rightly judge of the Methods of computing them; or rather it is convenient that you should imagine, that the Question is proposed of those *Data* and *Quæsita* given and sought Quantities, by which you think it is most easy for you to make out your Equation.

Having therefore any Problem proposed, compare the Quantities which it involves, and making no Difference between the given and sought ones, consider how they depend one upon another, that you may know what Quantities if they are assumed, will, by proceeding synthetically, give the rest. To do which, there is no need that you should at first of all consider how they may be deduced from one another Algebraically; but this general Consideration will suffice, that they may be some how or other deduced by a direct Connexion with one another. For Example; If the Question be put of the Diameter of the Circle AD, [See *Fig.* 8.] and the three Lines AB, BC, and CD inscribed in a Semi-circle, and from the rest given you are to find BC; at first Sight it is manifest, that the Diameter AD determines the Semi-circle, and then that the Lines AB and CD by Inscription determine the Points B and C, and consequently the Quantity sought BC, and that by a direct Connexion; and yet after what Manner BC is to be had from these *Data* or given Quantities, is not so evident to be found by an Analysis. The same Thing is also to be understood of AB or CD if they were to be sought from the other *Data*. Now, if AD were to be found from the given Quantities AB, BC, and CD, it is is equally evident it could not be done Synthetically; for the Distance of the Points A and D depends on the Angles B and C, and those Angles on the Circle in which the given Lines are to be inscribed, and that Circle is not given without knowing the Diameter A D. The Nature of the Thing therefore requires, that A D be sought, not Synthetically, but by assuming it as given to make thence a Regression to the Quantities given.

When you shall have thoroughly perceived the different Orderings of the Process by which the Terms of the Question may be explained, *make Use of any of the Synthetical Methods by assuming Lines as given, from which the Process to others seems very easy, and the Regression to them very difficult.* For the Computation, though it may proceed through various Mediums, yet will begin from those Lines; and will be sooner performed by supposing the Question to be such, as if it was proposed of those *Data*, and some Quantity sought that would easily come out from them, than by thinking of the Question as it is really proposed. Thus, in the proposed Example, if from the rest of the Quantities given you were to find A D. Since I perceive that it cannot be done Synthetically, but yet provided it was given, I could proceed in my Ratiocination in a direct Connexion from that to other Things, I assume A D as given, and then I begin to compute as if it was given indeed, and some of the other Quantities, *viz.* some of the given ones, as A B, B C, or C D, were sought. And by this Method, by carrying on the Computation from the Quantities assumed after this Way to the others, as the Relations of the Lines to one another direct, there will always be obtained an Equation between two Values of some one Quantity, whether one of those Values be a Letter set down as a Representation or Name at the Beginning of the Work for that Quantity, and the other a Value of it found out by Computation, or whether both be found by a Computation made after different Ways.

But when you have compared the Terms of the Question thus generally, there is more Art and Invention required to find out the particular Connexions or Relations of the Lines that shall accommodate them to Computation. For those Things, which to a Person that does not so thoroughly consider them, may seem to be immediately, and by a very near Relation connected together, when we have a Mind to express that Relation Algebraically, require a great deal more round-about Proceeding, and oblige you to begin your Schemes anew

anew, and carry on your Computation Step by Step; as may appear by finding BC from AD, AB, and CD. For you are only to proceed by such Propositions or Enunciations that can fitly be represented in Algebraick Terms, whereof in particular you have some from *Eucl. Ax.* 19. *Prop.* 4. *Book* 6. and *Prop.* 47. of the first.

In the first Place therefore, the Calculus may be assisted by the Addition and Subtraction of Lines, so that from the Values of the Parts you may find the Values of the Whole, or from the Value of the Whole and one of the Parts you may obtain the Value of the other Part.

In the second Place, the Calculus is promoted by the Proportionality of Lines; for we suppose (as above) that the Rectangle of the mean Terms, divided by either of the Extremes, gives the Value of the other; or, which is the same Thing, if the Values of all four of the Proportionals are first had, we make an Equality between the Rectangles of the Extremes and Means. But the Proportionality of Lines is best found out by the Semilarity of Triangles, which, as it is known by the Equality of their Angles, the Analyst ought in particular to be conversant in comparing them, and consequently not to be ignorant of *Eucl. Prop.* 5, 13, 15, 29, and 32 of the first Book, and of *Prop.* 4, 5, 6, 7, and 8 of the sixth Book, and of the 20, 21, 22, 27, and 31 of the third Book of his *Elem.* To which also may be added the 3d *Prop.* of the sixth Book, wherein, from the Proportion of the Sides is inferred the Equality of the Angles, and *è contra.* Sometimes likewise the 36 and 37th *Prop.* of the third Book will do the same Thing.

In the third Place, the Calculus is promoted by the Addition or Subtraction of Squares, *viz.* In right angled Triangles we add the Squares of the lesser Sides to obtain the Square of the greatest, or from the Square of the greatest Side we subtract the Square of one of the lesser, to obtain the Square of the other.

And on these few Foundations (if we add to them *Prop.* 1. of the 6th *Elem.* when the Business relates to Superficies, as also some Propositions taken out of the 11th and 12th of *Euclid*, when Solids come in Question) the whole Analytick Art, as to right-lined Geometry, depends. Moreover, all the Difficulties of Problems may be reduced to the sole Composition of Lines out of Parts, and the Similarity of Triangles; so that there is no Occasion to make use of other

Theorems; because they may all be resolved into these two, and consequently into the Solutions that may be drawn from them. And, for an Instance of this, I have subjoined a Problem about letting fall a Perpendicular upon the Base of an oblique-angled Triangle, which is solved without the Help of the 47th *Prop.* of the first Book of *Euclid*. But although it may be of Use not to be ignorant of the most simple Principles on which the Solutions of Problems depend, and though by only their Help any Problems may be solved; yet, for Expedition sake, it will be convenient not only that the 47th *Prop.* of the first Book of *Euclid*, whose Use is most frequent, but also that other *Theorems* should sometimes be made Use of.

As if, for Example, a Perpendicular being let fall upon the Base of an oblique-angled Triangle, the Question were (for the sake of promoting Algebraick Calculus) to find the Segments of the Base; here it would be of Use to know, that the Difference of the Squares of the Sides is equal to the double Rectangle under the Base, and the Distance of the Perpendicular from the Middle of the Basis.

If the Vertical Angle of any Triangle be bisected, it will not only be of Use to know, that the Base is divided in Proportion to the Sides, but also that the Difference of the Rectangles made by the Sides, and the Segments of the Base is equal to the Square of the Line that bisects the Angle.

If the Problem relate to Figures inscribed in a Circle, this Theorem will frequently be of Use, *viz.* That in any quadrilateral Figure inscribed in a Circle, the Rectangle of the Diagonals is equal to the Sum of the Rectangles of the opposite Sides.

The Analyst may observe several Theorems of this Nature in his Practice, and reserve them for his Use; but let him use them sparingly, if he can, with equal Facility, or not much more Difficulty, deduce the Solution from more simple Principles of Computation. Wherefore let him take especial Notice of the three Principles first proposed, as being more known, more simple, more general, but a few, and yet sufficient for all Problems, and let him endeavour to reduce all Difficulties to them before others.

But that these Theorems may be accommodated to the Solution of Problems, the *Schemes* are oft times to be farther constructed, and that most frequently, by producing out some of of the Lines till they cut others, or become of an aſſigned Length; or by drawing from ſome remarkable Point, Lines parallel or perpendicular to others, or by conjoining ſome remarkable Points; as alſo ſometimes by conſtructing after other Methods, according as the State of the Problem, and the Theorems which are made uſe of to ſolve it, ſhall require. As for Example, If two Lines that do not meet each other, make given Angles with a certain third Line, perhaps we produce them ſo, that when they concur, or meet, they ſhall form a Triangle, whoſe Angles, and conſequently the Ratio's of their Sides, ſhall be given; or, if any Angle is given, or be equal to any one, we often complete it into a Triangle given in Specie, or ſimilar to ſome other, and that by producing ſome of the Lines in the Scheme, or by drawing a Line ſubtending an Angle. If the Triangle be an oblique angled one, we often reſolve it into two right angled ones, by letting fall a Perpendicular. If the Buſineſs concerns multilateral or many ſided Figures, we reſolve them into Triangles, by drawing Diagonal Lines; and ſo in others; always aiming at this End, *viz. that the Scheme may be reſolved either into given, or ſimilar, or right angled Triangles.* Thus, in the Example propoſed, [See *Fig.* 9.] I draw the Diagonal BD, and the Trapezium ABCD may be reſolved into the two Triangles, ABD a right angled one, and BDC an oblique angled one. Then I reſolve the oblique angled one into two right angled Triangles, by letting fall a Perpendicular from any of its Angles, B, C or D, upon the oppoſite Side; as from B upon CD produced to E, that BE may meet it perpendicularly. But ſince the Angles BAD and BCD make in the mean while two right ones (by 22 *Prop.* 3. *Elem.*) as well as BCE and BCD, I perceive the Angles BAD and BCE to be equal; conſequently the Triangles BCE and DAB to be ſimilar. And ſo I ſee that the Computation (by aſſuming AD, AB, and BC as if CD were ſought) may be thus carried on, *viz.* AD and AB (by reaſon of the right-angled Triangle ABD) give you BD. AD, AB, BD, and BC (by reaſon of the ſimilar Triangles AED and CEB) give BE and CE. BD and BE (by reaſon of the right angled Triangle BED) give ED; and ED — EC gives CD. Whence there will be obtained an Equation between the Value of CD ſo found out, and the Algebraick Letter that was put for it. We may alſo (and for the greateſt Part it is better ſo to do, than to follow the Work too far in one continued Series) begin the

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Computation from different Principles, or at least promote it by divers Methods to any one and the same Conclusion, that at length there may be obtained two Values of any the same Quantity, which may be made equal to one another. Thus, AD, AB, and BC, give BD, BE, and CE as before; then CD + CE gives ED; and lastly, BD, and ED (by reason of the right angled Triangle BED) give BE. You might also very well form the Computation thus, that the Values of those Quantities should be sought between which any other known Relation interceeds, and then that Relation will bring it to an Equation. Thus, since the Relation between the Lines BD, DC, BC, and CE, is manifest from the 12th *Prop.* of the second Book of the *Elem. viz.* that $BDq - BCq - CDq$ is $= 2CD \times CE$: I seek BDq from the assumed AD and AB; and CE from the assumed AD, AB, and BC. And, lastly, assuming CD I make $BDq - BCq - CDq = 2CD \times CE$. After such Ways, and led by these Sorts of Consultations, you ought always to take care of the Series of the Analysis, and of the Scheme to be constructed in order to it, at once.

Hence, I believe, it will be manifest what Geometricians mean, when they bid you imagine that to be already done which is sought. For making no Difference between the known and unknown Quantities, you may assume any of them to begin your Computation from, as much as if all had indeed been known by a previous Solution, and you were no longer to consult the Solution of the Problem, but only the Proof of that Solution. Thus, in the first of the three Ways of computing already described, although perhaps AD be really sought, yet I imagine CD to be the Quantity sought, as if I had a mind to try whether its Value derived from AD will coincide with its Quantity before known. So also in the two last Methods, I do not propose, as my Aim, any Quantity to be sought, but only some how or other to bring out an Equation from the Relations of the Lines: And, for sake of that Business, I assume all the Lines AD, AB, BC, and CD as known, as much as if (the Question being before solved) the Business was to enquire whether such and such Lines would tisfy the Conditions of it, by agreeing with any Equations which the Relations of the Lines can exhibit. I entered upon the Business at first Sight after this Way, and with such Sort of Consultations; but when I arrive at an Equation, I change my Method, and endeavour to to find the Quantity sought by the Reduction and Solution of that Equation. Thus, lastly, we assume often more Quantities as known, than what are expressed in the State of the Question. Of this you may see an eminent Example in the 55th of the following Problems, where I have assumed a, b, and c, in the Equation $aa + bx + cx^2 = yy$ for determining the Conick Section; as also the other Lines r, s, t, v, of which the Problem, as it is proposed, hints nothing. For you may assume any Quantities by the Help whereof it is possible to come to Equations; only taking this Care, that you obtain as many Equations from them as you assume Quantities really unknown.

After you have consulted your Method of Computation, and drawn up your Scheme, give Names to the Quantities that enter into the Computation, (that is, from which being assumed, the Values of others are to be derived, until at last you come to an Equation) chusing such as involve all the Conditions of the Problem, and seem accommodated before others to the Business, and that shall render the Conclusion (as far as you can guess) more simple, but yet not more than what shall be sufficient for your Purpose. Wherefore, do not give proper Names to Quantities which may be denominated from Names already given. Thus, from a whole Line and its Parts, from the three Sides of a right angled Triangle, and from three or four Proportionals, some one of the least considerable we leave without a Name, because its Value may be derived from the Names of the rest. As in the Example already brought, if I make $AD = x$, and $AB = a$, I denote BD by no Letter, because it is the third Side of a right angled Triangle ABD, and consequently its Value is $\sqrt{xx - aa}$. Then if I call $BC = b$, since the Triangles DAB and BCE are similar, and thence the Lines AD : AB :: BC : CE proportional, to three whereof, *viz.* to AD, AB, and BC there are already Names given; for that reason I leave the fourth CE without a Name, and in its room I make Use of $\frac{ab}{x}$ discovered from the foregoing Proportionality. And so if DC be called c, I give no Name to DE, because from its Parts, DC and CE, or c and $\frac{ab}{x}$, its Value $c + \frac{ab}{x}$ comes out. [*See Figure* 10.]

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But while I am talking of these Things, the Problem is almost reduced to an Equation. For, after the aforesaid Letters are set down for the Species of the principal Lines, there remains nothing else to be done, but that out of those Species the Values of other Lines be made out according to a preconceived Method, until after some foreseen Way they come out to an Equation. And I see nothing wanting in this Case, except that by means of the right angled Triangles BCE and BDE I can bring out a double Value of BE, *viz.* $BCq - CEq$ (or $bb - \frac{aabb}{xx}$) $= BEq$; as also $BDq - DEq$ (or $xx - aa - cc - \frac{2abc}{x} - \frac{aabb}{xx}$) $= BEq$. And hence (blotting out on both Sides $\frac{aabb}{xx}$) I shall have the Equation $bb = xx - aa - cc - \frac{2abc}{x}$; which being reduced, becomes $x^3 = \begin{matrix} +aa \\ +bb \\ +cc \end{matrix} x + 2abc$.

But since I have reckoned up several Methods for the Solution of this Problem, and those not much unlike one another in the precedent Paragraphs, of which that taken from *Prop.* 12. of the second Book of the *Elem.* being something more elegant than the rest, we will here subjoin it. Make therefore $AD = x$, $AB = a$, $BC = b$, and $CD = c$, and you will have $BDq = xx - aa$, and $CE = \frac{ab}{x}$ as before. These Species therefore being substituted in the Theorem $BDq - BCq - CDq = 2\,CD \times CE$, there will arise $xx - aa - bb - cc = \frac{2abc}{x}$; and after Reduction, $x^3 = \begin{matrix} aa \\ +bb \\ +cc \end{matrix} x + 2abc$, as before.

But that it may appear how great a Variety there is in the Invention of Solutions, and that it is not very difficult for a prudent Geometrician to light upon them; I have thought fit to shew other Ways of doing the same Thing. And having drawn the Diagonal BD, if in room of the Perpendicular BE, which before was let fall from the Point B upon the Side DC, you now let fall a Perpendicular from the Point D upon

upon the Side B C, or from the Point C upon the Side B D, by which the oblique angled Triangle B C D may any how be resolved into two right angled Triangles, you may come almost by the same Methods I have already described to an Equation. And there are other Methods very different from these.

As if there are drawn two Diagonals, A C and B D, [*See Figure* 11.] B D will be given by assuming A D and A B; as also A C by assuming A D and C D; then, by the known Theorem of Quadrilateral Figures inscribed in a Circle, *viz.* That $AD \times BC + AB \times CD$ is $= AC \times BD$, you will obtain an Equation. [*See Figure* 11.] The Names therefore of the Lines A D, A B, B C, C D, remaining, *viz.* x, a, b, c; B D will be $= \sqrt{xx - aa}$, and $AC = \sqrt{xx - cc}$, by the 47th *Prop.* of the first *Elem.* and these Species of the Lines being substituted in the Theorem we just now mentioned, there will come out $xb + ac = \sqrt{xx - cc} \times \sqrt{xx - aa}$. The Parts of which Equation being squared and reduced, you will again have $x^3 = \begin{matrix} +aa \\ +bb \\ +cc \end{matrix} x + 2abc$.

But, moreover, that it may be manifest after what Manner the Solutions drawn from that Theorem may be thence reduced to only the Similarity of Triangles; erect B H perpendicular to B C, and meeting A C in H, and there will be formed the Triangles B C H, B D A similar, by reason of the right Angles at B, and equal Angles at C and D, (by the 21. 3. *Elem.*); as also the Triangles B C D, B H A similar, by reason of the equal Angles both at B, (as may appear by taking away the common Angle D B H from the two right ones) as also at D and A (by 21. 3. *Elem.*) You may see therefore, that from the Proportionality B D : A D : : H C, there is given the Line H C; as also A H from the Proportionality B D : C D : : A B : A H. Whence, since $AH + HC = AC$, you have an Equation. The Names therefore aforesaid of the Lines remaining, *viz.* x, a, b, c, as also the Values of the Lines A C and B D, *viz.* $\sqrt{xx - cc}$ and $\sqrt{xx - aa}$, the first Proportionality will give $HC = \frac{bx}{\sqrt{xx - aa}}$, and the second will give $AH = \frac{ac}{\sqrt{xx - aa}}$. Whence, by reason of

AH + HC = AC, you will have $\frac{bx+ac}{\sqrt{xx-aa}} = \sqrt{xx-cc}$; an Equation which (by multiplying by $\sqrt{xx-aa}$, and by squaring) will be reduced to a Form often described in the preceeding Pages.

But that it may yet farther appear what a Plenty of Solutions may be found, produce B C and A D [*See Figure* 12.] till they meet in F, and the Triangles A B F and C D F will be similar, because the Angle at F is common, and the Angles A B F and CDF (while they compleat the Angle CDA to two right ones, by 13, 1. and 22, 3 *Elem.*) are equal. Wherefore, if besides the four Terms which compose the Question, there was given A F, the Proportion A B : A F :: CD : C F would give C F. Also A F — A D would give D F, and the Proportion CD : D F :: A B : B F would give B F; whence (since B F — C F is = B C) there would arise an Equation. But since there are assumed two unknown Quantities A D and D F as if they were given, there remains another Equation to be found. I let fall therefore B G at right Angles upon A F, and the Proportion A D : A B :: A B : AG will give A G; which being had, the Theorem borrowed from the 13, 2 *Eucl. viz.* that $BFq + 2FAG$ is $= ABq + AFq$ will give another Equation. a, b, c, x, remaining therefore as before, and making $AF = y$, you will have (by insisting on the Steps already laid down) $\frac{cy}{a} = CF$. $y - x = DF$. $\frac{\overline{y-x} \times a}{c} = BF$. And thence $\frac{\overline{y-x} \times a}{c} - \frac{cy}{a} = b$, the first Equation. Also $\frac{aa}{x}$ will be $= AG$, and consequently $\frac{aayy - 2a^2xy + a^2x^2}{cc} + \frac{2aay}{x} = aa + yy$ for the second Equation. Which two, by Reduction, will give the Equation sought, *viz.* The Value of y found by the first Equation is $\frac{abc + aax}{aa - cc}$, which being substituted in the second, will give an Equation, from which rightly ordered will come out $x^3 = \begin{matrix} +aa \\ +bb \\ +cc \end{matrix} x + 2abc$, as before.

And so, if A B and D C are produced till they meet one another, the Solution will be much the same, unless perhaps it be something easier. Wherefore I will rather subjoyn another Specimen of this Problem drawn from a Fountain very unlike the former, *viz.* by seeking the Area of the Quadrilateral Figure proposed, and that doubly. I draw therefore the Diagonal B D, and the Quadrilateral Figure may be resolved into two Triangles. Then using the Names of the Lines x, a, b, c, as before, I find $BD = \sqrt{xx - aa}$, and thence $\frac{1}{2}a\sqrt{xx - aa}$ ($= \frac{1}{2}AB \times BD$) the Area of the Triangle A B D. Moreover, having let fall B E perpendicularly upon C D you will have (by reason of the similar Triangles A B D, B C E) $AD : BD :: BC : BE$, and consequently $BE = \frac{b}{x}\sqrt{xx - aa}$. Wherefore also $\frac{bc}{2x}\sqrt{xx - aa}$ ($= \frac{1}{2}CD \times BE$) will be the Area of the Triangle B C D. Now, by adding these Area's, there will arise $\frac{ax + bc}{2x}\sqrt{xx - aa}$, the Area of the whole Quadrilateral. After the same Way, by drawing the Diagonal A C, and seeking the Area's of the Triangles A C D and A C B, and adding them, there will again be obtained the Area of the Quadrilateral Figure $\frac{cx + ba}{2x}\sqrt{xx - cc}$. Wherefore, by making these Area's equal, and multiplying both by $2x$, you will have $\overline{ax + bc}\sqrt{xx - aa} = \overline{cx + ba}\sqrt{xx - cc}$, an Equation which, by squaring and dividing by $aax - ccx$, will be reduced to the Form already often found out, $x^3 \begin{matrix} + aa \\ + bb \\ + cc \end{matrix} x + 2abc$.

Hence it may appear how great a Plenty of Solutions may be had, and that some Ways are much more neat than others. Wherefore, if the Method you take from your first Thoughts, for solving a Problem, be but ill accommodated to Computation, you must again consider the Relations of the Lines, until you shall have hit on a Way as fit and elegant as possible. For those Ways that offer themselves at first Sight, may often create sufficient Trouble if they are made use of. Thus, in the Problem we have been upon, it would not have been more

more difficult to have fallen upon the following Method than upon one of the precedent ones. [*See Figure* 13.] Having let fall B R and C S perpendicular to A D, as also C T to B R, the Figure will be resolved into right angled Triangles. And it may be seen, that A D and A B give A R, A D and C D give S D, A D — A R — S D gives R S or T C. Also A B and A R give B R, C D and S D give C S or T R, and B R — T R gives B T. Lastly, B T and T C give B C, whence an Equation will be obtained. But if any one should go to compute after this Rate, he would fall into larger and more perplexed Algebraick Terms than are any of the former, and more difficult to be brought to a final Equation.

So much for the Solution of Problems in right lined Geometry; unless it may perhaps be worth while to note moreover, that when Angles, or Positions of Lines, expressed by Angles, enter the State of the Question, Lines, or the Proportions of Lines, ought to be used instead of Angles, *viz.* such as may be derived from given Angles by a Trigonometrical Calculation; or from which being found, the Angles sought will come out by the same Calculus. Several Instances of which may be seen in the following Pages.

As for what belongs to the Geometry of Curve Lines, we use to denote them, either by describing them by the local Motion of right Lines, or by using Equations indefinitely expressing the Relation of right Lines disposed according to some certain Law, and ending at the Curve Lines. The Antients did the same by the Sections of Solids, but less commodiously. But the Computations that regard Curves described after the first Way, are no otherwise performed than in the precedent Pages. [*See Figure* 14.] As if A K C be a Curve Line described by K the Vertical Point of the Square A K ϕ, whereof one Leg A K freely slides through the Point A given by Position, while the other K ϕ of a determinate Length is carried along the right Line A D also given by Position, and you are to find the Point C in which any right Line C D given also by Position shall cut this Curve: I draw the right Lines A C, C F, which may represent the Square in the Position sought, and the Relation of the Lines (without any Difference or Regard of what is given or sought, or any Respect had to the Curve) being considered, I perceive the Dependency of the others upon C F and any of these four, *viz.* B C, B F, A F, and A C to be Synthetical; two whereof I therefore assume, as C F $= a$, and C B

CB $= x$, and beginning the Computation from thence, I presently obtain BF $= \sqrt{aa - xx}$, and AB $= \frac{xx}{\sqrt{aa - xx}}$, by reason of the right Angle CBF, and that the Lines BF : BC : : BC : AB are continual Proportionals. Moreover, from the given Position of CD, AD is given, which I therefore call b; there is also given the Ratio of BC to BD, which I make as d to e, and you have BD $= \frac{ex}{d}$, and AB $= b - \frac{ex}{d}$. Therefore $b - \frac{ex}{d}$ is $= \frac{xx}{\sqrt{aa - xx}}$, an Equation which (by squaring its Parts and multiplying by $aa - xx$, &c.) will be reduced to this Form, $x^4 =$

$$\frac{2bdex^3 \begin{matrix} -bbdd \\ +aaee \end{matrix} xx - 2aabdex + aabbdd}{dd + ee}$$

Whence, lastly, from the given Quantities a, b, d, and e, there may be found x, by Rules hereafter to be given, and at that Interval or Distance x or BC, a right Line drawn parallel to AD will cut CD in the Point sought C.

But if we do not use Geometrical Descriptions but Equations to denote the Curve Lines by, the Computations will thereby become as much shorter and easier, as the gaining of those Equations can make them. [See *Fig.* 15.] As if the Intersection C of the given Ellipsis ACE with the right Line CD given by Position, be sought. To denote the Ellipsis, I take some known Equation proper to it, as $rx - \frac{r}{q}xx = yy$, where x is indefinitely put for any Part of the Axis Ab or AB, and y for the Perpendicular bc or BC terminated at the Curve; and r and q are given from the given Species of the Ellipsis. Since therefore CD is given by Position, AD will be also given, which call a; and BD will be $a - x$; also the Angle ABC will be given, and thence the Ratio of BD to BC, which call 1 to e, and BC (y) will be $= ea - ex$, whose Square $eeaa - 2eeax + eexx$ will be equal to $rx - \frac{r}{q}xx$. And thence by Reduction

duction there will arise $xx = \frac{2aeex + rx - aaee}{ee + \frac{r}{q}}$, or

$$x = \frac{aee + \frac{1}{2}r \pm e\sqrt{ar + \frac{rr}{4ee} - \frac{aar}{q}}}{ee + \frac{r}{q}}.$$

Moreover, although a Curve be denoted by a Geometrical Description, or by a Section of a Solid, yet thence an Equation may be obtained, which shall define the Nature of the Curve, and consequently all the Difficulties of Problems proposed about it may be reduced hither.

Thus, in the former Example, [See *Fig.* 14.] if A B be called x, and B C y, the third Proportional B F will be $\frac{yy}{x}$, whose Square, together with the Square of B C, is equal to C F q, that is, $\frac{y^4}{xx} + yy = aa$; or $y^4 + xxyy = aaxx$. And this is an Equation by which every Point C of the Curve A K C, agreeing or corresponding to any Length A B of the Base (and consequently the Curve it self) is defined, and from whence therefore you may obtain the Solutions of Problems proposed concerning this Curve.

After the same Manner almost, when a Curve is not given in Specie, but proposed to be determined, you may feign an Equation at Pleasure, that may generally contain its Nature; and assume this to denote it as if it was given, that from its Assumption you can any Way come to Equations by which the Assumptions may at length be determined: Examples whereof you have in some of the following Problems, which I have collected for a more full Illustration of this Doctrine, and for the Exercise of Learners, and which I now proceed to deliver.

PROBLEM I.

Having a finite right Line BC *given, from whose Ends the two right Lines* BA, CA *are drawn in the given Angles* ABC, ACB; *to find* AD *the Height of their Concourse* A, *above the given Line* BC. [*See* Figure 16.]

Make BC $= a$, and AD $= y$; and since the Angle ABD is given, there will be given (from the Table of Sines or Tangents) the Ratio between the Lines AD and BD which make as d to e. Therefore $d : e :: \text{AD}\ (y) : \text{BD}$. Wherefore $\text{BD} = \frac{ey}{d}$. In like manner by reason of the given Angle ACD there will be given the Ratio between AD and DC, which make as d to f, and you will have $\text{DC} = \frac{fy}{d}$.

But $\text{BD} + \text{DC} = \text{BC}$, that is, $\frac{ey}{d} + \frac{fy}{d} = a$. Which reduced, by multiplying both Parts of the Equation by d, and dividing by $e + f$ becomes $y = \frac{ad}{e+f}$.

PROBLEM II.

The Sides AB, AC *of the Triangle* ABC *being given, and also the Base* BC, *which the Perpendicular* AD *let fall from the Vertical Angle cuts in* D, *to find the Segments* BD *and* DC. [*See* Figure 17.]

Let $\text{AB} = a$, $\text{AC} = b$, $\text{BC} = c$, and $\text{BD} = x$, and DC will $= c - x$. Now since $\text{AB}q - \text{BD}q\ (aa - xx) = \text{AD}q$; and $\text{AC}q - \text{DC}q\ (bb - cc + 2cx - xx) = \text{AD}q$; you will have $aa - xx = bb - cc + 2cx - xx$; which by Reduction becomes $\frac{aa - bb + cc}{2c} = x$.

But that it may appear that all the Difficulties of all Problems may be resolved by only the Proportionality of Lines, without the Help of the 47 of 1 *Eucl.* although not without round-about Methods, I thought fit to subjoin the following Solution of this Problem over and above. From the Point

Point D let fall the Perpendicular D E upon the Side A B, and the Names of the Lines, already given, remaining, you will have A B : B D :: B D : B E.

$a : x :: x \frac{xx}{a}$. And B A — B E $\left(a - \frac{xx}{a}\right) =$ E A. Also E A : A D :: A D : A B, and consequently E A × A B $(aa - xx) =$ A D q. And so, by reasoning about the Triangle A C D, there will be found again A D $q = bb - cc + 2cx - xx$. Whence you will obtain as before $x = \frac{aa - bb + cc}{2c}$.

PROBLEM III.

The Area and Perimeter of the right angled Triangle A B C *being given, to find the Hypothenuse* B C. [*See* Figure 18.]

Let the Perimeter be called a, the Area bb, make B C $= x$, and A C $= y$; then will be A B $= \sqrt{xx - yy}$; whence again the Perimeter (B C + A C + A B) is $x + y + \sqrt{xx - yy}$, and the Area ($\frac{1}{2}$ A C × A B) is $\frac{1}{2}y\sqrt{xx - yy}$. Therefore $x + y + \sqrt{xx - yy} = a$, and $\frac{1}{2}y\sqrt{xx - yy} = bb$.

The latter of these Equations gives $\sqrt{xx - yy} = \frac{2bb}{y}$; wherefore I write $\frac{2bb}{y}$ for $\sqrt{xx - yy}$ in the former Equation, that the Asymmetry may be taken away; and there comes out $x + y + \frac{2bb}{y} = a$, or multiplying by y, and ordering the Equation $yy = ay - xy - 2bb$. Moreover from the Parts of the former Equation I take away $x + y$ and there remains $\sqrt{xx - yy} = a - x - y$, and squaring the Parts to take away again the Asymmetry, there comes out $xx - yy = aa - 2ax - 2ay + xx + 2xy + yy$, which ordered and divided by 2 becomes $yy = ay - xy + ax - \frac{1}{2}aa$. Lastly, making an Equality between the two Values of yy, I have $ay - xy - 2bb = ay - xy + ax - \frac{1}{2}aa$, which reduced becomes $\frac{1}{2}a - \frac{2bb}{a} = x$.

The

The same otherwise.

Let $\frac{1}{2}$ the Perimeter be $= a$, the Area $= bb$, and $BC = x$, and it will be $AC + AB = 2a - x$. Now since xx (BCq) is $= ACq + ABq$, and $4bb = 2AC \times AB$, $xx + 4bb$ will be $= ACq + ABq + 2AC \times AB =$ to the Square of $\overline{AC + AB} =$ to the Square of $\overline{2a - x} = 4aa - 4ax + xx$. That is, $xx + 4bb = 4aa - 4ax + xx$, which reduced becomes $a - \frac{bb}{a} = x$.

PROBLEM IV.

Having given the Perimeter and Perpendicular of a right angled Triangle, to find the Triangle. [*See* Figure 67.]

Let C be the right Angle of the Triangle ABC and CD a Perpendicular let fall thence to the Base AB. Let there be given $AB + BC + AC = a$, and $CD = b$. Make the Base $AB = x$, and the Sum of the Sides will be $a - x$. Put y for the Difference of the Legs, and the greater Leg AC will be $= \frac{a - x + y}{2}$; the less $BC = \frac{a - x - y}{2}$. Now from the Nature of a right angled Triangle you have $ACq + BCq = ABq$, that is $\frac{aa - 2ax + xx + yy}{2} = xx$. And also $AB : AC :: BC : DC$; therefore $AB \times DC = AC \times BC$, that is $bx = \frac{aa - 2ax + xx - yy}{4}$. By the former Equation yy is $= xx + 2ax - aa$. By the latter $yy = xx - 2ax + aa - 4bx$. And consequently $xx + 2ax - aa = xx - 2ax + aa - 4bx$. And by Reduction $4ax + 4bx = 2aa$, or $x = \frac{aa}{2a + 2b}$.

Geometrically thus. In every right angled Triangle, as the Sum of the Perimeter and Perpendicular is to the Perimeter, so is half the Perimeter to the Base.

Subtract $2x$ from a, and there will remain $\frac{ab}{a+b}$ the Excess of the Sides above the Base. Whence again, as in every right angled Triangle, the Sum of the Perimeter and Perpendicular is to the Perimeter, so is the Perpendicular to the Excess of the Sides above the Base.

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PROBLEM V.

Having given the Base A B *of a right angled Triangle, and the Sum of the Perpendicular, and the Legs* C A + C B + C D; *to find the Triangle.*

Let be $CA + CB + CD = a$, $AB = b$, $CD = x$, and $AC + CB$ will be $= a - x$. Put $AC - CB = y$, and AC will be $= \frac{a - x + y}{2}$, and $CB = \frac{a - x - y}{2}$. But $ACq + CBq$ is $= ABq$; that is $\frac{aa - 2ax + xx + yy}{2} = bb$.

Moreover it is $AC \times CB = AB \times CD$, that is $\frac{aa - 2ax + xx - yy}{4} = bx$. Which being compared, you have $2bb - aa + 2ax - xx = yy = aa - 2ax + xx - 4bx$. And by Reduction, $xx = 2ax + 2bx - aa + bb$, and $x = a + b - \sqrt{2ab + 2bb}$.

Geometrically thus. In any right-angled Triangle, from the Sum of the Legs and Perpendicular subtract the mean Proportional between the said Sum and the Double of the Base, and there will remain the Perpendicular.

The same otherwise.

Make $CA + CB + CD = a$, $AB = b$, and $AC = x$, and BC will be $= \sqrt{bb - xx}$, $CD = \frac{x\sqrt{bb - xx}}{b}$. And $x + CB + CD = a$, or $CB + CD = a - x$. And therefore $\frac{b + x}{b}\sqrt{bb - xx} = a - x$. And the Parts being squared and multiplied by bb, there will be made $-x^4 - 2bx^3 + 2b^3x + b^4 = aabb - 2abbx + bbxx$. Which Equation being ordered, by Transposition of Parts, after this Manner, $x^4 + 2bx^3 \begin{matrix} +3bb \\ -2ab \end{matrix} xx \begin{matrix} +2b^3 \\ +2abb \end{matrix} x \begin{matrix} +b^4 \\ +2ab^3 \\ +aabb \end{matrix} = \begin{matrix} 2bb \\ 2ab \end{matrix} xx \begin{matrix} +4b^3 \\ +4abb \end{matrix} x \begin{matrix} +2b^4 \\ +3ab^3 \end{matrix}$ and extracting the Roots on

both

both Sides, there will arise $xx + bx + bb + ab = x + b\sqrt{2ab + 2bb}$. And the Root being again extracted $x = -\frac{1}{2}b + \sqrt{\frac{1}{2}bb + \frac{1}{2}ab} \pm \sqrt{b\sqrt{\frac{1}{2}bb + \frac{1}{2}ab} - \frac{1}{4}bb - \frac{1}{2}ab}$.

The Geometrical Construction. [*See* Figure 68.]

Take therefore $AB = \frac{1}{2}b$, $BC = \frac{1}{2}a$, $CD = \frac{1}{2}AB$, AE a mean Proportional between b and AC, and EF on both sides a mean Proportional between b and DE, and BF, BF will be the two Legs of the Triangle.

PROBLEM VI.

Having given in the right-angled Triangle ABC, *the Sum of the Sides* AC + BC, *and the Perpendicular* CD, *to find the Triangle.*

Let be $AC + BC = a$, $CD = b$, $AC = x$, and BC will be $= a - x$, $AB = \sqrt{aa - 2ax + 2xx}$. Moreover $CD : AC :: BC : AB$. Therefore again $AB = \frac{ax - xx}{b}$. Wherefore $ax - xx = b\sqrt{aa - 2ax + 2xx}$; and the Parts being squared and ordered $x^4 - 2ax^3 \begin{matrix} + aa \\ - 2bb \end{matrix} xx + 2abbx - aabb = 0$ Add to both Parts $aabb + b^4$, and there will be made $x^4 - 2ax^3 \begin{matrix} + aa \\ - 2bb \end{matrix} xx + 2abbx + b^4 = aabb + b^4$. And the Root being extracted on both Sides, $xx - ax - bb = -b\sqrt{aa + bb}$, and the Root being again extracted $x = \frac{1}{2}a \pm \sqrt{\frac{1}{4}aa + bb - b\sqrt{aa + bb}}$.

The Geometrical Construction. [*See* Figure 69.]

Take $AB = BC = \frac{1}{2}a$. At C erect the Perpendicular $CD = b$. Produce DC to E, so that DE shall be $= DA$. And between CD and CE take a mean Proportional CF. And let a Circle GH described from the Center F and the Radius BC, cut the right Line BC in G and H, and BG and BH will be the two Sides of the Triangle.

The

The same otherwise.

Let be $AC+BC=a$, $AC-BC=y$, $AB=x$, and $DC=b$, and $\frac{a+y}{2}$ will be $=AC$, $\frac{a-y}{2}=BC$, $\frac{aa+yy}{2}=ACq+BCq=ABq=xx$. $\frac{aa-yy}{4b}=\frac{AC\times BC}{DC}=AB=x$. Therefore $2xx-aa=yy=aa-4bx$, and $xx=aa-2bx$, and the Root being extracted $x=-b+\sqrt{bb+aa}$. Whence in the Construction above CE is the Hypothenuse of the Triangle sought. But the Base and Perpendicular, as well in this as the Problem above being given, the Triangle is thus expeditiously constructed. [*See Figure* 70.] Make a Parallelogram CG, whose Side CE shall be the Basis of the Triangle, and the other Side CF the Perpendicular. And upon CE describe a Semicircle, cutting the opposite Side FG in H. Draw CH, EH, and CHE will be the Triangle sought.

PROBLEM VII.

In a right-angled Triangle, having given the Sum of the Legs, and the Sum of the Perpendicular and Base, to find the Triangle.

Let the Sum of the Legs AC and BC be a, the Sum of the Base AB and of the Perpendicular CD be b, the Leg $AC=x$, the Base $AB=y$, and BC will be $=a-x$, $CD=b-y$, $aa-2ax+2xx=ACq+BCq=ABq=yy$, $ax-xx=AC\times BC=AB\times CD=by-yy=by-aa+2ax-2xx$, and $by=aa-ax+xx$. Make its Square $a^4-2a^3x+3aaxx-2ax^3+x^4$ equal to $yy\times bb$, that is, equal to $aabb-2abbx+2bbxx$. And ordering the Equation, there will come out $x^4-2ax^3\begin{matrix}+3aa\\-2bb\end{matrix}xx\begin{matrix}-2a^3\\+2abb\end{matrix}x\begin{matrix}+a^4\\-aabb\end{matrix}=0$. Add to each Side of the Equation b^4-aabb, and there will come out $x^4-2ax^3\begin{matrix}+3aa\\-2bb\end{matrix}xx\begin{matrix}-2a^3\\+2abb\end{matrix}x\begin{matrix}+a^4\\-2aabb\\+b^4\end{matrix}=b^4-aabb$. And the Root being extracted on both Sides $xx-ax+aa-bb=-b\sqrt{bb-aa}$, and the Root being again extracted $x=\frac{1}{2}a+\sqrt{bb-\frac{3}{4}aa-b\sqrt{bb-aa}}$.

The

The Geometrical Construction.

Take R a mean Proportional between $b + a$ and $b - a$, and S a mean Proportional between R and $b - $R, and T a mean Proportional between $\frac{1}{2}a + $S; and $\frac{1}{2}a - $S; and $\frac{1}{2}a + $T and $\frac{1}{2}a - $T will be the Sides of the Triangle.

PROBLEM VIII.

Having given the Area, Perimeter, and one of the Angles A *of any Triangle* A B C, *to determine the rest.* [*See Figure* 19.]

Let the Perimeter be $= a$, and the Area $= bb$, and from either of the unknown Angles, as C, let fall the Perpendicular C D to the opposite Side A B; and by reason of the given Angle A, A C will be to C D in a given Ratio, suppose as d to e. Call therefore A C $= x$, and C D will be $= \frac{ex}{d}$, by which divide the Double of the Area, and there will come out $\frac{2bbd}{ex} = $ AB. Add AD (*viz.* $\sqrt{ACq - CDq}$, or $\frac{x}{d} \times \sqrt{dd - ee}$) and there will come out BD $= \frac{2bbd}{ex} + \frac{x}{d} \times \sqrt{dd - ee}$; to the Square whereof add CDq, and there will arise $\frac{4b^4dd}{eexx} + xx + \frac{4bb}{e}\sqrt{dd - ee} = $ BCq. Moreover from the Perimeter take away AC and AB, and there will remain $a - x - \frac{2bbd}{ex} = $ B C, the Square whereof $aa - 2ax + xx - \frac{4abbd}{ex} + \frac{4bbd}{e} + \frac{4b^4dd}{eexx}$ make equal to the Square before found; and neglecting the Equivalents, you will have $\frac{4bb}{e}\sqrt{dd - ee} = aa - 2ax - \frac{4abbd}{ex} + \frac{4bbd}{e}$. And this, by assuming $4af$ for the given Terms

$aa +$

$aa + \frac{4bbd}{e} - \frac{4bb}{e}\sqrt{dd-ee}$, and by reducing, becomes

$xx = 2fx - \frac{2bbd}{e}$, or $x = f \pm \sqrt{ff - \frac{2bbd}{e}}$.

The same Equation would have come out also by seeking the Leg A B; for the Sides A B and A C are indifferently alike to all the Conditions of the Problem. Wherefore if

A C be made $= f - \sqrt{ff - \frac{2bbd}{e}}$, A B will be $= f +$

$\sqrt{ff - \frac{2bbd}{e}}$, and reciprocally; and the Sum of these $2f$ subtracted from the Perimeter, leaves the third Side B C $= a - 2f$.

PROBLEM IX.

Having given the Altitude, Base, and Sum of the Sides, to find the Triangle.

Let the Altitude CD be $= a$, half the Basis A B $= b$, half the Sum of the Sides $= c$, and their Semi-difference $= z$; and the greater Side as BC will be $= c + z$, and the lesser A C $= c - z$. Subtract C Dq from C Bq, and also from A Cq, and hence will BD be $= \sqrt{cc + 2cz + zz - aa}$, and thence AD $= \sqrt{cc - 2cz + zz - aa}$. Subtract also A B from BD, and AD will again be $= \sqrt{cc + 2cz + zz - aa} - 2b$. Having now squared the Values of A D, and ordered the Terms, there will arise $bb + cz = b\sqrt{cc + 2cz + zz - aa}$. Again, by squaring and reducing into Order, you will obtain $cczz - bbzz = bbcc - bbaa - b^4$. And $z = b \times$

$\sqrt{1 - \frac{aa}{cc - bb}}$. Whence the Sides are given.

PROBLEM X.

Having given the Base AB, *and the Sum of the Sides* AC + BC, *and also the Vertical Angle* C, *to determine the Sides.* [*See* Figure 20.]

Make the Base $= a$, half the Sum of the Sides $= b$, and half the Difference $= x$, and the greater Side BC will be $= b + x$, and the lesser AC $= b - x$. From either of the unknown Angles A let fall the Perpendicular AD to the opposite Side BC, and by reason of the given Angle C there will be given the Ratio of AC to CD, suppose as d to e, and then CD will be $= \frac{eb - ex}{d}$. Also, by 13. 2 *Elem.* $\frac{ACq - ABq + BCq}{2BC}$ that is $\frac{2bb + 2xx - aa}{2b + 2x} = CD$; and so you have an Equation between the Values of CD. And this reduced, x becomes $= \sqrt{\frac{daa + 2ebb - 2dbb}{2d + 2e}}$, whence the Sides are given.

If the Angles at the Base were sought, the Conclusion would be more neat, as draw EC bisecting the given Angle and meeting the Base in E; and it will be AB : AC + BC (:: AE : AC) :: Sine Angle ACE : Sine Angle AEC. And if from the Angle AEC, and also from its Complement BEC you subtract $\frac{1}{2}$ the Angle C, there will be left the Angles ABC and BAC.

PROBLEM XI.

Having the Sides of a Triangle given, to find the Angles. [*See* Figure 72.]

Let the given Sides AB be $= a$, AC $= b$, BC $= c$, to find the Angle A. Having let fall to AB the Perpendicular CD, which is opposite to that Angle, you will have in the first Place, $bb - cc = ACq - BCq = ADq - BDq = \overline{AD + BD} \times \overline{AD - BD} = AB \times \overline{2AD - AB} = 2AD \times a - aa$. And consequently $\frac{1}{2}a + \frac{bb - cc}{2a} = AD$. Whence comes out this *first Theorem.*

I. As AB to AC + BC so AB − BC to a fourth Proportional N. $\frac{AB + N}{2} = AD$. As AC to AD so Radius to the Cosine of the **Angle A.**

Moreover $DCq = ACq - ADq =$

$$\frac{2aabb + 2aacc + 2bbcc - a^4 - b^4 - c^4}{4aa} =$$

$$\frac{\overline{a+b+c} \times \overline{a+b-c} \times \overline{a-b+c} \times \overline{-a+b+c}}{4aa}.$$ Whence having multiplied the Roots of the Numerator and Denominator by b, there is made this *second Theorem.*

II. As $2ab$ to a mean Proportional between $\overline{a+b+c} \times \overline{a+b-c}$ and $\overline{a-b+c} \times \overline{-a+b+c}$, so is Radius to the Sine of the Angle A.

Moreover on AB take AE = AC, and draw CE, and the Angle ECD will be equal to half the Angle A. Take AD from AE, and there will remain $DE = b - \frac{1}{2}a - $

$$\frac{bb - cc}{2a} = \frac{cc - aa + 2ab - bb}{2a} = \frac{\overline{c+a-b} \times \overline{c-a+b}}{2a}.$$

Whence $DEq = \frac{\overline{c+a-b} \times \overline{c+a-b} \times \overline{c-a+b} \times \overline{c-a+b}}{4aa}.$

And hence is made the *third* and *fourth Theorem, viz.*

III. As $2ab$ to $\overline{c+a-b} \times \overline{c-a+b}$ (so AC to **DE**) so Radius to the versed Sine of the Angle A.

IV. And, as a mean Proportional between $a + b + c$, **and** $a + b - c$ to a mean Proportional between $c + a - b$, and $c - a + b$ (so CD to DE) so Radius to the Tangent of half the Angle A, or the Co-tangent of half the Angle to Radius.

Besides, CEq is $= CDq + DEq = \frac{2abb + bcc - baa - b^3}{a}$

$= \frac{b}{a} \times \overline{c+a-b} \times \overline{c-a+b}$. Whence the *fifth* and *sixth Theorem.*

V. As a mean Proportional between $2a$ and $2b$ to a mean Proportional between $c + a - b$, and $c - a + b$, or as 1 to a mean Proportional between $\frac{c+a-b}{2a}$, and $\frac{c-a+b}{2b}$ (so AC

to

to $\frac{1}{2}$ CE or CE to DE) so Radius to the Sine of $\frac{1}{2}$ the Angle A.

VI. And as a mean Proportional between $2a$ and $2b$ to a mean Proportional between $a+b+c$ and $a+b-c$ (so CE to CD) so Radius to the Cosine of half the Angle A.

But if besides the Angles, the Area of the Triangle be also sought, multiply CDq by $\frac{1}{4}ABq$, and the Root, *viz.* $\frac{1}{4}\sqrt{\overline{a+b+c}\times\overline{a+b-c}\times\overline{a-b+c}\times\overline{-a+b+c}}$ will be the Area sought.

PROBLEM XII.

Having the Sides and Base of any right lined Triangle given, to find the Segments of the Base, the Perpendicular, the Area, and the Angles. [See Figure 40.]

Let there be given the Sides AC, BC, and the Base AB of the Triangle ABC. Bisect AB in I, and take on it (being produced on both Sides) AF and AE equal to AC, and BG and BH equal to BC. Join CE, CF; and from C to the Base let fall the Perpendicular CD. And $ACq - BCq$ will be $= ADq + CDq - CDq - BDq$ $= ADq - BDq = \overline{AD+BD}\times\overline{AD-BD} = AB\times$ $2DI$. Therefore $\frac{ACq - BCq}{2AB} = DI$. And $2AB : AC + BC :: AC - BC : DI$. Which is a Theorem for determining the Segments of the Base.

From IE, that is, from $AC - \frac{1}{2}AB$, take away DI, and there will remain $DE = \frac{BCq - ACq + 2AC\times AB - ABq}{2AB}$, that is $= \frac{\overline{BC+AC-AB}\times\overline{BC-AC+AB}}{2AB}$, or $=$ $\frac{HE\times EG}{2AB}$. Take away DE from FE, or $2AC$, and there will remain $FD = \frac{ACq + 2AC\times AB + ABq - BCq}{2AB}$; that is $= \frac{\overline{AC+AB+BC}\times\overline{AC+AB-BC}}{2AB}$, or $=$

 FG

$\frac{FG \times FH}{2AB}$. And since CD is a mean Proportional between DE and DF, and CE a mean Proportional between DE and EF, and CF a mean Proportional between DF and EF, CD will be $= \frac{\sqrt{FG \times FH \times HE \times EG}}{2AB}$, $CE = \sqrt{\frac{AC \times HE \times EG}{AB}}$, and $CF = \sqrt{\frac{AC \times FG \times FH}{AB}}$. Multiply CD into $\frac{1}{2}AB$, and you will have the Area $= \frac{1}{4}\sqrt{FG \times FH \times HE \times EG}$. But for determining the Angle A, there come out several Theorems:

1. As $2AB \times AC : HE \times EG$ ($:: AC : DE$) $::$ Radius : versed Sine of the Angle A.

2. $2AB \times AC : FG \times FH$ ($:: AC : FD$) $::$ Radius : versed Cosine of A.

3. $2AB \times AC : \sqrt{FG \times FH \times HE \times EG}$ ($:: AC : CD$) $::$ Radius : Sine of A.

4. $\sqrt{FG \times FH} : \sqrt{HE \times EG}$ ($:: CF : CE$) $::$ Radius : Tangent of $\frac{1}{2}$ A.

5. $\sqrt{HE \times EG} : \sqrt{FG \times FH}$ ($:: CE : FC$) $::$ Radius : Cotangent of $\frac{1}{2}$ A.

6. $2\sqrt{AB \times AC} : \sqrt{HE \times EG}$ ($:: FE : CE$) $::$ Radius : Sine of $\frac{1}{2}$ A.

7. $2\sqrt{AB \times AC} : \sqrt{FG \times FH}$ ($:: FE : FC$) $::$ Radius : Cosine of $\frac{1}{2}$ A.

PROBLEM XIII.

To subtend the given Angle CBD *with the given right Line* CD; *so that if* AD *be drawn from the End of that right Line* D *to the Point* A, *given on the right Line* CB *produced, the Angle* ADC *shall be equal to the Angle* ABD. [*See* Figure 71.]

Make $CD = a$, $AB = b$, $BD = x$, and it will be $BD : BA :: CD : DA = \frac{ab}{x}$. Let fall the Perpendicular DE, and BE will be $= \frac{BDq - ADq + BAq}{2BA} = \frac{xx - \frac{aabb}{xx} + bb}{2b}$. By reason of the given Triangle DBA, make $BD : BE :: b : c$, and you will have again $BE = \frac{cx}{b}$, therefore $xx - \frac{aabb}{xx} + bb = 2cx$. And $x^4 - 2cx^3 + bbxx - aabb = 0$.

PROBLEM XIV.

To find the Triangle ABC, *whose three Sides* AB, AC, BC, *and its Perpendicular* DC *are in Arithmetical Progression.* [*See* Figure 46.]

Make $AC = a$, $BC = x$, and DC will be $= 2x - a$, and $AB = 2a - x$. Also AD will be $(= \sqrt{ACq - DCq}) = \sqrt{4ax - 4xx}$, and BD $(= \sqrt{BCq - DCq}) = \sqrt{4ax - 3xx - aa}$. And so again $AB = \sqrt{4ax - 4xx} + \sqrt{4ax - 3xx - aa}$. Wherefore $2a - x = \sqrt{4ax - 4xx} + \sqrt{4ax - 3xx - aa}$, or $2a - x - \sqrt{4ax - 4xx} = \sqrt{4ax - 3xx - aa}$. And the Parts being squared, $4aa - 3xx - \overline{4a + 2x} \times \sqrt{4ax - 4xx} = 4ax - 3xx - aa$, or $5aa - 4ax = \overline{4a - 2x} \times \sqrt{4ax - 4xx}$. And the Parts being again squared, and the Terms rightly disposed, $16x^4 - 80ax^3 + 144$

$+144aaxx - 104a^3x + 25a^4 = 0$. Divide this Equation by $2x - a$, and there will arise $8x^3 - 36axx + 54aax - 25a^3 = 0$, an Equation by the Solution whereof x is given from a, being any how assumed. a and x being had, make a Triangle, whose Sides shall be $2a - x$, a and x; and a Perpendicular let fall upon the Side $2a - x$ will be $2x - a$.

If I had made the Difference of the Sides of the Triangle to be d, and the Perpendicular to be x, the Work would have been something neater; this Equation at last coming out, *viz.* $x^3 = 24ddx - 48d^3$.

PROBLEM XV.

To find a Triangle ABC, *whose three Sides* AB, AC, BC, *and the Perpendicular* CD *shall be in a Geometrical Progression.*

Make $AC = x$, $BC = a$; and AB will be $= \frac{xx}{a}$.

And $CD = \frac{aa}{x}$. And $AD \left(= \sqrt{ACq - CDq}\right) = \sqrt{xx - \frac{a^4}{xx}}$; and $BD \left(= \sqrt{BCq - DCq}\right) = \sqrt{aa - \frac{a^4}{xx}}$

and consequently $\frac{xx}{a} (= AB) = \sqrt{xx - \frac{a^4}{xx}} + \sqrt{aa - \frac{a^4}{xx}}$, or $\frac{xx}{a} - \sqrt{aa - \frac{a^4}{xx}} = \sqrt{xx - \frac{a^4}{xx}}$; and the Parts of the Equation being squared, $\frac{x^4}{aa} - \frac{2xx}{a} \times \sqrt{aa - \frac{a^4}{xx}} + aa - \frac{a^4}{xx} = xx - \frac{a^4}{xx}$, that is, $x^4 - aaxx + a^4 = 2aax\sqrt{xx - aa}$. And the Parts being again squared, $x^8 - 2aax^6 + 3a^4x^4 - 2a^6xx + a^8 = 4a^4x^4 - 4a^6xx$. That is, $x^8 - 2aax^6 - a^4x^4 + 2a^6xx + a^8 = 0$. Divide this Equation by $x^4 - aaxx - a^4$, and there will arise $x^4 - aaxx - a^4$. Wherefore x^4 is $= aaxx + a^4$. And extracting the Root $xx = \frac{1}{2}aa + \sqrt{\frac{5}{4}a^4}$, or $x = a\sqrt{\frac{1}{2} + \sqrt{\frac{5}{4}}}$. Take therefore a or BC, of any Length,

Length, and make B C : A C :: A C : A B :: 1 : $\sqrt{\frac{1}{2}} + \sqrt{\frac{5}{4}}$, and the Perpendicular D C of a Triangle A B C made of these Sides, will be to the Side B C in the same Ratio.

The same otherwise. [*See* Figure 47.]

Since A B : A C :: B C : D C. I say the Angle A C B is a right one. For if you deny it, draw C E, making the Angle E C B a right one. Therefore the Triangles B C E, D B C are similiar by 8. 6 *Elem.* and consequently E B : E C :: B C : D C, that is, E B : E C :: A B : A C. Draw A F perpendicular to C E, and by reason of the parallel Lines A F, B C, E B will be : E C :: A E : F E :: A B : F C. Therefore by 9. 5 *Elem.* A C is = F C, that is, the Hypothenuse of a right-angled Triangle is equal to the Side, contrary to the 19. 1 *Elem.* Therefore the Angle E C B is not a right one; wherefore it is necessary A C B should be a right one. Therefore A Cq + B Cq is = A Bq. But A Cq = A B × B C, therefore A B × B C + B Cq = A Bq, and extracting the Root A B = $\frac{1}{2}$ B C + $\sqrt{\frac{5}{4}}$ B Cq. Wherefore take B C : A B :: 1 : $\frac{1+\sqrt{5}}{2}$, and A C a mean Proportional between B C and A B, and a Triangle being made of these Sides, A B, A C, B C, D C will be continually Proportionals.

Problem XVI.

To make the Triangle A B C *upon the given Base* A B, *whose Vertex* C *shall be in the right Line* E C *given in Position, and the Base an Arithmetical Mean between the Sides.* [*See* Figure 48.]

Let the Base A B be bisected in F, and produced until it meet the right Line given in Position E C in E, and let fall to it the Perpendicular C D; and making A B = a, F E = b, and B C — A B = x, B C will be = $a + x$, A C = $a - x$; and by the 13. 2 *Elem.* B D ($= \frac{BCq - ACq + ABq}{2AB}$) = $2x + \frac{1}{2}a$. And consequently, F D = $2x$, D E = $b + 2x$, and C D ($= \sqrt{CBq - BDq}$) = $\sqrt{\frac{3}{4}aa - 3xx}$. But by reason of the given Positions of the right Lines C E and A B, the Angle C E D is given; and consequently the Ratio of D E to C D, which if it be put as d to e, will give the Proportion

portion $d : e :: b + 2x : \sqrt{\frac{3}{4}aa - 3xx}$. Whence the Means and Extremes being multiplied by each other, there arises the Equation $eb + 2ex = d\sqrt{\frac{3}{4}aa - 3xx}$, the Parts whereof being squared and rightly ordered, you have $xx = \frac{\frac{3}{4}d^2a^2 - eebb - 4eebx}{4ee + 3dd}$, and the Root being extracted $x = \frac{-2eeb + d\sqrt{3eeaa - 3eebb + \frac{9}{4}ddaa}}{4ee + 3dd}$. But x being given, there is given $BC = a + x$, and $AC = a - x$.

Problem XVII.

Having given the Sides of any Parallelogram AB, BD, DC, and AC, *and one of the Diagonals* BC, *to find the other Diagonal* AD. [*See* Figure 21.]

Let E be the Concourse of the Diagonals, and to the Diagonal BC let fall the Perpendicular AF, and by the 13. 2 *Elem.* $\frac{ACq - ABq + BCq}{2BC} = CF$, and also $\frac{ACq - AEq + ECq}{2EC} = CF$. Wherefore since EC is $= \frac{1}{2}BC$, and $AE = \frac{1}{2}AD$ it will be $\frac{ACq - ABq + BCq}{2BC} = \frac{ACq - \frac{1}{4}ADq + \frac{1}{4}BCq}{BC}$, and having reduced, $AD = \sqrt{2ACq + 2ABq - BCq}$.

Whence, by the by, in any Parallelogram, the Sum of the Squares of the Sides is equal to the Sum of the Squares of the Diagonals.

Problem XVIII.

Having given the Angles of the Trapezium ABCD, *also its Perimeter and Area, to determine the Sides.* [See Figure 22.]

Produce any two of the Sides AB and DC till they meet in E, and let AB be $= x$, and $BC = y$, and because all the Angles are given, there are given the Ratio's of BC to CE and BE, which make d to e and f; and CE will be be $= \frac{ey}{d}$, and BE $= \frac{fy}{d}$, and consequently AE $= x + \frac{fy}{d}$. There are also given the Ratio's of AE to AD and to DE; which make as g and as h to d; and AD will be $= \frac{dx + fy}{g}$ and ED $= \frac{dx + fy}{h}$, and consequently CD $= \frac{dx + fy}{h} - \frac{ey}{d}$, and the Sum of all the Sides $x + y + \frac{dx + fy}{g} + \frac{dx + fy}{h} - \frac{ey}{d}$; which, since it is given, call it a, and the Terms will be abbreviated by writing $\frac{p}{r}$ for the given Quantity $1 + \frac{d}{g} + \frac{d}{h}$, and $\frac{q}{r}$ for the given $1 + \frac{f}{g} + \frac{f}{h} - \frac{e}{d}$, and you will have the Equation $\frac{px + qy}{r} = a$.

Moreover, by reason of all the Angles being given, there is given the Ratio of BCq to the Triangle BCE, which make as m to n, and the Triangle BCE will be $= \frac{n}{m}yy$. There is also given the Ratio of AEq to the Triangle ADE; which make as m to d; and the Triangle ADE will be $= \frac{ddxx + 2dfxy + ffyy}{dm}$. Wherefore, since the Area AC, which is the Difference of these Triangles, is given, let it be bb, and $\frac{ddxx + 2dfxy + ffyy - dnyy}{dm}$ will be $= bb$. And so you have two Equations, from the Reduction whereof all is determined, *viz.* The former Equation gives $\frac{ra - qy}{p} = x$, and by writing $\frac{ra - qy}{p}$ for x in the last, there comes out

I

$drraa$

$\frac{drraa - 2dqray + dqqyy}{ppm} + \frac{2afry - 2fqyy}{pm} + \frac{ffyy - dnyy}{dm} = bb$. And the Terms being abbreviated by writing s for the given Quantity $\frac{dqq}{pp} - \frac{2fq}{p} + \frac{ff}{d} - n$, and st for the given $+\frac{adqr}{pp} - \frac{afr}{p}$, and stv for the given $bbm - \frac{drraa}{pp}$, there arises $yy = 2ty + tv$, or $y = t + \sqrt{st + tv}$.

Problem XIX.

To surround the Fish-pond ABCD *with a Walk* ABC DEFGH *of a given Area, and of the same Breadth every where.* [*See* Figure 23.]

Let the Breadth of the Walk be x, and its Area aa. And, letting fall the Perpendiculars AK, BL, BM, CN, CO, DP, DQ, AI, from the Points A, B, C, D, to the Lines EF, FG, GH, and HE, to divide the Walk into the four Trapezia IK, LM, NO, PQ, and into the four Parallelograms AL, BN, CP, DI, of the Latitude x, and of the same Length with the Sides of the given Trapezium. Let therefore the Sum of the Sides (AB + BC + CD + DA) be $= b$, and the Sum of the Parallelograms will be $= bx$.

Moreover, having drawn AE, BF, CG, DH; since AI is $=$ AK, the Angle AEI will be $=$ Angle AEK $= \frac{1}{2}$ IEK, or $\frac{1}{2}$ DAB. Therefore the Angle AEI is given, and consequently the Ratio of AI to IE, which make as d to e, and IE will be $= \frac{ex}{d}$. Multiply this into $\frac{1}{2}$ AI, or $\frac{1}{2}x$, and the Area of the Triangle AEI will be $= \frac{exx}{2d}$. But by reason of equal Angles and Sides, the Triangles AEI and AEK are equal, and consequently the Trapezium IK ($= 2$ Triang. AEI) $= \frac{exx}{d}$. In like manner, by putting

putting BL : LF :: $d : f$, and CN : NG :: $d : g$, and DP : PH :: $d : h$, (for those Ratio's are also given from the given Angles, B, C, and D) you will have the Trapezium LM $= \frac{fxx}{d}$, NO $= \frac{gxx}{d}$, and PQ $= \frac{hxx}{d}$. Wherefore $\frac{cxx}{d} + \frac{fxx}{d} + \frac{gxx}{d} + \frac{hxx}{d}$, or $\frac{pxx}{d}$, by writing p for $e + f + g + h$ will be equal to the four Trapeziums IK + LM + NO + PQ; and consequently $\frac{pxx}{d} + bx$ will be equal to the whole Walk aa. Which Equation, by dividing all the Terms by $\frac{p}{d}$, and extracting its Root, x will become $= \frac{-db + \sqrt{bbdd + 4aapd}}{2p}$. And the Breadth of the Walk being thus found, it is easy to describe it.

Problem XX.

From the given Point C, *to draw the right Line* CF, *which together with two other right Lines* AE *and* AF *given by Position, shall comprehend or constitute the Triangle* AEF *of a given Magnitude.* [*See* Figure 24.]

Draw CD parallel to AE, and CB and EG perpendicular to AF, and let AD $= a$, CB $= b$, and AF $= x$, and the Area of the Triangle AEF be cc, and by reason of the proportional Quantities DF : AF (:: DC : AE) :: CB : EG; that is, $a + x : x :: b : \frac{bx}{a+x}$ it will be $\frac{bx}{a+x} =$ EG. Multiply this into $\frac{1}{2}$ AF, and there will come out $\frac{bxx}{2a+2x}$, the Quantity of the Area AEF, which is $= cc$. And so the Equation being ordered xx will be $= \frac{2ccx + 2cca}{b}$ or $x = \frac{cc + \sqrt{c^4 + 2ccab}}{b}$.

After

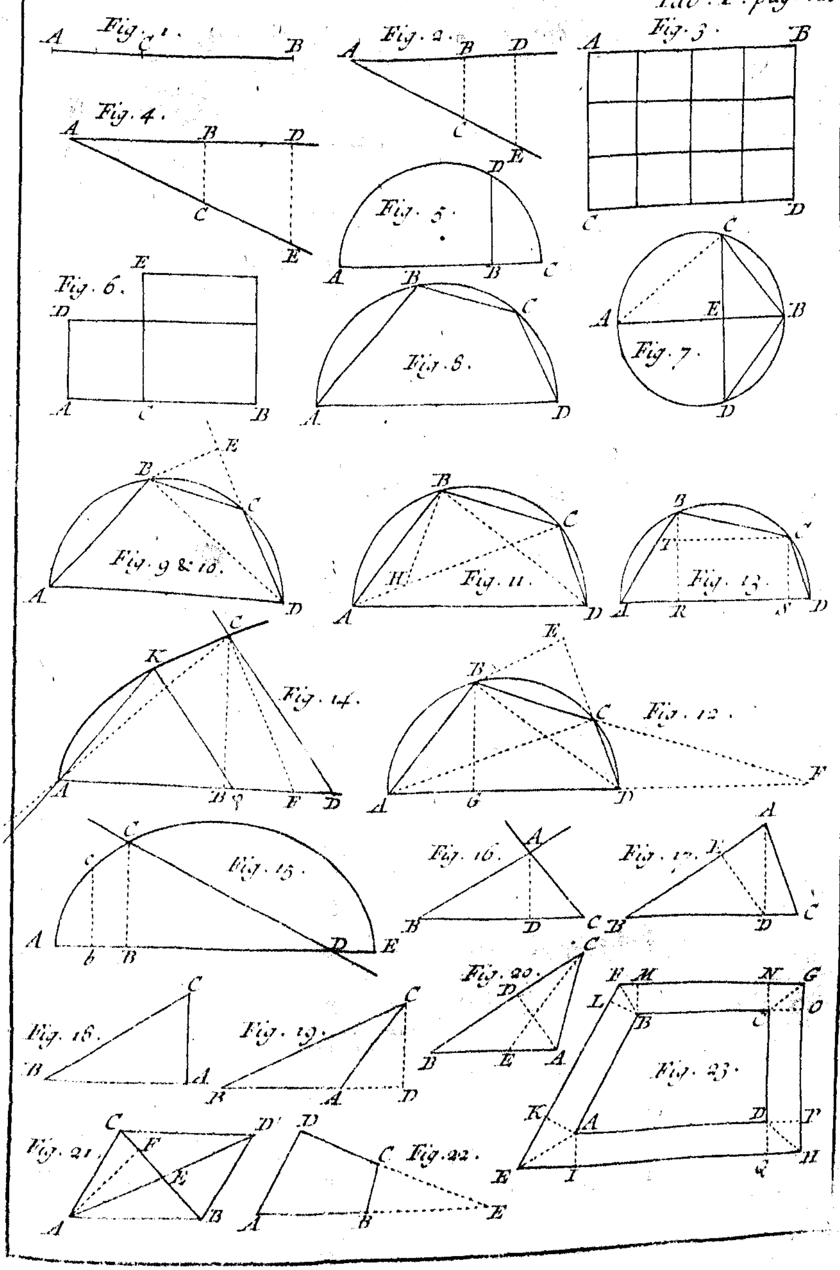
Fig. 1.
Fig. 2.
Fig. 3.
Fig. 4.
Fig. 5.
Fig. 6.
Fig. 7.
Fig. 8.
Fig. 9 & 10.
Fig. 11.
Fig. 12.
Fig. 13.
Fig. 14.
Fig. 15.
Fig. 16.
Fig. 17.
Fig. 18.
Fig. 19.
Fig. 20.
Fig. 21.
Fig. 22.
Fig. 23.

After the same manner a right Line may be drawn through a given Point, which shall divide any Triangle or Trapezium in a given Ratio.

PROBLEM XXI.

To determine the Point C *in the given right Line* D F, *from which the right Lines* A C *and* B C *drawn to two other Points* A *and* B *given by Position, shall have a given Difference.* [*See* Figure 25.] [*See* PROB. xlv.]

From the given Points let fall the Perpendiculars A D and B F to the given right Line, and make $AD = a$, $BF = b$, $DF = c$, $DC = x$, and A C will be $= \sqrt{aa + xx}$, $FC = x - c$, and $BC = \sqrt{bb + xx - 2cx + cc}$. Now let their given Difference be d, A C being greater than B C then $\sqrt{aa + xx} - d$ will be $= \sqrt{bb + xx - 2cx + cc}$. And squaring the Parts $aa + xx + dd - 2d\sqrt{aa + xx} = bb + xx - 2cx + cc$. And reducing, and for Abbreviation sake, writing $2ee$ instead of the given Quantities $aa + dd - bb - cc$, there will come out $ee + cx = d \times \sqrt{aa + xx}$. And again, having squared the Parts $e^4 + 2ceex + ccxx = ddaa + ddxx$. And the Equation being reduced $xx = \frac{2eecx + e^4 - aadd}{dd - cc}$, or

$$x = \frac{eec + \sqrt{e^4dd - aad^4 + aaddcc.}}{dd - cc.}$$

The Problem will be resolved after the same Way, if the Sum of the Lines A C and B C, or the Sum or the Difference of their Squares, or the Proportion or Rectangle, or the Angle comprehended by them be given: Or also, if instead of the right Line D C, you make use of the Circumference of a Circle, or any other Curve Line, so the Calculation (in this last Case especially) relates to the Line that joyns the Points A and B.

PROBLEM XXII.

Having the three right Lines AD, AE, BF, *given by Position, to draw a fourth* DF, *whose Parts* DE *and* EF, *intercepted by the former, shall be of given Lengths.* [*See* Figure 49.]

Let fall EG perpendicular to BF, and draw EC parallel to AD, and the three right Lines given by Position meeting in A, B, and H, make $AB = a$, $BH = b$, $AH = c$, $ED = d$, $EF = e$, and $HE = x$. Now, by reason of the similar Triangles ABH, ECH, it is AH : AB :: HE : EC $= \frac{ax}{c}$, and AH : HB :: HE : CH $= \frac{bx}{c}$. Add HB, and there comes $CB = \frac{bx + bc}{c}$. Moreover, by reason of the similar Triangles FEC, FDB it is ED : CB : : EF : CF $= \frac{ebx + ebc}{dc}$. Lastly, by the 12 and 13. 2 *Elem.* you have $\frac{ECq - EFq}{2FC} + \frac{1}{2}FC = (CG) = \frac{HEq - ECq}{2CH} - \frac{1}{2}CH$; that is,

$$\frac{\frac{aaxx}{cc} - ee}{\frac{2ebx + 2ebc}{dc}} + \frac{ebx + ebc}{2dc} = \frac{xx - \frac{aaxx}{cc}}{\frac{2bx}{c}} - \frac{bx}{2c}.$$ Or

$$\frac{aadxx - eedcc}{ebx + ebc} + \frac{ebx}{d} + \frac{ebc}{d} = \frac{ccx - aax - bbx}{b}.$$

Here, for Abbreviation sake, for $\frac{cc - aa - bb}{b} - \frac{eb}{d}$ write m, and you will have $\frac{aadxx - eedcc}{ebx + ebc} + \frac{ebc}{d} = mx$; and all the Terms being multiplied by $x + c$, there will come out $\frac{aadxx - eedc}{eb} - \frac{ebcx}{d} + \frac{ebcc}{d} = mxx + mcx$. Again,

for

for $\frac{aad}{eb} - m$ write p, and for $mc + \frac{ebc}{d}$ write $2pq$, and for $-\frac{ebcc}{d} + \frac{eedcc}{eb}$ write prr, and xx will become $= 2qx + rr$, and $x = q \pm \sqrt{qq + rr}$. Having found x or H E, draw E C parallel to A B, and take F C : B C :: e : d, and having drawn F E D, it will satisfy the Conditions of the Question.

PROBLEM XXIII.

To determine the Point Z, from which if four right Lines Z A, Z B, Z C, and Z D *are drawn at given Angles to four right Lines given by Position,* viz. F A, E B, F C, G D, *the Rectangle of two of the given Lines* Z A *and* Z B, *and the Sum of the other two* Z C *and* Z B *may be given.* [*See* Figure 26.]

From among the Lines chuse one, as F A, given by Position, as also another, Z A, not given by Position, and which is drawn to it, from the Lengths whereof the Point Z may be determined, and produce the other Lines given by Position until they meet these, or be produced farther out if there be Occasion, as you see here. And having made E A $= x$, and A Z $= y$, by reason of the given Angles of the Triangle A E H, there will be given the Ratio of A E to A H, which make as p to q, and A H will be $= \frac{qx}{p}$. Add A Z, and Z H will be $= y + \frac{qx}{p}$. And thence, since by reason of the given Angles of the Triangle H Z B, there is given the Ratio of H Z to B Z, if that be made as n to p you will have Z B $= \frac{py + qx}{n}$.

Moreover, if the given E F be called a, A F will be $= a - x$, and thence, if by reason of the given Angles of the Triangle A F I, A F be made to A I in the same Ratio as p to r, A I will become $= \frac{ra - rx}{p}$. Take this from A Z and there will remain I Z $= y - \frac{ra - rx}{p}$. And by reason of the

the given Angles of the Triangle ICZ, if you make IZ to ZC in the same Ratio as m to p, ZC will become $= \frac{py - ra + rx}{m}$.

After the same Way, if you make EG $= b$. AG : AK :: $l : s$, and ZK : ZD :: $p : l$, there will be obtained ZD $= \frac{sb - sx - ly}{p}$.

Now, from the State of the Question, if the Sum of the two Lines ZC and ZD, *viz.* $\frac{py - ra + rx}{m} + \frac{sb - sx - ly}{p}$ be made equal to any given Quantity f; and the Rectangle of the other two $\frac{pyy + qxy}{n}$ be made $= gg$, you will have two Equations for determining x and y. By the latter there comes out $x = \frac{ngg - pyy}{qy}$, and by writing this Value of x in the room of that in the former Equation, there will come out $\frac{py - ra}{m} + \frac{rngg - rpyy}{mqy} + \frac{sb - ly}{p} - \frac{sngg - spyy}{pqy} = f$; and by Reduction $yy =$ $\frac{apqry - bmqsy + fmpqy + ggmns - ggnpr}{ppq - ppr - mlq + mps}$; and for Abbreviation sake, writing $2h$ for $\frac{apqr - bmqs + fmpq}{ppq - ppr - mlq + mps}$, and kk for $\frac{ggmns - ggpnr}{ppq - ppr - mlq + mps}$, you will have $yy = 2hy + kk$, or $y = h \pm \sqrt{hh + kk}$. And since y is known by means of this Equation, the Equation $\frac{ngg - pyy}{qy} = x$ will give x. Which is sufficient to determine the Point Z.

After the same Way a Point may be determined from which other right Lines may be drawn to more or fewer right Lines given by Position, so that the Sum, or Difference, or Rectangle of some of them may be given, or may be made equal to the Sum, or Difference, or Rectangle of the rest, or that they may have any other assigned Conditions.

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PROBLEM XXIV.

To subtend the right Angle E A F *with the right Line* E F *given in Magnitude, which shall pass through the given Point* C, *equidistant from the Lines that comprehend the right Angle (when they are produced).* [*See* Figure 27.]

Complete the Square A B C D, and bisect the Line E F in G. Then call C B or C D, a; E G or F G, b; and C G, x; and C E will be $= x - b$, and C F $= x + b$. Then since C F q — B C q = B F q, B F will be $= \sqrt{xx + 2bx + bb - aa}$. Lastly, by reason of the similar Triangles C D E, F B C, C E : C D :: C F : B F, or $x - b : a :: x + b : \sqrt{xx + 2bx + bb - aa}$. Whence $ax + ab = \overline{x - b} \times \sqrt{xx + 2bx + bb - aa}$. Each Part of which Equation being squared, and the Terms that come out being reduced into Order, there comes out $x^4 = \begin{matrix} 2aa \\ +2bb \end{matrix} xx \begin{matrix} +2aabb \\ -b^4. \end{matrix}$ And extracting the Root as in Quadratick Equations, there comes out $xx = aa + bb + \sqrt{a^4 + 4aabb}$; and consequently $x = \sqrt{aa + bb + \sqrt{a^4 + 4aabb}}$. And C G being thus found, gives C E or C F, which, by determining the Point E or F, satisfies the Problem.

The same otherwise.

Let C E be $= x$, C D $= a$, and E F $= b$; and C F will be $= x + b$, and B F $= \sqrt{xx + bb + 2bx - aa}$. And then since C E : C D :: C F : B F, or $x : a :: x + b : \sqrt{xx + 2bx + bb - aa}$, $ax + ab$ will be $= x \times \sqrt{xx + 2bx + bb - aa}$. The Parts of this Equation being squared, and the Terms reduced into Order, there will come out $x^4 + 2bx^3 \begin{matrix} +bb \\ -2aa \end{matrix} xx - 2aabx - aabb = 0$, a Biquadratick Equation, the Investigation of the Root of which is more difficult than in the former Case. But it may be thus investigated; put $x^4 + 2bx^3 \begin{matrix} +bb \\ -2aa \end{matrix} xx - 2aabx + a^4 = aabb + a^4$, and extracting the Root on both Sides $xx + bx - aa = \pm a\sqrt{aa + bb}$.

Hence

Hence I have an Opportunity of giving a Rule for the Election of Terms for the Calculus.

Viz. *When there happens to be such an Affinity or Similitude of the Relation of two Terms to the other Terms of the Question, that you should be obliged in making Use of either of them to bring out Equations exactly alike; or that both, if they are made Use of together, shall bring out the same Dimensions and the same Form (only excepting perhaps the Signs* + *and* — *) in the final Equation (which will be easily seen) then it will be the best Way to make Use of neither of them, but in their room to chuse some third, which shall bear a like Relation to both, as suppose the half Sum, or half Difference, or perhaps a mean Proportional, or any other Quantity related to both indifferently and without a like.*

Thus, in the precedent Problem, when I see the Line EF alike related to both AB and AD, (which will be evident if you also draw EF in the Angle BAH) and therefore I can by no Reason be perswaded why ED should be rather made Use of than BF, or AE rather than AF, or CE rather than CF for the Quantity sought: Wherefore, in the room of the Points E and F, from whence this Ambiguity comes, (in the former Solution) I made Use of the intermediate Point G, which has a like Relation to both the Lines AB and AD. Then from this Point G, I did not let fall a Perpendicular to AF for finding the Quantity sought, because I might by the same Reason have let one fall to AD. And therefore I let it fall upon neither CB nor CD, but proposed CG for the Quantity sought, which does not admit of a like; and so I obtained a Biquadratick Equation without the odd Terms.

I might also (taking Notice that the Point G lies in the Periphery of a Circle described from the Center A, by the Radius EG) have let fall the Perpendicular GK upon the Diagonal AC, and have sought AK or CK, (as which bear also a like Relation to both AB and AD) and so I should have fallen upon a Quadratick Equation, *viz.* $yy = -\frac{1}{2}ey + \frac{1}{2}bb$, making $AK = y$, $AC = e$, and $EG = b$. And AK being so found, there must have been erected the Perpendicular KG meeting the aforesaid Circle in G, through which CF would pass.

Taking particular Notice of this Rule in *Prob.* IX. and X. where the like Sides BC and AC of the Triangle were to be determined, I rather sought the Semi-difference than either of them. But the Usefulness of this Rule will be more evident from the XXVIIIth Problem.

PRO-

PROBLEM XXV.

To a Circle described from the Center C, *and with the Radius* CD, *to draw a Tangent* DB, *the Part whereof* PB *placed between the right Lines given by Position,* AP *and* AB *shall be of a given Length.* [*See* Figure 50.]

From the Center C to either of the right Lines given by Position, as suppose to AB, let fall the Perpendicular CE, and produce it till it meets the Tangent DB in H. To the same AB let fall also the Perpendicular PG, and making $EA = a$, $EC = b$, $CD = c$, $BP = d$, and $PG = x$, by reason of the similar Triangles PGB, CDH, you will have $GB\ (\sqrt{dd - xx}) : PB :: CD : CH = \frac{cd}{\sqrt{dd - xx}}$. Add EC, and you will have $EH = b + \frac{cd}{\sqrt{dd - xx}}$. Moreover PG is $: GB :: EH : EB = \frac{b}{x}\sqrt{dd - xx} + \frac{cd}{x}$. Farther because of the given Angle PAG, there is given the Ratio of PG to AG, which being made as e to f, AG will be $= \frac{fx}{e}$. Add EA and BG, and you will have, lastly, $EB = a + \frac{fx}{e} + \sqrt{dd - xx}$. Therefore it is $\frac{cd}{x} + \frac{b}{x}\sqrt{dd - xx} = a + \frac{fx}{e} + \sqrt{dd - xx}$, and by Transposition of the Terms, $a + \frac{fx}{e} - \frac{cd}{x} = \frac{b - x}{x}\sqrt{dd - xx}$. And the Parts of the Equation being squared, $aa + \frac{2afx}{e} - \frac{2acd}{x} + \frac{ffxx}{ee} - \frac{2cdf}{e} + \frac{ccdd}{xx} = \frac{bbdd}{xx} - bb - \frac{2bdd}{x} + 2bx + dd - xx$. And by a due Reduction

$$\frac{x^4 \begin{matrix} +2aef \\ -2bee \end{matrix} x^3 \begin{matrix} +aaee \\ +bbee \\ -ddee \\ -2cdef \end{matrix} xx \begin{matrix} +2bddee \\ -2acdee \end{matrix} x \begin{matrix} +ccddee \\ -bbddee \end{matrix}}{ee + ff} = 0.$$

PRO-

Problem XXVI.

To find the Point D, from which three right Lines DA, DB, DC, let fall perpendicular to so many other right Lines AE, BF, CF, given in Position, shall obtain a given Ratio to one another. [*See* Figure 44.]

Of the right Lines given in Position, let us suppose one as BF be produced, as also its Perpendicular BD, till they meet the rest AE and CF, *viz.* BF in E and F, and BD in H and G. Now let EB be $= x$, and EF $= a$; and BF will be $= a - x$. But since, by reason of the given Position of the right Lines EF, EA, and FC, the Angles E and F, and consequently the Proportions of the Sides of the Triangles EBH and FBG are given; let EB be to BH as d to e; and BH will be $= \frac{ex}{d}$, and EH $(= \sqrt{EBq + BHq} = \sqrt{xx + \frac{eexx}{dd}}$, that is, $\frac{x}{d} \times \sqrt{dd+ee}$. Let also BF be to BG as d to f; and BG will be $= \frac{fa - fx}{d}$, and FG $(= \sqrt{BFq + BGq}) =$ $\sqrt{\frac{aadd - 2axdd + xxdd + ffaa - 2ffax + ffxx}{dd}}$, that is, $= \frac{a-x}{d}\sqrt{dd+ff}$. Besides, make BD $= y$, and HD will be $= \frac{ex}{d} - y$, and GD $= \frac{fa - fx}{d} - y$; and so, since AD is : HD (: : EB : EH) : : $d : \sqrt{dd+ee}$, and DC : GD (: : BF : FG) : : $d : \sqrt{dd+ff}$, AD will be $= \frac{ex - dy}{\sqrt{dd+ee}}$, and DC $= \frac{fa - fx - dy}{\sqrt{dd+ff}}$. Lastly, by reason of the given Proportions of the Lines BD, AD, DC, let BD : AD : : $\sqrt{dd+ee} : h - d$, and $\frac{hy - dy}{\sqrt{dd+ee}}$ will be

(=

$(=AD)=\frac{ex-dy}{\sqrt{dd+ee}}$, or $hy=ex$. Let alſo $BD:DC::\sqrt{dd+ff}:k-d$, and $\frac{ky-dy}{\sqrt{dd+ff}}$ will be $(=DC)=\frac{fa-fx-dy}{\sqrt{dd+ff}}$ or $ky=fa-fx$. Therefore $\frac{ex}{h}$ $(=y)$ is $=\frac{fa-fx}{k}$; and by Reduction $\frac{fha}{ek+fh}=x$. Wherefore take $EB:EF::h:\frac{ek}{f}+h$, then $BD:EB::e:h$, and you will have the Point ſought D.

PROBLEM XXVII.

To find the Point D, from which three right Lines DA, DB, DC, drawn to the three given Points, A, B, C, ſhall have a given Ratio among themſelves. [*See* Figure 45.]

Of the given three Points join any two of them, as ſuppoſe A and C, and let fall the Perpendicular BE from the third B, to the Line that conjoins A and C, as alſo the Perpendicular DF from the Point ſought D; and making $AE=a$, $AC=b$, $EB=c$, $AF=x$, and $FD=y$; and ADq will be $=xx+yy$. $FC=b-x$, CDq $(=FCq+FDq)=bb-2bx+xx+yy$. $EF=x-a$, and BDq $(=EFq+EB+FDq)=xx-2ax+aa+cc+2cy+yy$. Now, ſince AD is to CD in a given Ratio, let it be as d to e; and CD will be $=\frac{e}{d}\sqrt{xx+yy}$. Since alſo AD is to BD in a given Ratio, let that be as d to f, and BD will be $=\frac{f}{d}\sqrt{xx+yy}$. And, conſequently it is $\frac{eexx+eeyy}{dd}$ $(=CDq)=bb-2bx+xx+yy$, and $\frac{ffxx+ffyy}{dd}$ $(=BDq)=xx-2ax+aa+cc+2cy+yy$. In

In which if, for Abbreviation ſake, you write p for $\frac{dd-ee}{d}$, and q for $\frac{dd-ff}{d}$, there will come out $bb-2bx+\frac{p}{d}xx+\frac{p}{d}yy=0$, and $aa+cc-2ax+2cy+\frac{q}{d}xx+\frac{q}{d}yy=0$. And by the former you have $\frac{2bqx-bbq}{p}=\frac{q}{d}xx+\frac{q}{d}yy$. Wherefore, in the latter, for $\frac{q}{d}xx+\frac{q}{d}yy$, write $\frac{2bqx-bbq}{p}$, and there will come out $\frac{2bqx-bbq}{p}+aa+cc-2ax+2cy=0$. Again, for Abbreviation ſake, write m for $a-\frac{bq}{p}$, and $2cn$ for $\frac{bbq}{p}-aa-cc$, and you will have $2mx+2cn=2cy$, and the Terms being divided by $2c$, there ariſes $\frac{mx}{c}+n=y$. Wherefore, in the Equation $bb-2bx+\frac{p}{d}xx+\frac{p}{d}yy=0$, for yy write the Square of $\frac{mx}{c}+n$, and you will have $bb-2bx+\frac{p}{d}xx+\frac{pmm}{dcc}xx+\frac{2pmn}{dc}x+\frac{pnn}{d}=0$. Where, laſtly, if, for Abbreviation ſake, you write $\frac{b}{r}$ for $\frac{p}{d}+\frac{pmm}{dcc}$, and $\frac{sb}{r}$ for $b-\frac{pmn}{dc}$, and $\frac{tbb}{r}$ for $bb+\frac{pnn}{d}$, you will have $xx=2sx-tb$. And having extracted the Root $x=s\pm\sqrt{ss-tb}$. Having found x, the Equation $\frac{mx}{c}+n=y$ will give y; and from x and y given, that is, AF and FD, the given Point D is determined.

PRO-

PROBLEM XXVIII.

So to inscribe the right Line D C *of a given Length in the given Conick Section* D A C, *that it may pass through the Point* G *given by Position.* [*See* Figure 28.]

Let A F be the Axis of the Curve, and from the Points D, G, and C, let fall to it the Perpendiculars D H, G E, and C B. Now to determine the Position of the right Line D C, it may be proposed to find out the Point D or C; but since these are related, and so alike, that there would be the like Operation in determining either of them, whether I were to seek CG, CE, or AB; or their likes, DG, D H, or A H; therefore I look after a third Point, that regards D and C alike, and at the same time determines them. And I see F to be such a Point.

Now let A E be $=a$, E G $=b$, D C $=c$, and E F $=z$; and besides since the Relation between A B and B C is had in the Equation, I suppose, given for determining the Conick Section, let A B $=x$, B C $=y$, and F B will be $=x-a+z$. And because G E : E F :: C B : F B, F B will again be $=\frac{yz}{b}$. Therefore, $x-a+z=\frac{yz}{b}$.

These Things being thus laid down, take away x, by the Equation that denotes the Curve. As if the Curve be a Parabola expressed by the Equation $rx=yy$, write $\frac{yy}{r}$ for x; and there will arise $\frac{yy}{r}-a+z=\frac{yz}{b}$, and extracting the Root $y=\frac{rz}{2b}\pm\sqrt{\frac{rrzz}{4bb}+ar-rz}$. Whence it is evident, that $\sqrt{\frac{rrzz}{bb}+4ar-4rz}$ is the Difference of the double Value of y, that is, of the Lines $+$ B C and $-$ D H, and consequently (having let fall D K perpendicular upon C B) that Difference is equal to C K. But F G : G E :: D C : C K, that is, $\sqrt{bb+zz}:b::c:\sqrt{\frac{rrzz}{bb}+4ar-4rz}$.

And

And by multiplying the Squares of the Means, and also the Squares of the Extreams into one another, and ordering the Products, there will arise $z^4 =$

$$\frac{4bbrz^3 \begin{matrix} -4abbr \\ -bbrr \end{matrix} zz + 4b^4rz \begin{matrix} -4ab^4r \\ +b^4cc \end{matrix}}{rr},$$ an Equation

of four Dimensions, which would have risen to one of eight Dimensions if I had sought either CG, or CB, or AB.

PROBLEM XXIX.

To multiply or divide a given Angle by a given Number. [*See* Figure 29.]

In any Angle FAG inscribe the Lines AB, BC, CD, DE, &c. of any the same Length, and the Triangles ABC, BCD, CDE, DEF, &c. will be *Isosceles*, and consequently by the 32. 1. *Eucl.* the Angle CBD will be = Angle A + ACB = 2 Angle A, and the Angle DCE = Angle A + ADC = 3 Angle A, and the Angle EDF = A + AED = 4 Angle A, and the Angle FEG = 5 Angle A, and so onwards. Now, making AB, BC, CD, &c. the Radii of equal Circles, the Perpendiculars BK, CL, DM, &c. let fall upon AC, BD, CE, &c. will be the Sines of those Angles, and AK, BL, CM, DN, &c. will be their Sines Complement to a right one; or making AB the Diameter, the Lines AK, BL, CM, &c. will be Chords. Let therefore AB = $2r$, and AK = x, then work thus:

$$AB : AK :: AC : AL.$$

$$2r : x :: 2x : \frac{xx}{r}.$$

And $\left\{\begin{matrix} AL - AB \\ \frac{xx}{r} - 2r \end{matrix}\right\}$ = BL, the Duplication.

$$AB : AK :: AD\,(2AL - AB) : AM.$$

$$2r : x :: \frac{2xx}{r} - 2r : \frac{x^3}{rr} - x.$$

And

$$\text{And} \left\{ \begin{matrix} AM - AC \\ \frac{x^3}{rr} - 3x \end{matrix} \right\} = CM, \text{ the Triplication.}$$

$$AB : AK :: AE\,(2AM - AC) : AN.$$

$$2r : x :: \frac{2x^3}{rr} - 4x : \frac{x^4}{r^3} - \frac{2xx}{r}.$$

$$\text{And} \left\{ \begin{matrix} AN - AD \\ \frac{x^4}{r^3} - \frac{4xx}{r} + 2r \end{matrix} \right\} = DN, \text{ the Quadruplication:}$$

$$AB : AK :: AF\,(2AN - AD) : AO.$$

$$2r : x :: \frac{2x^4}{r^3} - \frac{6xx}{r} + 2r : \frac{x^5}{r^4} - \frac{3x^3}{rr} + x.$$

$$\text{And} \left\{ \begin{matrix} AO - AE \\ \frac{x^5}{r^4} - \frac{5x^3}{rr} + 5x \end{matrix} \right\} = EO, \text{ the Quintuplication.}$$

And ſo onwards. Now if you would divide an Angle into any Number of Parts, put q for B L, C M, D N, *&c.* and you will have $xx - 2rr = qr$ for the Biſection; $xxx - 3rrx = qr^2$ for the Triſection; $xxxx - 4rrxx + 2r^4 = qr^3$ for the Quadriſection; $xxxxx - 5r^2x^3 + 5r^4x = qr^4$ for the Quinquiſection, *&c.*

Problem XXX.

To determine the Poſition of a Comet's Courſe that moves uniformly in a right Line, as B D, *from three Obſervations.* [*See* Figure 30.]

Suppoſe A to be the Eye of the Spectator, B the Place of the Comet in the firſt Obſervation, C in the ſecond, and D in the third; the Inclination of the Line *BD* to the Line *AB* is to be found. From the Obſervations therefore there are given the Angles B A C, B A D; and conſequently if B H be drawn Perpendicular to A B, and meeting *A C* and A D in E and F, aſſuming any how *A B*, there will be given B E and *B F*, *viz.* the Tangents of the Angles in reſpect of the Radius A B. Make therefore A B $= a$, B E $= b$, and B F $= c$. Moreover, from the given Intervals of of the Obſervations, there will be given the Ratio of B C to B D, which, if it be made as b to e, and D G be drawn parallel to A C, ſince B E is to B G in the ſame Ratio, and B E was called b, B G will be $= e$, and conſequently G F $= e - c$. Farther, if you let fall D H perpendicular to B G, by reaſon of the Triangles A B F and D H F being like, and alike divided by the Lines A E and D G, F E will be : A B :: F G : H D, that is, $c - b : a :: e - c : \frac{ae - ac}{c - b} =$ H D. Moreover, F E will be : F B :: F G : F H, that is, $c - b : c :: e - c : \frac{ce - cc}{c - b} =$ F H; to which add B F, or c, and B H will be $= \frac{ce - cb}{c - b}$. Wherefore $\frac{ce - cb}{c - b}$ is to $\frac{ae - ac}{c - b}$ (or $ce - cb$ to $ae - ac$, or $\frac{ce - cb}{e - c}$ to a) as B H to H D; that is, as the Tangent of the Angle H D B, or A B K to the Radius. Wherefore ſince a is ſuppoſed to be the Radius, $\frac{ce - cb}{e - c}$ will be the Tangent of the Angle A B K, and therefore by reſolving them into an Analogy, it will be as $e - c$ to $e - b$, (or G F to G E) ſo c (or the Tangent of the Angle B A F) to the Tangent of the Angle A B K.

Say therefore, as the Time between the firſt and ſecond Obſervation to the Time between the firſt and third, ſo the Tangent of the Angle B A E to a fourth Proportional. Then as the Difference between that fourth Proportional and the Tangent of the Angle B A F, to the Difference between the ſame fourth Proportional and the Tangent of the Angle B A E, ſo the Tangent of the Angle B A F to the Tangent of the Angle A B K.

PRO-

Problem XXXI.

Rays of Light from any shining or lucid Point diverging to a refracting Spherical Surface, to find the Concourse of each of the refracted Rays with the Axis of the Sphere passing through that lucid Point. [*See* Figure 31.]

Let A be that lucid Point, and BV the Sphere, the Axis whereof is AD, the Center C, and the Vertex V; and let AB be the incident Ray, and BD the refracted Ray; and having let fall to those Rays the Perpendiculars CE and CF, as also BG perpendicular to AD, and having drawn BC, make $AC = a$, VC or $BC = r$, $CG = x$, and $CD = z$, and AG will be $= a - x$, $BG = \sqrt{rr - xx}$, $AB = \sqrt{aa - 2ax + rr}$; and by reason of the similar Triangles ABG and ACE, $CE = \frac{a\sqrt{rr - xx}}{\sqrt{aa - 2ax + rr}}$. Also $GD = z + x$, $BD = \sqrt{zz + 2zx + rr}$; and by reason of the similar Triangles DBG and DCF, $CF = \frac{z\sqrt{rr - xx}}{\sqrt{zz + 2zx + rr}}$. Besides since the Ratio of the Sines of Incidence and Refraction, and consequently of CE to CF, is given, suppose that Ratio to be as a to f, and $\frac{fa\sqrt{rr - xx}}{\sqrt{aa - 2ax + rr}}$ will be $= \frac{az\sqrt{rr - xx}}{\sqrt{zz + 2zx + rr}}$; and multiplying cross-ways, and dividing by $a\sqrt{rr - xx}$, it will be $f\sqrt{zz + 2zx + rr} = z\sqrt{aa - 2xa + rr}$, and by squaring and reducing the Terms into Order, $zz = \frac{2ffxz + ffrr}{aa - 2ax + rr - ff}$. Then for the given $\frac{ff}{a}$ write p, and q for the given $a + \frac{rr}{a} - p$, and zz will be $= \frac{2pxz + prr}{q - 2x}$, and $z = \frac{px + \sqrt{ppxx - 2prrx + pqrr}}{q - 2x}$. Therefore z is found; that is, the Length of CD, and consequently the Point sought D, where the refracted Ray BD meets with the Axis. Q. E. F.

Here

Here I made the incident Rays to diverge, and fall upon a thicker Medium; but changing what is requisite to be changed, the Problem may be as easily resolved when the Rays converge, or fall from a thicker Medium into a thinner one.

PROBLEM XXXII.

If a Cone be cut by any Plane, to find the Figure of the Section. [*See* Figure 32 *and* 33.]

Let ABC be a Cone standing on a circular Base BC, and IEM its Section sought; and let KILM be any other Section parallel to the Base, and meeting the former Section in HI; and ABC a third Section, perpendicularly bisecting the two former in EH and KL, and the Cone in the Triangle ABC. And producing EH till it meet AK in D; and having drawn EF and DG, parallel to KL, and meeting AB and AC in F and G, call EF $=a$, DG $=b$, ED $=c$, EH $=x$, and HI $=y$; and by reason of the similar Triangles EHL, EDG, ED will be : DG :: EH : HL $=\frac{bx}{c}$. Then by reason of the similar Triangles DEF, DHK, DE will be : EF :: DH : ($c-x$ in the thirty second Figure, and $c+x$ in the thirty third Figure) HK $=\frac{ac\mp ax}{c}$. Lastly, since the Section KIL is parallel to the Base, and consequently circular, HK × HL will be $=$ HIq, that is, $\frac{ab}{c}x\mp\frac{ab}{cc}xx=yy$, an Equation which expresses the Relation between EH (x) and HI (y), that is, between the Axis and the Ordinate of the Section EIM; which Equation, since it expresses an Ellipse in the thirty second Figure, and an Hyperbola in the thirty third Figure, it is evident, that that Section will be Elliptical or Hyperbolical.

Now if ED no where meets AK, being parallel to it, then HK will be $=$ EF (a), and thence $\frac{ab}{c}x$ (HK × HL) $=yy$, an Equation expressing a Parabola.

PRO-

PROBLEM XXXIII.

If the right Line XY *be turned about the Axis* AB, *at the Distance* CD, *with a given Inclination to the Plane* DCB, *and the Solid* PQRUTS, *generated by that Circumrotation, be cut by any Plane as* INQLK, *to find the Figure of the Section.* [*See* Figure 34.]

Let BHQ, or GHO be the Inclination of the Axis AB to the Plane of the Section; and let L be any Concourse of the right Line XY with that Plane. Draw DF parallel to AB, and let fall the Perpendiculars LG, LF, LM, to AB, DF, and HO, and join FG and MG. And having called CD $= a$, CH $= b$, HM $= x$, and ML $= y$, by reason of the given Angle GHO, making MH : HG :: $d : e$, $\frac{ex}{d}$ will be $=$ GH, and $b + \frac{ex}{d} =$ GC or FD. Moreover, by reason of the given Angle LDF (*viz.* the Inclination of the right Line XY to the Plane GCDF) putting FD : FL :: $g : h$, it will be $\frac{hb}{g} + \frac{hex}{dg} =$ FL, to whose Square add FGq (DCq, or aa) and there will come out GL$q = aa + \frac{hhbb}{gg} + \frac{2hhbex}{dgg} + \frac{hheexx}{ddgg}$. Hence subtract MG$q$ (HMq − HGq, or $xx - \frac{ee}{dd}xx$) and there will remain $\frac{aagg + hhbb}{gg} + \frac{2hhbe}{dgg}x + \frac{hhee - ddgg + eegg}{ddgg} \times xx$ ($=$ MLq) $= yy$: an Equation that expresses the Relation between x and y, that is, between HM the Axis of the Section, and ML its Ordinate. And therefore, since in this Equation x and y ascend only to two Dimensions, it is evident, that the Figure INQLK is a Conick Section. As for Example, if the Angle MHG is greater than the Angle LDF, this Figure will be an Ellipse; but if less, an Hyperbola; and if equal, either a Parabola, or (the Points C and H moreover coinciding) a Parallelogram.

PROBLEM XXXIV.

If you erect AD *of a given Length perpendicular to* AF, *and* ED, *one Leg of a Square* DEF, *pass continually through the Point* D, *while the other Leg* EF *equal to* AD *slide upon* AF, *to find the Curve* HIC, *which the Leg* EF *describes by its middle Point* C. [*See* Figure 35.]

Let EC or CF be $=a$, the Perpendicular CB $=y$, AB $=x$, and on account of the similar Triangles FBC, FEG, it will be BF $(\sqrt{aa-yy})$: BC + CF $(y+a)$: : EF $(2a)$: EG + GF (AG + GF) or AF. Wherefore $\frac{2ay+2aa}{\sqrt{aa-yy}}$ (= AF = AB + BF) $= x + \sqrt{aa-yy}$. Now by multiplying by $\sqrt{aa-yy}$ there is made $2ay + 2aa = aa - yy + x\sqrt{aa-yy}$, or $2ay + aa + yy = x \times \sqrt{aa-yy}$, and by squaring the Parts, divided by $\sqrt{a+y}$, and ordering them, there comes out $y^3 + 3ayy \begin{matrix} +3aa \\ +xx \end{matrix} y \begin{matrix} +a^3 \\ -axx \end{matrix} = 0$.

The same otherwise. [*See* Figure 36.]

On BC take at each End BI, and CK equal to CF, and draw KF, HI, HC, and DF; whereof let HC and DF meet AF, and IK in M and N, and upon HC let fall the Perpendicular IL; and the Angle K will be $= \frac{1}{2}$ BCF $= \frac{1}{2}$ EGF = GFD = AMH = MHI = CIL; and consequently the right-angled Triangles KBF, FBN, HLI, and ILC will be similar. Make therefore FC $=a$, HI $=x$, and IC $=y$; and BN $(2a-y)$ will be : BK (y) : : LC : LH : : CIq (yy) : HIq (xx), and consequently $2axx - yxx = y^3$. From which Equation it is easily inferred, that this Curve is the Cissoid of the Antients, belonging to a Circle, whose Center is A, and its Radius AH.

Tab: II. p: 135.

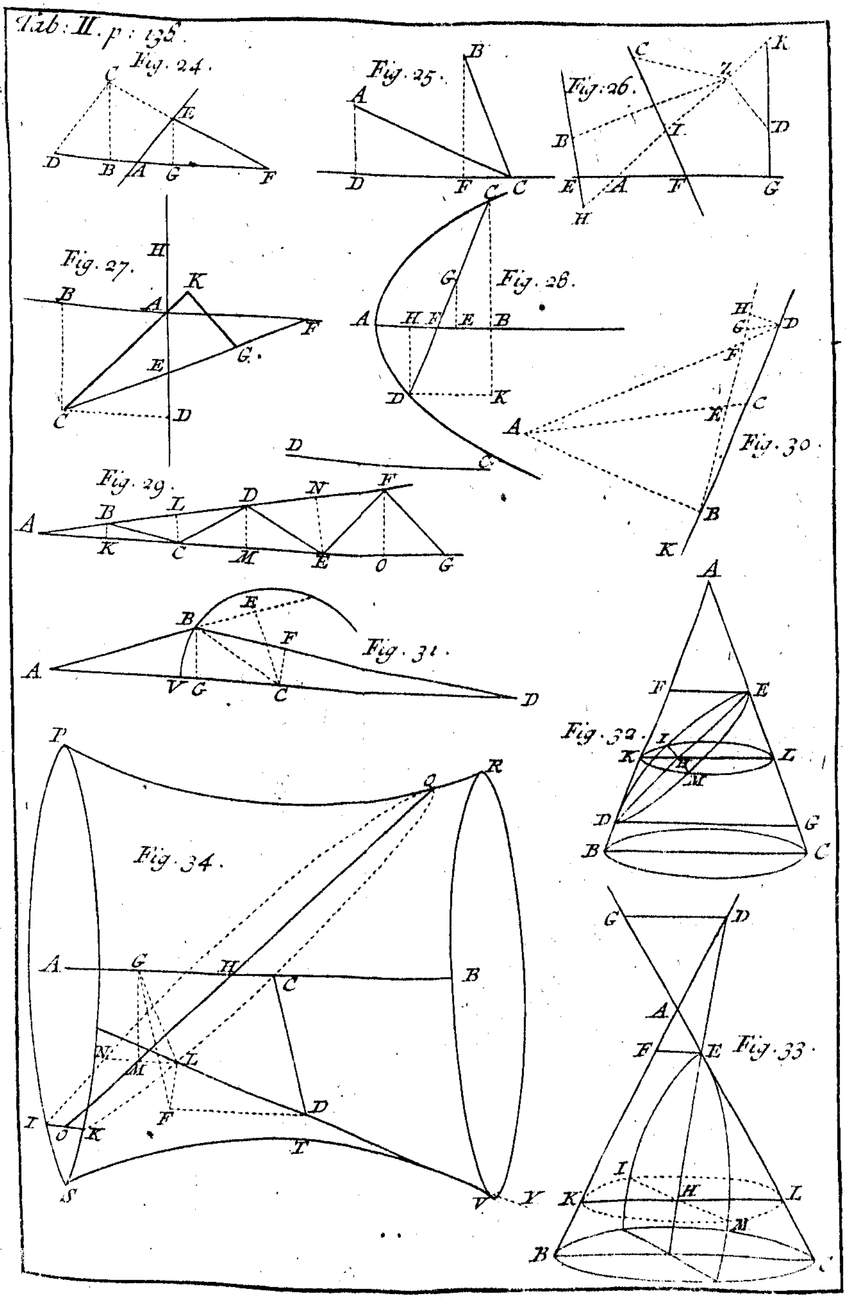


Fig. 24.
Fig. 25.
Fig. 26.
Fig. 27.
Fig. 28.
Fig. 29.
Fig. 30.
Fig. 31.
Fig. 32.
Fig. 33.
Fig. 34.

PROBLEM XXXV.

If a right Line E D *of a given Length subtending the given Angle* E A D, *be so moved, that its Ends* D *and* E *always touch the Sides* A D *and* A E *of that Angle; let it be proposed to determine the Curve* F C G, *which any given Point* C *in that right Line* E D *describes.* [*See* Figure 37.]

From the given Point C draw C B parallel to E A; and make A B $= x$, B C $= y$, C E $= a$, and C D $= b$, and by reason of the similar Triangles D C B, D E A, it will be E C : A B :: C D : B D; that is, $a : x :: b :$ B D $= \frac{bx}{a}$. Besides, having let fall the Perpendicular C H, by reason of the given Angle D A E, or D B C, and consequently of the given Ratio of the Sides of the right-angled Triangle B C H, you will have $a : e ::$ B C : B H, and B H will be $= \frac{ey}{a}$. Take away this from B D, and there will remain H D $= \frac{bx - ey}{a}$. Now in the Triangle B C H, because of the right Angle B H C, it is B Cq — B Hq = C Hq; that is $yy - \frac{eeyy}{aa}$ = C Hq. In like manner, in the Triangle C D H, because of the right Angle C H D, it is C Dq — C Hq = H Dq; that is, $bb - yy + \frac{eeyy}{aa}$ (= H Dq = $\frac{bx - ey}{a}q$.) $= \frac{bbxx - 2bexy + eeyy}{aa}$; and by Reduction $yy = \frac{2be}{aa} \times xy + \frac{aabb - bbxx}{aa}$. Where, since the unknown Quantities rise but to two Dimensions, it is evident that the Curve is a Conick Section. Then extracting the Root, you will have $y = \frac{bex \pm b\sqrt{eexx - aaxx + a^4}}{aa}$. Where, in the Radical Term, the Coefficient of xx is $ee - aa$. But it was $a : e ::$ B C : B H; and B C is necessarily a greater Line than B H, *viz.* the Hypothenuse of a right-angled Triangle is greater than the Side of it; therefore a is greater than e, and $ee - aa$ is a negative Quantity, and consequently the Curve will be an Ellipsis.

PROBLEM XXXVI.

If the Ruler EBD, *forming a right Angle, be so moved, that one Leg of it,* EB, *continually subtends the right Angle* EAB, *while the End of the other Leg,* BD, *describes some Curve Line, as* FD; *to find that Line* FD, *which the Point* D *describes.* [*See* Figure 38.]

From the Point D let fall the Perpendicular DC to the Side AC; and making $AC = x$, and $DC = y$, and $EB = a$, and $BD = b$. In the Triangle BDC, by reason of the right Angle at C, BCq is $= BDq - DCq = bb - yy$. Therefore $BC = \sqrt{bb - yy}$; and $AB = x - \sqrt{bb - yy}$. Besides, by reason of the similar Triangles BEA, DBC, it is BD : DC :: EB : AB; that is, $b : y :: a : x - \sqrt{bb - yy}$. Wherefore $bx - b\sqrt{bb - yy} = ay$, or $bx - ay = b\sqrt{bb - yy}$. And the Parts being squared and duly reduced $yy = \frac{2abxy + b^4 - bbxx}{aa + bb}$. And extracting the Root $y = \frac{abx \pm bb\sqrt{aa + bb - xx}}{aa + bb}$. Whence it is again evident, that the Curve is an Ellipsis.

This is so where the Angles EBD and EAB are right; but if those Angles are of any other Magnitude, as long as they are equal, you may proceed thus: [*See* Figure 39.] Let fall DC perpendicular to AC as before, and draw DH, making the Angle DHA equal to the Angle HAE, suppose Obtuse, and calling $EB = a$, $BD = b$, $AH = x$, and $HD = y$; by reason of the similar Triangles EAB, BHD, BD will be : DH :: EB : AB; that is, $b : y :: a : AB = \frac{ay}{b}$. Take this from AH and there will remain $BH = x - \frac{ay}{b}$. Besides, in the Triangle DHC, by reason of all the Angles given, and consequently the Ratio of the Sides given, assume DH to HC in any given Ratio, suppose as b to e; and since DH is y, HC will be $\frac{ey}{b}$, and $HB \times HC =$

exy

$\frac{exy}{b} - \frac{aeyy}{bb}$. Lastly, by the 12, 2 *Elem.* in the Triangle BHD, it is $BDq = BHq + DHq + 2BH \times HC$; that is, $bb = xx - \frac{2axy}{b} + \frac{aayy}{bb} + yy + \frac{2exy}{b} - \frac{2aeyy}{bb}$. and extracting the Root $x = \frac{ay - ey \pm \sqrt{eeyy - bbyy + bbbb}}{b}$. Where when b is greater than e, that is, when $ee - bb$ is a negative Quantity, it is again evident, that the Curve is an Ellipse.

PROBLEM XXXVII.

In the given Angle PAB *having any how drawn the right Lines,* BD, PD, *in a given Ratio, on this Condition, that* BD *shall be parallel to* AP, *and* PD *terminated at the given Point* P *in the right Line* AP; *to find the Locus of the Point* D. [*See* Figure 41.]

Draw CD parallel to AB, and DE perpendicular to AP; and make $AP = a$, $CP = x$, and $CD = y$, and let BD be to PD in the same Ratio as d to e, and AC or BD will be $= a - x$, and $PD = \frac{ea - ex}{d}$. Moreover, by reason of the given Angle DCE, let the Ratio of CD to CE be as d to f, and CE will be $= \frac{fy}{d}$, and $EP = x - \frac{fy}{d}$. But by reason of the Angles at E being right ones, it is $CDq - CEq (= EDq) = PDq - EPq$; that is, $yy - \frac{ffyy}{dd} = \frac{eeaa - 2eeax + eexx}{dd} - xx + \frac{2fxy}{d} - \frac{ffyy}{dd}$; and blotting out on each Side $-\frac{ffyy}{dd}$, and the Terms being rightly disposed, $yy = \frac{2fxy}{d} + \frac{eeaa - 2eeax + eexx - ddxx}{dd}$, and extracting the Root

$y =$

$$y = \frac{fx}{d} \pm \frac{\sqrt{eeaa - 2eeax \begin{matrix} +ce \\ -dd \\ +ff \end{matrix} xx}}{d}.$$

Where, since x and y in the last Equation ascends only to two Dimensions, the Place of the Point D will be a Conick Section, and that either an Hyperbola, Parabola, or Ellipse, as $ee - dd + ff$, (the Co-efficient of xx in the last Equation) is greater, equal to, or less than nothing.

Problem XXXVIII.

The two right Lines VE and VC *being given in Position, and cut any how in* C *and* E *by another right Line,* PE *turning about the Pole,* P *given also in Position; if the intercepted Line* CE *be divided into the Parts* CD, DE, *that have a given Ratio to one another, it is proposed to find the Place of the Point* D. [*See* Figure 42.]

Draw VP, and parallel to it, DA, and EB meeting VC in A and B. Make $VP = a$, $VA = x$, and $AD = y$, and since the Ratio of CD to DE is given, or conversely of CD to CE, that is, the Ratio of DA to EB, let it be as d to e, and EB will be $= \frac{ey}{d}$. Besides, since the Angle EVB is given, and consequently the Ratio of EB to VB, let that Ratio be as e to f, and VB will be $= \frac{fy}{d}$. Lastly, by reason of the similar Triangles CEB, CDA, CPV, it is EB : CB :: DA : CA :: VP : VC, and by Composition EB + VP : CB + VC :: DA + VP : CA + VC; that is, $\frac{ey}{d} + a : \frac{fy}{d} :: y + a : x$, and multiplying together the Means and Extremes $eyx + dax = fyy + fay$. Where since the indefinite Quantities x and y ascend only to two Dimensions, it follows, that the Curve VD, in which the Point D is always found, is a Conick Section, and that an Hyperbola, because one of the indefinite Quantities, *viz.* x is only of one Dimension, and in the Term exy is multiplied by the other indefinite one y.

PRO-

PROBLEM XXXIX.

If two right Lines, A C *and* B C, *in any given Ratio, are drawn from the two Points* A *and* B *given in Position, to a third Point* C, *to find the Place of* C, *the Point of Concourse.* [*See* Figure 43.]

Join A B, and let fall to it the Perpendicular C D; and making A B $= a$, A D $= x$, D C $= y$, A C will be $= \sqrt{xx + yy}$, B D $= x - a$, and B C $(= \sqrt{BDq + DCq}) = \sqrt{xx - 2ax + aa + yy}$. Now since there is given the Ratio of A C to B C, let that be as d to e; and the Means and Extremes being multiplied together, you will have $e\sqrt{xx + yy} = d\sqrt{xx - 2ax + aa + yy}$, and by Reduction $\sqrt{\frac{ddaa - 2ddax}{ee - dd} - xx} = y$. Where since xx is Negative, and affected only by Unity, and also the Angle A D C a right one, it is evident, that the Curve in which the Point C is placed is a Circle, *viz.* in the right Line A B take the Points E and F, so that $d : e :: $ A E : B E :: A F : B F, and E F will be the Diameter of this Circle.

And hence from the Converse this Theorem comes out, that in the Diameter of any Circle E F produced, having given any how the two Points A and B on this Condition, that A E : A F :: B E : B F, and having drawn from these Points the two right Lines A C and B C, meeting the Circumference in any Point C; A C will be to B C in the given Ratio of A E to B E.

PROBLEM XL.

If a lucid Point, as A, *dart forth Rays towards a refracting plain Surface, as* C D; *to find the Ray* A C, *whose refracted Part* C B *will strike the given Point* B. [*See* Figure 51.]

From that lucid Point let fall the Perpendicular A D to the refracting Plane, and let the refracted Ray B C meet with it, being produced out on both Sides, in E; and a Perpendicular let fall from the Point B in F, and draw B D; and making A D $= a$, D B $= b$, B F $= c$, D C $= x$, make the Ratio of the Sines of Incidence and Refraction, that is, is, of the Sines of the Angles CAD, CED, to be d to e, and since EC and AC (as is known) are in the same Ratio, and AC is $\sqrt{aa+xx}$, EC will be $=\frac{d}{e}\sqrt{aa+xx}$.

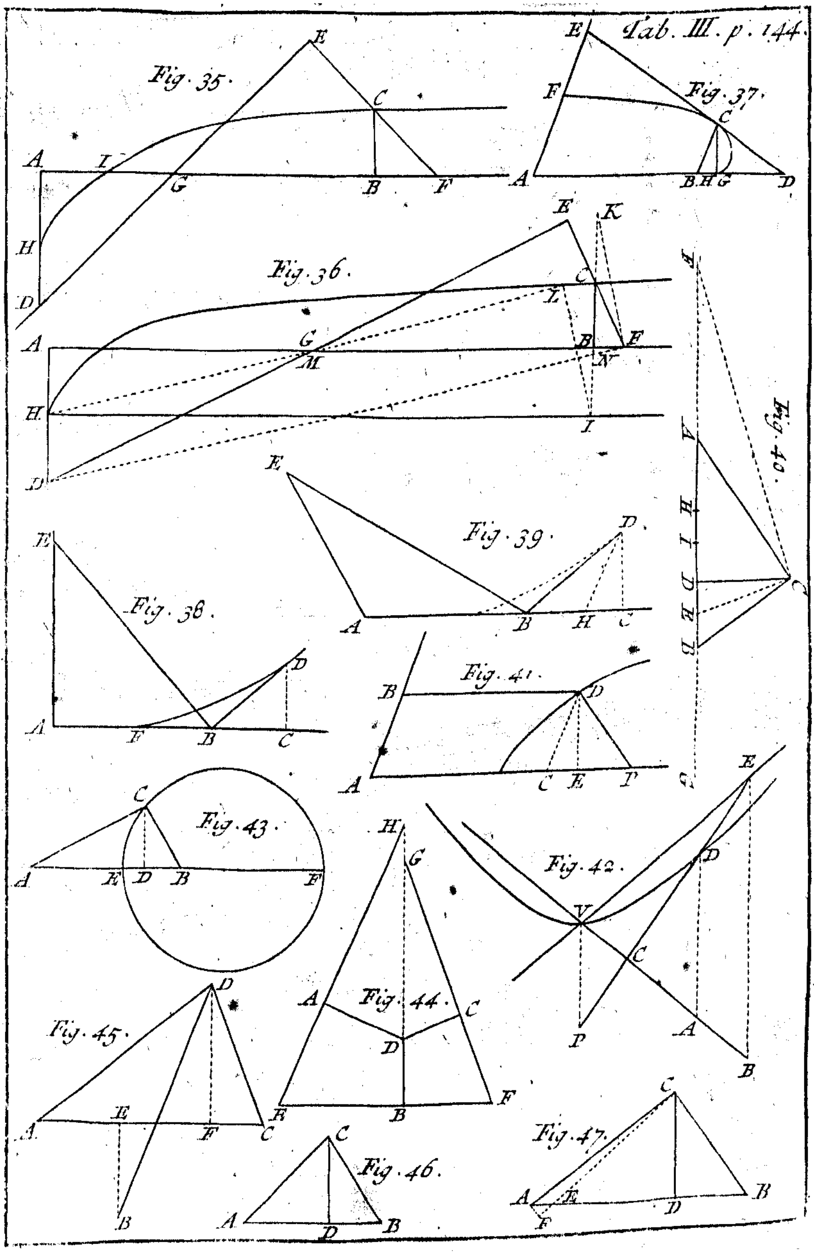
Besides ED ($=\sqrt{ECq-CDq}$) is $=\sqrt{\frac{ddaa+ddxx}{ee}-xx}$, and DF $=\sqrt{bb-cc}$, and EF $=\sqrt{bb-cc}+\sqrt{\frac{ddaa+ddxx}{ee}-xx}$. Lastly, because of the similar Triangles ECD, EBF, it is ED : DC :: EF : FB, and multiplying the Values of the Means and Extremes into one another, $c\sqrt{\frac{ddaa+ddxx}{ee}-xx}=x\sqrt{bb-cc}+x\times\sqrt{\frac{ddaa+ddxx}{ee}-xx}$, or $\overline{c-x}\sqrt{\frac{ddaa+ddxx}{ee}-xx}=x\sqrt{bb-cc}$, and the Parts of the Equation being squared and duly dispofed.

$$x^4-2cx^3=\frac{\begin{matrix}+ddcc\\+ddaaxx\\-eebb\end{matrix}-2ddaacx+ddaacc}{dd-ee}=0.$$

Problem XLI.

To find the Locus or Place of the Vertex of a Triangle D, *whose Base* AB *is given, and the Angles at the Base* DAB, DBA, *have a given Difference.* [*See Figure* 52.]

Where the Angle at the Vertex, or (which is the same Thing) where the Sum of the Angles at the Base is given, *Euclid* [in 29.3.] has taught us, that the Locus of the Vertex is in the Circumference of a Circle; but we have proposed the finding the Place when the Difference of the Angles at the Base is given. Let the Angle DBA be greater than the Angle DAB, and let ABF be their given Difference, the right Line BF meeting AD in F. Moreover, let fall the per-


Tab. III. p. 144.
Fig. 35.
Fig. 36.
Fig. 37.
Fig. 38.
Fig. 39.
Fig. 40.
Fig. 41.
Fig. 42.
Fig. 43.
Fig. 44.
Fig. 45.
Fig. 46.
Fig. 47.

Perpendicular D E to B F, as also D C perpendicular to A B meeting B F in G. And making A B $= a$, A C $= x$, and C D $= y$, B C will be $= a - x$. Now since in the Triangle B C G there are given all the Angles, there will be given the Ratio of the Sides B C and G C, let that be as d to a, and C G will be $= \frac{aa - ax}{d}$; take away this from D C or y, and there will remain D G $= \frac{dy - aa + ax}{d}$. Besides because of the similar Triangles B G C, and D G E, it is B G : B C :: D G : D E. But in the Triangle B G C, it is $a : d ::$ C G : B C. And consequently $aa : dd ::$ C Gq : B Cq, and by compounding $aa + dd : dd ::$ B Gq : B Cq, and extracting the Roots $\sqrt{aa + dd} : d$ (:: B G : B C) :: D G : D E. Therefore D E $= \frac{dy - aa + ax}{\sqrt{aa + dd}}$. Moreover since the Angle A B F is the Difference of the Angles B A D and A B D, and consequently the Angles B A D and F B D are equal, the right-angled Triangles C A D and E B D will be similar, and therefore the Sides proportional or D A : D C :: D B : D E. But D C is $= y$. D A $(= \sqrt{ACq + DCq}) = \sqrt{xx + yy}$. D B $(= \sqrt{BCq + DCq}) = \sqrt{aa - 2ax + xx + yy}$, and above D E was $= \frac{dy - aa + ax}{\sqrt{aa + dd}}$. Wherefore $\sqrt{xx + yy} : y :: \sqrt{aa - 2ax + xx + yy} : \frac{dy - aa + ax}{\sqrt{aa + dd}}$; and the Squares of the Means and Extremes being multiplied by each other

$$aayy - 2axyy + xxyy + y^4 = \frac{ddxxyy + ddy^4}{aa + dd}$$

$$\frac{-2aadxxy - 2aady^3 + 2adyx^3 + 2adxy^3 + a^4x^2 + a^4yy}{aa + dd}$$

$$\frac{-2a^3x^3 - 2a^3xyy + aax^4 + a^2x^2y^2}{aa + dd}.$$ Multiply all the

the Terms by $aa+dd$, and reduce those Terms that come out into due Order, and there will arise

$$x^4 \begin{matrix} -2a \\ +\frac{2d}{a}y \end{matrix} x^3 \begin{matrix} -2dy \\ +aa \end{matrix} xx \begin{matrix} +\frac{2d}{a}y^3 \\ +\frac{2dd}{a}yy \end{matrix} x \begin{matrix} -ddyy \\ -2dy^3 \\ -y^4 \end{matrix} = 0.$$

Divide this Equation by $xx - ax \begin{matrix} +dy \\ +yy \end{matrix}$, and there will arise $xx \begin{matrix} -a \\ +\frac{2d}{a}y \end{matrix} x \begin{matrix} -yy \\ -dy \end{matrix} = 0$; there come out therefore two Equations in the Solution of this Problem: The first, $xx - ax \begin{matrix} +dy \\ +yy \end{matrix} = 0$ is to a Circle, *viz.* the Place of the Point D, where the Angle F B D is taken on the other Side of the right Line B F than what is described in the Figure, the Angle A B F being the Sum of the Angles D A B and D B A at the Base, and so the Angle A D B at the Vertex being given. The last, *viz.* $xx \begin{matrix} -a \\ +\frac{2d}{a}y \end{matrix} x \begin{matrix} -yy \\ -dy \end{matrix} = 0$, is an Hyperbola, the Place of the Point D, where the Angle F B D obtains the Situation from the right Line B F, which we described in the Figure; that is, so that the Angle A B F may be the Difference of the Angles D A B, D B A, at the Base. And this is the Determination of the Hyperbola: Bisect A B in P; draw P Q, making the Angle B P Q equal to half the Angle A B F: To this draw the Perpendicular P R, and P Q and P R will be the Asymptotes of this Hyperbola, and B a Point through which the Hyperbola will pass.

Hence arises this *Theorem*. Any Diameter, as A B, of a right-angled Hyperbola, being drawn, and having drawn the right Lines A D, B D, A H, B H, from it's Ends to any two Points D and H of the Hyperbola; these right Lines will make equal Angles D A H, D B H at the Ends of the Diameter.

The same after a shorter Way.

At Prob. xxiv. I laid down a *Rule* about the most commodious Election of Terms to proceed with in the Calculus of of Problems, where there happens any *Ambiguity in* the Election of such *Terms*. Here the Difference of the Angles at the Base is indifferent in respect to both Angles; and in the Construction of the Scheme, it might equally have been added to the lesser Angle D A B, by *drawing* from A a right Line *parallel to* B F, or subtracted from the greater Angle D B A, by drawing the right Line B F. Wherefore I neither add nor substract it, but add half of it to one of the Angles, and substract half of it from the other. Then since it is also doubtful whether A C or B C must be made Use of for the indefinite Term whereon the Ordinate D C stands, I use neither of them; but I bisect A B in P, and I make use of P C; or rather, having drawn M P Q [*See Figure* 53.] making, on both Sides, the Angles A P Q, B P M equal to half the Difference of the Angles at the Base, so that it, with the right Lines A D, B D, may make the Angles D Q P, D M P equal; I let fall to M Q the Perpendiculars A R, B N, D O, and I use D O for the Ordinate, and P O for the indefinite Line it stands on. I make therefore P O $= x$, D O $= y$, A R or B N $= b$, and P R or P N $= c$. And by reason of the similar Triangles B N M, D O M, B N will be : D O :: M N : M O. And by Division D O $-$ B N $(y - b)$: D O (y) :: M O $-$ M N (O N or $c - x$) : M O. Wherefore M O $= \frac{cy - xy}{y - b}$. In like Manner on the other Side, by reason of the similar Triangles A R Q, D O Q, A R will be : D O :: R Q : Q O, and by Composition D O $+$ A R $(y + b)$: D O (y) :: Q O $+$ R Q (O R or $c + x$) : Q O. Wherefore Q O $= \frac{cy + xy}{y + b}$. Lastly by reason of the equal Angles D M Q, D Q M, M O and Q O are equal, that is, $\frac{cy - xy}{y - b} = \frac{cy + xy}{y + b}$. Divide all by y, and multiply by the Denominators, and there will arise $cy + cb - xy - xb = cy - cb + xy - xb$, or $cb = xy$, the most noted Equation that expresses the Hyperbola.

Moreover, the Locus or Place of the Point D might have been found without an Algebraick Calculus; for from what we have said above, $DO - BN : ON :: DO : MO$ $(QO) :: DO + AR : OR$. That is, $DO - BN : DO + BN :: ON : OR$. And mixtly, $DO : BN :: \frac{ON + OR}{2} (NP) : \frac{OR - ON}{2} (OP)$. And consequently, $DO \times OP = BN \times NP$.

Problem XLII.

To find the Locus or Place of the Vertex of a Triangle whose Base is given, and one of the Angles at the Base differs by a given Angle from being double of the other.

In the last Scheme of the former Problem, let ABD be that Triangle, AB its Base bisected in P, APQ or BPM the third of the given Angle, by which DBA exceeds the double of the Angle DAB; and the Angle DMQ will be double of the Angle DQM. To PM let fall the Perpendiculars AR, BN, DO, and bisect the Angle DMQ by the right Line MS meeting DO in S; and the Triangles DOQ, SOM will be similar; and consequently $OQ : OM :: OD : OS$, and by dividing $OQ - OM : OM :: SD : OS ::$ (by the 3. of the 6th *Elem.*) $DM : OM$. Wherefore (by the 9. of the 5th *Elem.*) $OQ - OM = DM$. Now making $PO = x$, $OD = y$, AR or $BN = b$, and PR or $PN = c$, you will have, as in the former Problem, $OM = \frac{cy - xy}{y - b}$, and $OQ = \frac{cy + xy}{y + b}$, and consequently $OQ - OM = \frac{-2bcy + 2xyy}{yy - bb}$. Make now $DOq + OMq = DMq$, that is, $yy + \frac{cc - 2cx + xx}{yy - 2by + bb} yy = \frac{4bbcc - 8bcxy + 4xxyy}{y^4 - 2bbyy + b^4} yy$, and by due Reduction there will at length arise

$$y^4 * \begin{matrix} +cc \\ -2bb \\ -2cx \\ -3xx \end{matrix}\, yy \begin{matrix} +2bxx \\ +4bcx \\ +2bcc \end{matrix}\, y \begin{matrix} +b^4 \\ -3bbcc \\ -2bbcx \\ +bbxx \end{matrix} = 0.$$

Divide

Divide all by $y - b$, and it will become

$$y^3 + byy \begin{matrix} -bb \\ +cc \\ -2cx \\ -3xx \end{matrix} y \begin{matrix} -b^3 \\ +3bcc \\ +2bcx \\ -bxx \end{matrix} = 0.$$ Wherefore the Point D is in a Curve of three Dimensions; which however becomes an Hyperbola when the Angle B P M vanishes or becomes nothing; or which is the same Thing, when one of the Angles at the Base D B A is double of the other D A B. For then B N or b vanishing, the Equation will become $yy = 3xx + 2cx - cc$.

And from the Construction of this Equation there comes this *Theorem.* [See *Figure* 54.] If to the Center C, and Asymptotes C S, C T, containing the Angle S C T of 120 Degrees, you describe any Hyperbola, as D V, whose Semi-Axis's are C V, C A; produce C V to B, so that V B shall be = V C, and from A and B you draw any how the right Lines A D, B D, meeting at the Hyperbola; the Angle B A D will be half the Angle A B D, but a third Part of the Angle A D E, which the right Line A D comprehends with B D produced. This is to be understood of the Hyperbola that passes through the Point V. But if the two right Lines A *d* and B *d*, drawn from the same Points A and B, meet in the opposite Hyperbola that passes through A, then of those two external Angles of the Triangle at the *Base*, that at *B* will be double of that at *A*.

PROBLEM XLIII.

To describe a Circle through two given Points that shall touch a right Line given in Position. [*See* Figure 55.]

Let A and B be the two given Points, and E F the right Line given in Position, and let it be required to describe a Circle A B E through those Points which shall touch that right Line F E. Join A B and bisect it in D. Upon D erect the Perpendicular D F meeting the right Line F E in F, and the Center of the Circle will fall upon this last drawn Line D F, as suppose in C. Join therefore C B; and on F E let fall the Perpendicular C E, and E will be the Point of Contact, and C B and C E equal, as being Radii of the Circle sought. Now since the Points A, B, D, and F, are given, let D B = a, and D F = b; and seek for D C to to determine the Center of the Circle, which therefore call x. Now in the Triangle C D B, because the Angle at D is a right one, you have $\sqrt{DBq + DCq}$, that is $\sqrt{aa + xx} = CB$. Also D F — D C, or $b - x = CF$. And since in the right-angled Triangle C F E the Angles are given, there will be given the Ratio of the Sides C F and C E. Let that be as d to e; and CE will be $= \frac{e}{d} \times CF$, that is, $= \frac{eb - ex}{d}$. Now put GB and CE (the Radii of the Circle sought) equal to one another, and you will have the Equation $\sqrt{aa + xx} = \frac{eb - ex}{d}$. Whose Parts being squared and multiplied by dd, there arises $aadd + ddxx = eebb - 2eebx + eexx$; or $xx = \frac{-2eebx - aadd + eebb}{dd - ee}$. And extracting the Root $x = \frac{-eeb + d\sqrt{eebb + eeaa - ddaa}}{dd - ee}$. Therefore the Length of D C, and consequently the Center C is found, from which a Circle is to be described through the Points A and B that shall touch the right Line F E.

PROBLEM XLIV.

To describe a Circle through a given Point that shall touch two right Lines given in Position. [See Figure 56.]

N. B. *This Proposition is resolved as* Prop. 43. *for the Point* A *being given, there is also given the other Point* B.

Suppose the given Point to be A; and let E F, F G, be the two right Lines given by Position, and A E G the Circle sought touching the same, and passing through that Point A. Let the Angle E F G be bisected by the right Line C F, and the Center of the Circle will be found therein. Let that be C; and having let fall the Perpendiculars C E, C G to E F and F G, E and G will be the Points of Contact. Now in the Triangles C E F, C G F, since the Angles at E and G are right ones, and the Angles at F are halves of the Angle EFG, all

all the Angles are given, and consequently the Ratio of the Sides CF and CE or CG. Let that be as d to e; and if for determining the Center of the Circle sought C, there be assumed $CF = x$, CE or CG will be $= \frac{ex}{d}$. Besides, let fall the Perpendicular AH to FC, and since the Point A is given, the right Lines AH and FH will be given. Let them be called a and b, and taking FC or x from FH or b, there will remain $CH = b - x$. To whose Square $bb - 2bx + xx$ add the Square of AH or aa, and the Sum $aa + bb - 2bx + xx$ will be ACq by the 47. 1. *Eucl.* because the Angle AHC is, by Supposition, a right one. Now make the Radii of the Circle AC and CG equal to each other; that is, make an equality between their Values, or between their Squares, and you will have the Equation $aa + bb - 2bx + xx = \frac{eexx}{dd}$. Take away xx from both Sides, and changing all the Signs, you will have $-aa - bb + 2bx = xx - \frac{eexx}{dd}$. Multiply all by dd, and divide by $dd - ee$, and it will become $\frac{-aadd - bbdd + 2bddx}{dd - ee} = xx$. The Root of which Equation being extracted, is $x = \frac{bdd - d\sqrt{eebb + eeaa - ddaa}}{dd - ee}$. Therefore the Length FC is found, and consequently the Point C, which is the Center of the Circle sought.

If the found Value x or FC be taken from b or HF, there will remain $HC = \frac{-eeb + d\sqrt{eebb + eeaa - ddaa}}{dd - ee}$ the same Equation which came out in the former Problem, for determining the Length of DC.

PRO-

Problem XLV.

To describe a Circle through two given Points, which shall touch another Circle given in Position. [*See* Problem 21, *and* Figure 57.]

Let A, B be the two Points given, EK the Circle given in Magnitude and Position, F its Center, ABE the Circle sought, passing through the Points A and B, and touching the other Circle in E, and let C be its Center. Let fall the Perpendiculars CD and FG to AB being produced, and draw CF cutting the Circles in the Point of Contact E, and draw also FH parallel to DG, and meeting CD in H. These being thus constructed, make AD or DB $=a$, DG or HF $=b$, GF $=c$, and EF (the Radius of the Circle given) $=d$, and DC $=x$; and CH will be ($=$ CD $-$ FG) $=x-c$, and CFq ($=$ CHq $+$ HFq) $=xx-2cx+cc+bb$, and CBq ($=$ CDq $+$ DBq) $=xx+aa$, and consequently CB or CE $=\sqrt{xx+aa}$. To this add EF, and you will have CF $=d+\sqrt{xx+aa}$, whose Square $dd+aa+xx+2d\sqrt{xx+aa}$, is equal to the Value of the same CFq found before, *viz.* $xx-2cx+cc+bb$. Take away from both Sides xx, and there will remain $dd+aa+2d\sqrt{xx+aa}=cc+bb-2cx$. Take away moreover $dd+aa$, and there will come out $2d\sqrt{xx+aa}=cc+bb-dd-aa-2cx$. Now, for Abbreviation sake, for $cc+bb-dd-aa$, write $2gg$, and you will have $2d\sqrt{xx+aa}=2gg-2cx$, or $d\sqrt{xx+aa}=gg-cx$. And the Parts of the Equation being squared, there will come out $ddxx+ddaa=g^4-2ggcx+ccxx$. Take from both Sides $ddaa$ and $ccxx$, and there will remain $ddxx-ccxx=g^4-ddaa-2ggcx$. And the Parts of the Equation being divided by $dd-cc$, you will have $xx=\frac{g^4-ddaa-2ggcx}{dd-cc}$. And by Extraction of the affected Root $x=\frac{-ggc+\sqrt{g^4dd-d^4aa+ddaacc}}{dd-cc}$.

Having

Having found therefore x, or the Length of DC, bisect AB in D, and at D erect the Perpendicular DC = $\frac{ggc + d\sqrt{g^4 - aadd + aacc}}{dd - cc}$. Then from the Center C, through the Point A or B, describe the Circle ABE; for that will touch the other Circle EK, and pass through both the Points A, B. Q. E. F.

PROBLEM XLVI.

To describe a Circle through a given Point which shall touch a given Circle, and also a right Line, both given in Position. [*See* Figure 58.]

Let the Circle to be described be BD, its Center C, and B a Point through which it is to be described, and AD the right Line which it shall touch; the Point of Contact D, and the Circle which it shall touch GEM, its Center F, and its Point of Contact E. Join CB, CD, CF, and CD will be perpendicular to AD, and CF will cut the Circles in the Point of Contact E. Produce CD to Q, so that DQ shall be = EF, and through Q draw QN parallel to AD. Lastly, from B and F to AD and QN, let fall the Perpendiculars BA, FN; and from C to AB and FN let fall the Perpendiculars CK, CL. And since BC is = CD or AK, BK will be ($=AB-AK$) $= AB - BC$, and consequently $BKq = ABq - 2AB \times BC + BCq$. Subtract this from BCq, and there will remain $2AB \times BC - ABq$ for the Square of CK. Therefore $AB \times \overline{2BC - AB}$ is = CKq; and for the same Reason it will be $FN \times \overline{2FC - FN} = CLq$, and consequently $\frac{CKq}{AB} + AB = 2BC$, and $\frac{CLq}{FN} + FN = 2FC$. Wherefore, if for AB, CK, FN, KL, and CL, you write a, y, b, c, and $c-y$, you will have $\frac{yy}{2a} + \frac{1}{2}a = BC$, and $\frac{cc - 2cy + yy}{2b} + \frac{1}{2}b = FC$. From FC take away BC, and there will remain $EF = \frac{cc - 2cy + yy}{2b} + \frac{1}{2}b - \frac{yy}{2a} - \frac{1}{2}a$.

Now,

X

Now, if the Points where F N being produced cuts the right Line A D, and the Circle G E M be marked with the Letters H, G, and M, and upon H G produced you take H R = A B, since H N (= D Q = E F) is = G F, by adding F H on both Sides, you will have F N = G H, and consequently A B − F N (= H R − G H) = G R, and A B − F N + 2 E F; that is, $a - b + 2$ E F = R M, and $\frac{1}{2}a - \frac{1}{2}b +$ E F = $\frac{1}{2}$ R M.

Wherefore, since above E F was $= \frac{cc - 2cy + yy}{2b} + \frac{1}{2}b - \frac{yy}{2a} - \frac{1}{2}a$, if this be written for E F you will have $\frac{1}{2}$ R M $= \frac{cc - 2cy + yy}{2b} - \frac{yy}{2a}$. Call therefore R M, d; and d will be $= \frac{cc - 2cy + yy}{b} - \frac{yy}{a}$. Multiply all the Terms by a and b, and there will arise $abd = acc - 2acy + ayy - byy$. Take away from both Sides $acc - 2acy$, and there will remain $abd - acc + 2acy = ayy - byy$. Divide by $a - b$, and there will arise $\frac{abd - acc + 2acy}{a - b} = yy$. And extracting the Root $y = \frac{ac}{a - b} \pm \sqrt{\frac{aabd - abbd + abcc}{aa - 2ab + bb}}$. Which Conclusions may be thus abbreviated; make $c : b :: d : e$, then $a - b : a :: c : f$; and $fe - fc + 2fy$ will be $= yy$, or $y = f \pm \sqrt{ff + fe - fc}$. Having found y or K C or A D, take A D $= f \pm \sqrt{ff + fe - fc}$, and at D erect the Perpendicular D C (= B C) $= \frac{KCq}{2AB} + \frac{1}{2}AB$; and from the Center C, at the Interval C B or C D, describe the Circle B D E, for this passing through the given Point B, will touch the right Line A D in D, and the Circle G E M in E. Q. E. F.

Hence also a Circle may be described which shall touch two given Circles, and a right Line given in Position. [See *Figure* 59.] For let the given Circles be R T, S V, their Centers B, F, and the right Line given in Position P Q. From the Center F, with the Radius F S − B R, describe the Circle

Circle E M. From the Point B to the right Line P Q let fall the Perpendicular B P, and having produced it to A, so that P A shall be = B R, through A draw A H parallel to P Q, and describe a Circle which shall pass through the Point B, and touch the right Line A H and the Circle E M. Let its Center be C; join B C, cutting the Circle R T in R; and the Circle R S described from the same Center C, and the Radius C R will touch the Circles R T, S V, and the right Line P Q, as is manifest by the Construction.

Problem XLVII.

To describe a Circle that shall pass through a given Point, and touch two other Circles given in Position and Magnitude. [*See* Figure 60.]

Let the given Point be A, and let the Circles given in Magnitude and Position be T I V, R H S, their Centers C and B; the Circle to be described A I H, its Center D, and the Points of Contact I and H. Join A B, A C, A D, D B, and let A B produced cut the Circle R H S in the Points R and S, and A C produced, cut the Circle T I V in T and V. And having from the Points D and C let fall the Perpendiculars D E to A B, and D F to A C meeting A B in G, and C K to A B; in the Triangle A D B, it will be $ADq - DBq + ABq = 2\,AE \times AB$, by the 13 of the 2 *Elem.* But $DB = AD + BR$, and consequently $DBq = ADq + 2\,AD \times BR + BRq$. Take away this from $ADq + ABq$, and there will remain $ABq - 2\,AD \times BR - BRq$ for $2\,AE \times AB$. Moreover $ABq - BRq$ is $= \overline{AB - BR} \times \overline{AB + BR} = AR \times AS$. Wherefore $AR \times AS - 2\,AD \times BR = 2\,AE \times AB$. And $\frac{AR \times AS - 2\,AB \times AE}{BR} = 2\,AD$.

And by a like Reasoning in the Triangle A D C, there will come out again $2\,AD = \frac{TAV - 2\,CAF}{CT}$.

Wherefore $\frac{RAS - 2\,BAE}{BR} = \frac{TAV - 2\,CAF}{CT}$. And $\frac{TAV}{CT} - \frac{RAS}{BR} + \frac{2\,BAE}{BR} = \frac{2\,CAF}{CT}$. And

 TAV

$\overline{\frac{TAV}{CT} - \frac{RAS}{BR} + \frac{2BAE}{BR}} \times \frac{CT}{2AC} = AF$. Whence since it is $AK : AC :: AF : AG$, AG will be $=$
$\overline{\frac{TAV}{CT} - \frac{RAS}{BR} + \frac{2BAE}{BR}} \times \frac{CT}{2AK}$. Take away this from AE, or $\frac{2KAE}{CT} \times \frac{CT}{2AK}$, and there will remain $GE =$
$\overline{\frac{RAS}{BR} - \frac{TAV}{CT} - \frac{2BAE}{BR} + \frac{2KAE}{CT}} \times \frac{CT}{2AK}$. Whence since it is $KC : AK :: GF : DE$; DE will be $=$
$\overline{\frac{RAS}{BR} - \frac{TAV}{CT} - \frac{2BAE}{BR} + \frac{2KAE}{CT}} \times \frac{CT}{2KC}$. Upon AB take AP, which let be to AB as CT to BR, and $\frac{2PAE}{CT}$ will be $= \frac{2BAE}{BR}$, and so $\frac{2PK \times AE}{CT} = \frac{2BAE}{BR} - \frac{2KAE}{CT}$, and so $DE = \overline{\frac{RAS}{BR} - \frac{TAV}{CT} - \frac{2PK \times AE}{CT}} \times \frac{CT}{2KC}$. Upon AB erect the Perpendicular $AQ = \overline{\frac{RAS}{BR} - \frac{TAV}{CT}} \times \frac{CT}{2KC}$, and in it take $QO = \frac{PK \times AE}{KC}$, and AO will be $= DE$.

Join DO, DQ, and CP, and the Triangles DOQ, CKP will be similar, because their Angles at O and K are right ones, and the Sides ($KC : PK :: AE$, or $DO : QO$) proportional. Therefore the Angles OQD, KPC are equal, and consequently QD is perpendicular to CP. Wherefore if AN be drawn parallel to CP, and meeting QD in N, the Angle ANQ will be a right one, and the Triangles AQN, PCK similar; and consequently $PC : KC :: AQ : AN$. Whence since AQ is $\overline{\frac{RAS}{BR} - \frac{TAV}{CT}} \times \frac{CT}{2KC}$, AN will be $\overline{\frac{RAS}{BR} - \frac{TAV}{CT}} \times \frac{CT}{2PC}$. Produce AN to M, so that NM shall be $= AN$, and AD will be $= DM$, and consequently the Circle will pass through the Point M.

Since

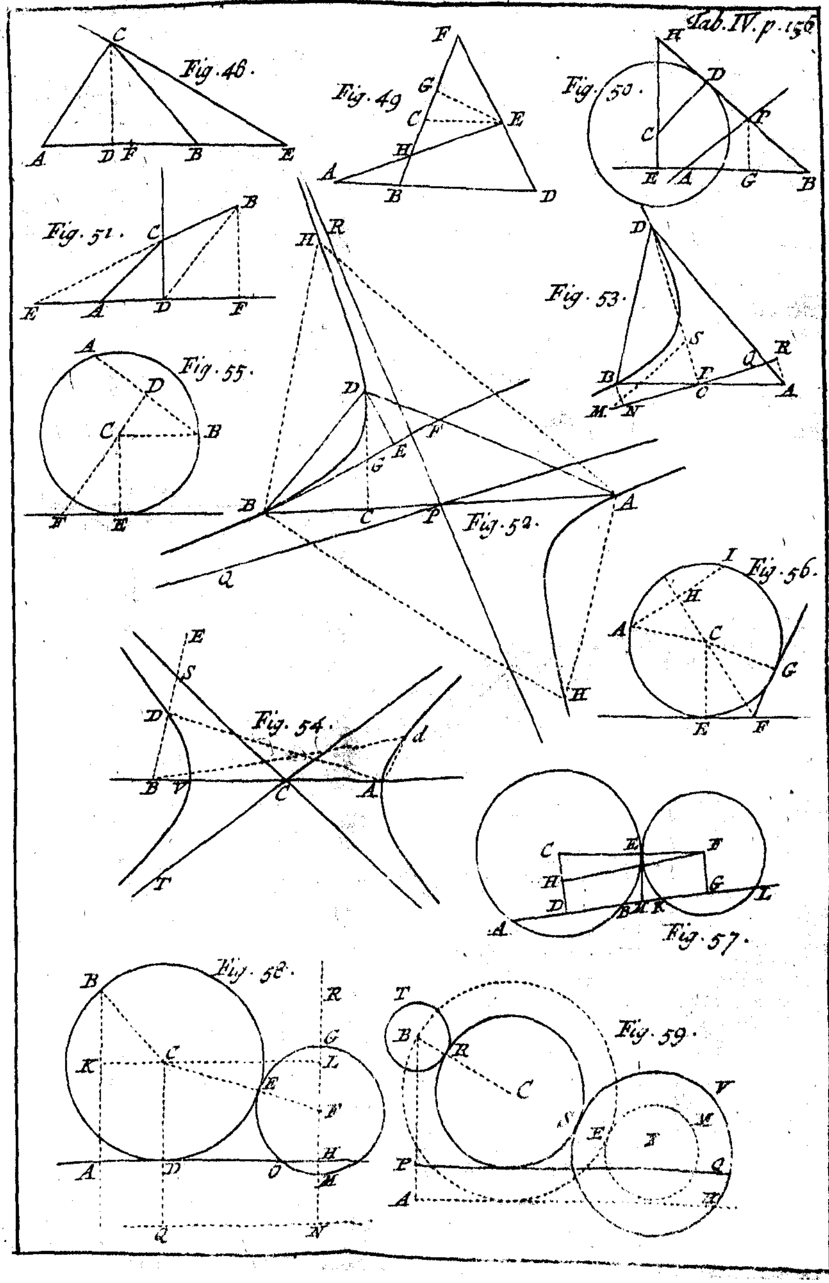


Fig. 48.
Fig. 49
Fig. 50.
Fig. 51.
Fig. 52.
Fig. 53.
Fig. 54.
Fig. 55.
Fig. 56.
Fig. 57.
Fig. 58.
Fig. 59.

Since therefore the Point M is given, there follows this Resolution of the Problem, without any farther Analysis.

Upon AB take AP, which must be to AB as CT to BR; join CP, and draw parallel to it AM, which shall be to $\frac{RAS}{BR} - \frac{TAV}{CT}$, as CT to PC; and by the Help of the 45th *Problem*, describe through the Points A and M the Circle AIHM, which shall touch either of the Circles TIV, RHS, and the same Circle shall touch both. Q. E. F.

And hence also a Circle may be described, which shall touch three Circles given in Magnitude and Position. For let the Radii of the given Circles be A, B, C, and their Center D, E, F. From the Centers E and F, with the Radii $B \pm A$ and $C \pm A$ describe two Circles, and let a third Circle which touches these two be also described, and let it pass through the Point D; let its Radius be G, and its Center H, and a Circle described on the same Center H, with the Radius $G \pm A$, shall touch the three former Circles, as was required.

PROBLEM XLVIII.

If at the Ends of the Thread D A E, *moving upon the fixed Tack* A, *there are hanged two Weights,* D and E, *whereof the Weight* E *slides through the oblique Line* B G; *to find the Place of the Weight* E, *where these Weights are in Æquilibrio.* [*See* Figure 63.]

Suppose the Problem done, and parallel to A D draw E F, which let be to A E as the Weight E to the Weight D. And from the Points A and F to the Line B G let fall the Perpendiculars A B, F G. Now since the Weights are, by Supposition, as the Lines A E and E F, express those Weights by those Lines, the Weight D by the Line E A, and the Weight E by the Line E F. Therefore the Body E, directed by the Force of its own Weight E F, tends towards F, and by the oblique Force E G tends towards G. And the same Body E by the direct Force A E of the Weight D is drawn towards A, and by the oblique Force B E is drawn towards B. Since therefore the Weights sustain each other in Æquilibrio, the Force by which the Weight E is drawn towards B, ought to be equal to the contrary Force by which it tends towards G, that is, B E ought to be equal to E G. But now the Ratio of A E to

E F

EF is given by the Hypothesis; and by reason of the given Angle FEG, there is also given the Ratio of FE to EG, to which BE is equal. Therefore there is given the Ratio of AE to BE. AB is also given in Length; and thence the Triangle ABE, and the Point E will easily be given, *Viz.* make AB $= a$, BE $= x$, and AE will be $= \sqrt{aa + xx}$; moreover, let AE be to BE in the given Ratio of d to e, and $e\sqrt{aa+xx}$ will be $= dx$. And the Parts of the Equation being squared and reduced, $eeaa = ddxx - eexx$, or $\frac{ea}{\sqrt{dd-ee}} = x$. Therefore the Length BE is found, which determines the Place of the Weight E. Q. E. F.

But if both Weights descend by oblique Lines, the Computation may be made thus. [See *Figure* 64.] Let CD and BE be oblique Lines given in Position, through which those Weights D and E descend. From the fixed Tack A to these Lines let fall the Perpendiculars AC, AB, and let the Lines EG, DH, erected from the Weights perpendicularly to the Horizon, meet them in the Points G and H; and the Force by which the Weight E endeavours to descend in a perpendicular Line, that is the whole Gravity of E, will be to the Force by which the same Weight endeavours to descend in the oblique Line BE, as GE to BE; and the Force by which it endeavours to descend in the oblique Line BE, will be to the Force by which it endeavours to descend in the Line AE, that is, to the Force by which the Thread AE is distended or stretched as BE to AE. And consequently the Gravity of E will be to the Tension of the Thread AE as GE to AE. And by the same reason the Gravity of D will be to the Tension of the Thread AD as HD to AD. Let therefore the Length of the whole Thread DA + AE be c, and let its Part AE be $= x$, and its other Part AD will be $= c - x$. And because AEq — ABq is $=$ BEq, and ADq — ACq $=$ CDq; let, moreover, AB be $= a$, and AC $= b$, and BE will be $= \sqrt{xx - aa}$ and CD $= \sqrt{xx - 2cx + cc - bb}$. Farther, since the Triangles BEG, CDH are given in Specie, let BE : EG :: f : E and CD : DH :: f : g, and EG will be $= \frac{E}{f}\sqrt{xx - aa}$,

and

and $DH = \frac{g}{f}\sqrt{xx - 2cx + cc - bb}$. Wherefore since $GE : AE ::$ Weight E : Tension of AE; and $HD : AD ::$ Weight D : Tension of AD; and those Tensions are equal, you will have $\frac{Ex}{\frac{E}{f}\sqrt{xx - aa}} =$ Tension of $AE =$ the Tension $AD = \frac{Dc - Dx}{\frac{g}{f}\sqrt{xx - 2cx + cc - bb}}$; from the Reduction of which Equation there comes out $gx x \sqrt{xx - 2cx + cc - bb} = \overline{Dc - Dx}\sqrt{xx - aa}$, or

$$\frac{gg}{-DD}x^4 \begin{matrix}-2ggc \\ +2DDc\end{matrix}\, x^3 \begin{matrix}+ggcc \\ -ggbb \\ -DDcc \\ +DDaa\end{matrix}\, xx - 2DDcaax + DDccaa = 0.$$

But if you desire a Case wherein this Problem may be constructed by a Rule and Compass, make the Weight D to the Weight E as the Ratio $\frac{BE}{EG}$ to the Ratio $\frac{CD}{DH}$, and g will become $= D$; and so in the Room of the precedent Equation you will have this, $-\frac{aa}{bb}xx - 2acx + aaacc = 0$, or $x = \frac{ac}{a+b}$.

Problem XLIX.

If on the String DACBF, *that slides about the two Tacks* A *and* B, *there are hung three Weights,* D, E, F; D *and* F *at the Ends of the String, and* E *at its middle Point* C, *placed between the Tacks: From the given Weights and Position of the Tacks to find the Situation of the Point* C, *where the middle Weight hangs, and where they are in Æquilibrio.* [*See* Figure 65.]

Since the Tension of the Thread AC is equal to the Tension of the Thread AD, and the Tension of the Thread BC to the Tension of the Thread BF, the Tensions of the Strings or Threads AC, BC, EC will be as the Weights D, E, F.

Then

Then take the Parts of the Thread CG, CH, CI, in the same Ratio as the Weights. Compleat the Triangle GHI. Produce IC till it meet GH in K, and GK will be = KH, and CK $= \frac{1}{2}$ CI, and consequently C the Center of Gravity of the Triangle GHI. For, draw PQ through C, perpendicular to CE, and perpendicular to that, from the Points G and H, draw GP, HQ. And if the Force by which the Thread AC by the Force of the Weight D draws the Point C towards A, be expressed by the Line GC, the Force by which that Thread will draw the same Point towards P, will be expressed by the Line CP; and the Force by which it draws it towards K, will be expressed by the Line GP. And in like manner, the Forces by which the Thread BC, by means of the Weight F, draws the same Point C towards B, Q, and K, will be expressed by the Lines CH, CQ, and HQ; and the Force by which the Thread CE, by means of the Weight E, draws that Point C towards E, will be expressed by the Line CI. Now since the Point C is sustained in Æquilibrio by equal Forces, the Sum of the Forces by which the Threads AC and BC do together draw C towards K, will be equal to the contrary Force by which the Thread EC draws that Point towards E; that is, the Sum GP + HQ will be equal to CI; and the Force by which the Thread AC draws the Point C towards P, will be equal to the contrary Force by which the Thread BC draws the same Point C towards Q; that is, the Line PC is equal to the Line CQ. Wherefore, since PG, CK, and QH are Parallel, GK will be also = KH, and CK ($= \frac{GP + HQ}{2}$) $= \frac{1}{2}$ CI. Which was to be shewn. It remains therefore to determine the Triangle GCH, whose Sides GC and HC are given, together with the Line CK, which is drawn from the Vertex C to the middle of the Base. Let fall therefore from the Vertex C to the Base CH the Perpendicular CL, and $\frac{GCq - CHq}{2GH}$ will be $= KL = \frac{GCq - KCq - GKq}{2GK}$. For 2GK write GH, and having rejected the common Divisor GH, and ordered the Terms, you will have $GCq - 2KCq + CHq = 2GKq$, or $\sqrt{\frac{1}{2}GCq - KCq + \frac{1}{2}CHq} = GK$. Having found GK, or KH, there are given together the Angles GCK, KCH, or or D A C, F B C. Wherefore, from the Points A and B in these given Angles D A C, F B C, draw the Lines A C, B C, meeting in the Point C; and C will be the Point sought.

But it is not always necessary to solve Questions that are of the same Kind, particularly by Algebra, but from the Solution of one of them you may most commonly infer the Solution of the other. As if now there should be proposed this Question.

The Thread A C D B *being divided into the given Parts* A C, C D, D B, *and its Ends being fastened to the two Tacks given in Position,* A *and* B; *and if at the Points of Division,* C *and* D, *there are hanged the two Weights* E *and* F; *from the given Weight* F, *and the Situation of the Points* C *and* D, *to know the Weight* E. [*See* Figure 66.]

From the Solution of the former Problem the Solution of this may be easily enough found. Produce the Lines A C, B D, till they meet the Lines D F, C E in G and H; and the Weight E will be to the Weight F, as D G to C H.

And hence by the by may appear a Method of making a Balance of only Threads, by which the Weight of any Body E may be known, from only one given Weight F.

PROBLEM L.

A Stone falling down into a Well, from the Sound of the Stone striking the Bottom, to determine the Depth of the Well.

Let the Depth of the Well be x, and if the Stone descends with an uniformly accelerated Motion through any given Space a in any given Time b, and the Sound passes with an uniform Motion through the same given Space a in the given Time d, the Stone will descend through the Space x in the Time $b\sqrt{\frac{x}{a}}$; but the Sound which is caused by the Stone striking upon the Bottom of the Well, will ascend through the same Space x, in the Time $\frac{dx}{a}$. For the Spaces described by descending heavy Bodies, are as the Squares of the Times of Descent; or the Roots of the Spaces, that is, $\sqrt{x}$ and $\sqrt{a}$ are as the Times themselves. And the Spaces x and a, through which the Sound passes, are as the Times of Passage. And

And the Sum of these Times $b\sqrt{\frac{x}{a}}$, and $\frac{dx}{a}$, is the Time of the Stone's falling to the Return of the Sound. This Time may be known by Observation. Let it be t, and you will have $b\sqrt{\frac{x}{a}} + \frac{dx}{a} = t$. And $b\sqrt{\frac{x}{a}} = t - \frac{dx}{a}$. And the Parts being squared, $\frac{bbx}{a} = tt - \frac{2tdx}{a} + \frac{ddxx}{aa}$. And by Reduction $xx = \frac{2adt + abb}{dd}x - \frac{aatt}{dd}$. And having extracted the Root $x = \frac{adt + \frac{1}{2}abb}{dd} - \frac{ab}{2dd} \times \sqrt{bb + 4dt}$.

PROBLEM LI.

Having given the Globe A, *and the Position of the Wall* DE, *and* BD *the Distance of the Center of the Globe* B *from the Wall; to find the Bulk of the Globe* B, *on this Condition, that if the Globe* A, (*whose Center is in the Line* BD, *which is perpendicular to the Wall, and produced out beyond* B) *be moved in free absolute Space, and where Gravity cannot act, with an uniform Motion towards* D, *until it strikes against the other quiescent Globe* B; *and that Globe* B, *after it is reflected from the Wall, shall meet the Globe* A *in the given Point* C. [*See* Figure 81.]

Let the Velocity of the Globe A before Reflection be a, and by *Problem* XII. p. 80. the Velocity of the Globe A will be after Reflection $= \frac{aA - aB}{A + B}$, and the Velocity of the Globe B after Reflection will be $= \frac{2aA}{A + B}$. Therefore the Velocity of the Globe A to the Velocity of the Globe B is as A − B to 2 A. On GD take $gD = GH$, *viz.* to the Diameter of the Globe B, and those Velocities will be as GC to $Gg + gC$. For when the Globe A struck upon the Globe B, the Point G, which being on the Surface of the Globe B is moved in the Line AD, will go through the

Space

Space Gg before that Globe B ſhall ſtrike againſt the Wall, and through the Space gC after it is reflected from the Wall; that is, through the whole Space $Gg+gC$, in the ſame Time wherein the Point F of the Globe A ſhall paſs through the Space GC, ſo that both Globes may again meet and ſtrike one another in the given Point C. Wherefore, ſince the Intervals BC and CD are given, make $BC=m$, $BD+CD=n$, and $BG=x$, and GC will be $=m+x$, and $Gg+gC=GD+DC-2gD=GB+BD+DC-2GH=x+n-4x$, or $=n-3x$. Above you had $A-B$ to $2A$, as the Velocity of the Globe A to the Velocity of the Globe B, and the Velocity of the Globe A to the Velocity of the Globe B, as GC to $Gg+gC$, and conſequently $A-B$ to $2A$, as GC to $Gg+gC$; therefore ſince GC is $=m+x$, and $Gg+gC=n-3x$, $A-B$ will be to $2A$ as $m+x$ to $n-3x$. Moreover, the Globe A is to the Globe B as the Cube of its Radius AF to the Cube of the others Radius GB; that is, if you make the Radius AF to be s, as s^3 to x^3; therefore $s^3-x^3:2s^3$ $(::A-B:2A)::m+x:n-3x$. And multiplying the Means and Extreams by one another, you will have this Equation, $s^3n-3s^3x-nx^3+3x^4=2ms^3+2s^3x$. And by Reduction $3x^4-nx^3-5s^3x\begin{smallmatrix}+s^3n\\-2s^3m\end{smallmatrix}=0$. From the Conſtruction of which Equation there will be given x, the Semi-Diameter of the Globe B; which being given, that Globe is alſo given. Q. E. F.

But note, when the Point C lies on contrary Sides of the Globe B, the Sign of the Quantity $2m$ muſt be changed, and written $3x^4-nx^3-5s^3x\begin{smallmatrix}+s^3n\\+2s^3m\end{smallmatrix}=0$.

If the Globe B were given, and the Globe A ſought on this Condition, that the two Globes, after Reflection, ſhould meet in C, the Queſtion would be eaſier; *viz.* in the laſt Equation found, x would be ſuppoſed to be given, and s to be ſought. Whereby, by a due Reduction of that Equation, the Terms $-5s^3x+s^3n-2s^3m$ being tranſlated to the contrary Side of the Equation, and each Part divided by $5x-n+2m$, there would come out $\frac{3x^4-nx^3}{5x-n+2m}=s^3$. Where s will be obtained by the bare Extraction of the Cube Root.

But if both Globes being given, you were to find the Point C, in which both would fall upon one another after Reflection: Since above it was $A - b$ to $2A$ as GC to $Gg + gC$, therefore by Inversion and Composition $3A - B$ will be to $A - B$ as $2Gg$ to the sought Distance GC.

Problem LII.

If two Globes, A and B, are joined together by a small Thread P Q, and the Globe B hanging on the Globe A; if you let fall the Globe A, so that both Globes may begin to fall together by the sole Force of Gravity in the same perpendicular Line P Q; and then the lower Globe B, after it is reflected upwards from the Bottom or Horizontal Plane F G, it shall meet the upper Globe A, as falling, in a certain Point D: From the given Length of the Thread P Q, and the Distance D F of that Point D from the Bottom, to find the Height P F, from which the upper Globe A ought to be let fall to cause this Effect. [*See* Figure 83.]

Let a be the Length of the Thread P Q. In the Perpendicular P Q R F, from F upwards take F E equal to Q R the Diameter of the lower Globe, so that when the lowest Point R of that Globe falls upon the Bottom in F, its upper Point Q shall possess the Place E; and let E D be the Distance through which that Globe, after it is reflected from the Bottom, shall, by ascending, pass, before it meets the upper falling Globe in the Point D. Therefore, by reason of the given Distance D F of the Point D from the Bottom, and the Diameter E F of the inferior Globe, there will be given their Difference D E. Let that be $= b$, and let the Height R F, or Q E, which that lower Globe describes by falling through it before it touches the Bottom, be $= x$, by reason it is unknown. And having found x, if to it you add E F and P Q, there will be had the Height P F, from which the upper Globe ought to fall to have the desired Effect.

Since therefore P Q is $= a$, and Q E $= x$, P E will be $= a + x$. Take away D E or b, and there will remain P D $= a + x - b$. But the Time of the Descent of the Globe A is as the Root of the Space described in falling, or $\sqrt{a + x - b}$, and the Time of the Descent of the other Globe B as the Root of the Space described by its falling, or $\sqrt{x}$, and the Time of its Ascent as the Difference of that

Root, and of the Root of the Space which it would describe by falling only from Q to D. For this Difference is as the Time of Descent from D to E, which is equal to the Time of Ascent from E to D. But that Difference is $\sqrt{x} - \sqrt{x-b}$. Whence the Time of Descent and Ascent together will be as $2\sqrt{x} - \sqrt{x-b}$. Wherefore, since this Time is equal to the Time of Descent of the upper Globe, it will be $\sqrt{a+x-b} = 2\sqrt{x} - \sqrt{x-b}$. The Parts of which Equation being squared, you will have $a+x-b = 5x - b - 4\sqrt{xx-bx}$, or $a = 4x - 4\sqrt{xx-bx}$; and the Equation being ordered, $4x - a = 4\sqrt{xx-bx}$; and squaring the Parts of that Equation again, there arises $16xx - 8ax + aa = 16xx - 16bx$, or $aa = 8ax - 16bx$; and dividing all by $8a - 16b$, you will have $\frac{aa}{8a-16b} = x$. Make therefore as $8a - 16b$ to a, so a to x, and you will have x or QE. Q. E. I.

But if from the given QE you are to find the Length of the Thread PQ or a; the same Equation $aa = 8ax - 16bx$, by extracting the affected Quadratick Root, would give $a = 4x - \sqrt{16xx - 16bx}$; that is, if you take QY a mean Proportional between QD and QE, PQ will be $= 4$ EY. For that mean Proportional will be $\sqrt{x \times x-b}$, or $\sqrt{xx-bx}$; which substracted from x, or QE, leaves EY, the Quadruple whereof is $4x - 4\sqrt{xx-bx}$.

But if from the given Quantities QE, or x, as also the Length of the Thread PQ, or a, there were sought the Point D in which the upper Globe falls upon the under one; the Distance DE, or b, of that Point from the given Point E, will be had from the precedent Equation $aa = 8ax - 16bx$ by transferring aa and $16bx$ to the contrary Sides of the Equation with the Signs changed, and by dividing the whole by $16x$. For there will arise $\frac{8ax - aa}{16x} = b$. Make therefore as $16x$ to $8x - a$, so a to b, and you will have b or DE.

Hitherto I have supposed the Globes tied together by a small Thread to be let fall together. Which, if they are let fall at different Times connected by no Thread, so that the

the upper Globe A, for Example, being let fall first, shall descend through the Space PT before the other Globe begins to fall, and from the given Distances PT, PQ, and DE, you are to find the Height PF, from which the upper Globe ought to be let fall, so that it shall fall upon the inferior or lower one at the Point D. Make PQ $= a$, DE $= b$, PT $= c$, and QE $= x$, and PD will be $= a + x - b$, as above. And the Times wherein the upper Globe, by falling, will describe the Spaces PT and TD, and the lower Globe by falling before, and then by re-ascending, will describe the Sum of the Spaces QE + ED will be as $\sqrt{PT}$, $\sqrt{PD} - \sqrt{PT}$, and $2\sqrt{QE} - \sqrt{QD}$; that is, as $\sqrt{c}$, $\sqrt{a+x-b} - \sqrt{c}$, and $2\sqrt{x} - \sqrt{x-b}$. But the two last Times, because the Spaces TD and QE + ED are described together, are equal. Therefore $\sqrt{a+x-b} - \sqrt{c} = 2\sqrt{x} - \sqrt{x-b}$. And the Parts being squared $a + c - 2\sqrt{ca+cx-cb} = 4x - 4\sqrt{xx-bx}$. Make $a + c = e$, and $a - b = f$, and by a due Reduction it will be $4x - e + 2\sqrt{cf+cx} = 4\sqrt{xx-bx}$, and the Parts being squared $ee - 8ex + 16xx + 4cf + 4cx + \overline{16x - 4e} \times \sqrt{cf+cx} = 16xx - 16bx$. And blotting out on both Sides $16xx$, and writing m for $ee + 4cf$, and also writing n for $8e - 16b - 4c$, you will have by due Reduction $\overline{16x - 4e} \times \sqrt{cf+cx} = nx - m$. And the Parts being squared you will have $256cfxx + 256cx^3 - 128cefx - 128cexx + 16ceef + 16ceex = nnxx - 2mnx + mm$. And having ordered the Equation

$$256cx^3 \begin{array}{l} +256cf \\ -128ce \\ -nn \end{array} xx \begin{array}{l} -128cef \\ +16cee \\ +2mn \end{array} x \begin{array}{l} +6ceef \\ -mm \end{array} = 0.$$

By the Construction of which Equation x or QE will be given, to which if you add the given Distances PQ and EF, you will have the Height PF, which was to be found.

PRO-

PROBLEM LIII.

If two quiescent Globes, the upper one A *and the under one* B, *are let fall at different Times; and the lower Globe begins to fall in the same Moment that the upper one, by falling, has described the Space* PT; *to find the Places* α, β, *which those falling Globes shall occupy when their Interval or Distance* $\pi\chi$ *is given.* [*See* Figure 84.]

Since the Distances PT, PQ, and $\pi\chi$ are given, call the first a, the second b, the third c, and for $P\pi$, or the Space that the superior Globe describes by falling before it comes to the Place sought α, put x. Now the Times wherein the upper Globe describes the Spaces PT, $P\pi$, $T\pi$, and the lower one the Space $Q\chi$, are as $\sqrt{PT}$, $\sqrt{P\pi}$, $\sqrt{P\pi}-\sqrt{PT}$, and $\sqrt{Q\chi}$. The latter two of which Times, because the Globes by falling together describe the Spaces $T\pi$ and $Q\chi$, are equal. Whence also $\sqrt{P\pi}-\sqrt{PT}$ will be equal to $\sqrt{Q\chi}$. $P\pi$ was $=x$, and $PT=a$, and by adding $\pi\chi$, or c, to $P\pi$, and subtracting PQ, or b, from the Sum, you will have $Q\chi=x+c-b$. Wherefore substituting these, you will have $\sqrt{x}-\sqrt{a}=\sqrt{x+c-b}$. And squaring both Sides of the Equation, there will arise $x+a-2\sqrt{ax}=x+c-b$. And blotting out on both Sides x, and ordering the Equation, you will have $a+b-c=2\sqrt{ax}$. And having squared the Parts, the Square of $a+b-c$ will be $=4ax$, and that Square divided by $4a$ will be $=x$, or $4a$ will be to $a+b-c$ as $a+b-c$ is to x. But from x found, or $P\pi$, there is given the Place sought, *viz.* α of the superior Globe. And by the Distance of the Places, there is also given the Place of the lower one β.

And hence, if you were to find the Point where the upper Globe, by falling, will at length fall upon the lower one; by putting the Distance $\pi\chi=0$, or by extirpating c, say $4a$ is to $a+b$ as $a+b$ is to x, or $P\pi$, and the Point π will be that sought.

And reciprocally, if that Point π, or χ, in which the upper Globe falls upon the under one, be given, and you are to find the Place T which the lower Point P of the upper falling Globe possessed, or was then in, when the lower Globe began to fall; because $4a$ is to $a+b$ as $a+b$ is to x; or multiplying the Means and Extremes together, $4ax=aa$

$= aa + 2ab + bb$, and by due ordering of the Equation $aa = 4ax - 2ab - bb$; extract the Square Root and you will have $a = 2x - b - 2\sqrt{xx - bx}$. Take therefore $V\pi$, a mean Proportional between $P\pi$ and $Q\pi$, and towards V take $VT = VQ$, and T will be the Point you seek. For $V\pi$ will be $= \sqrt{P\pi \times Q\pi}$, that is $= \sqrt{x \times \overline{x - b}}$, or $= \sqrt{xx - bx}$; the double whereof subtracted from $2x - b$, or from $2P\pi - PQ$, that is, from $PQ + 2Q\pi$, leaves $PQ - 2VQ$, or $PV - VQ$, that is, PT.

If, lastly, the lower of the Globes, after the upper has fallen upon the lower, and the lower, by their Shock upon one another, is accelerated, and the superior one retarded, the Places are required where, in falling, they shall acquire a Distance equal to a given right Line. In the first Place you must seek the Place where the upper one falls upon the lower one; then from the known Magnitudes of the Globes, as also from their Celerities where they fall on each other, you must find the Celerities they shall have immediately after Reflection, after the same Way as in *Probl.* XII. *p.* 80. Afterwards you must find the highest Places to which the Globes with these Celerities, if they were carried upwards, would ascend, and thence the Spaces will be known, which the Globes will describe by falling in any given Times after Reflection, as also the Difference of the Spaces; and reciprocally from that Difference assumed, you may go back Analytically to the Spaces described in falling.

As if the upper Globe falls upon the lower one at the Point π, [See *Figure* 85.] and after Reflection, the Celerity of the upper one downwards be so great, as if it were upwards, it would cause that Globe to ascend through the Space πN; and the Celerity of the lower one downwards was so great, as that, if it were upwards, it would cause the lower one to ascend through the Space πM; then the Times wherein the upper Globe would reciprocally descend through the Spaces $N\pi$, NG, and the inferior one through the Spaces $M\pi$, MH, would be as $\sqrt{N\pi}$, $\sqrt{NG}$, $\sqrt{M\pi}$, $\sqrt{MH}$; and consequently the Times wherein the upper Globe would run the Space πG, and the lower one πH, would be as $\sqrt{NG} - \sqrt{N\pi}$, to $\sqrt{MH} - \sqrt{M\pi}$. Make those Times equal, and $\sqrt{NG} - \sqrt{N\pi}$ will be $= \sqrt{MH} - \sqrt{M\pi}$. And, moreover, since there is given the Distance GH, put $\pi G + GH = \pi H$. And by the Reduction of these two Equations, the Problem will be solved. As if $M\pi$ is $= a$,

$N\pi = b$, $GH = c$, $\pi G = x$, you will have, according to the latter Equation, $x + c = \pi H$. Add $M\pi$, you will have $MH = a + c + x$. To πG add $N\pi$, and you will have $NG = b + x$. Which being found, according to the former Equation, $\sqrt{b+x} - \sqrt{b}$ will be $= \sqrt{a+c+x} - \sqrt{a}$. Write e for $a + c$, and $\sqrt{f}$ for $\sqrt{a} + \sqrt{b}$, and the Equation will become $\sqrt{b+x} = \sqrt{e+x} - \sqrt{f}$. And the Parts being squared $b + x = e + x + f - 2\sqrt{ef+fx}$, or $e + f - b = 2\sqrt{ef+fx}$. For $e + f - b$ write g, and you will have $g = 2\sqrt{ef+fx}$, and the Parts being squared, $gg = 4ef + 4fx$, and by Reduction $\frac{gg}{4f} - e = x$.

PROBLEM LIV.

If there are two Globes, A, B, whereof the upper one A falling from the Height G, strikes upon another lower one B rebounding from the Ground H upwards; and these Globes so part from one another by Reflection, that the Globe A returns by Force of that Reflection to its former Altitude G, and that in the same Time that the lower Globe B returns to the Ground H; then the Globe A falls again, and strikes again upon the Globe B, rebounding again back from the Ground; and after this rate the Globes always rebound from one another and return to the same Place: From the given Magnitude of the Globes, the Position of the Ground, and the Place G from whence the upper Globe falls, to find the Place where the Globes shall strike upon each other. [*See* Figure 86.]

Let e be the Center of the Globe A, and f the Center of the Globe B, d the Center of the Place G wherein the upper Globe is in its greatest Height, g the Center of the Place of the lower Globe where it falls on the Ground, a the Semi-Diameter of the Globe A, b the Semi-Diameter of the Globe B, c the Point of Contact of the Globes falling upon one another, and H the Point of Contact of the lower Globe and the Ground. And the Celerity of the Globe A, where it falls on the Globe B, will be the same which is generated by the Fall of the Globe from the Height de, and consequently is as $\sqrt{de}$. With this same Celerity the Globe A ought to be reflected upwards, that it may return to its former Place G. And the Globe B ought to be reflected

reflected with the same Celerity downwards wherewith it ascended, that it may return in the same Time to the Ground it took up in mounting from it. And that both these may come to pass, the Motion of the Globes in reflecting ought to be equal. But the Motions are compounded of the Celerities and Magnitudes together, and consequently the Product of the Bulk and Celerity of one Globe will be equal to the Product of the Bulk and Celerity of the other. Whence, if the Product of the Bulk and Celerity of one Globe be divided by the Bulk of the other Globe, you will have the Celerity of the other just before and after Reflection, or at the End of the Ascent, and at the Beginning of the Descent. Therefore this Celerity will be as $\frac{A\sqrt{de}}{B}$, or since the Globes are as the Cubes of the Radii as $\frac{a^3\sqrt{de}}{b^3}$. But as the Square of this Celerity to the Square of the Celerity of the Globe A just before Reflection, so is the Height to which the Globe B would ascend with this Celerity, if it was not hindered by meeting the Globe A falling upon it, to the Height ed from which the Globe B descends. That is, as $\frac{Aq}{Bq}de$ to de, or as Aq to Bq, or a^6 to b^6, so that first Height to x, if you put x for the latter Height cd. Therefore this Height, *viz.* to which B would ascend, if it was not hindered, is $\frac{a^6}{b^6}x$. Let that be fK. To fK add fg, or $dH - de - ef - gH$; that is, $p - x$, if for the given $dH - ef - gh$ you write p, and x for the unknown de; and you will have $Kg = \frac{a^6}{b^6}x + p - x$. Whence the Celerity of the Globe B, when it falls from K to the Ground, that is when it falls through the Space Kg, which its Center would describe in falling, will be as $\sqrt{\frac{a^6}{b^6}x + p - x}$. But that Globe falls from the Place Bcf to the Ground in the same Time that the upper Globe A ascends from the Place Ace to its greatest Height d, or on the other Hand falls from d to the Place Ace; and therefore since the Celerities of falling Bodies are

are equally augmented in equal Times, the Celerity of the Globe B, by falling to the Ground, will be augmented as much as is the whole Celerity which the Globe A acquires by falling in the ſame Time from d to e, or loſes by aſcending from e to d. Therefore, to the Celerity which the Globe B has in the Place Bcf, add the Celerity which the Globe A has in the Place Ace, and the Sum, which is as $\sqrt{de} + \frac{a^3\sqrt{de}}{b^3}$, or $\sqrt{x} + \frac{a^3}{b^3}\sqrt{x}$, will be the Celerity of the Globe B when it falls on the Ground. Therefore $\sqrt{x} + \frac{a^3}{b^3}\sqrt{x}$ will be equal to $\sqrt{\frac{a^6}{b^6}x + p - x}$. For $\frac{a^3 + b^3}{b^3}$ write $\frac{r}{s}$, and for $\frac{a^6 - b^6}{b^6}$, $\frac{rt}{ss}$, and that *Equation* will become $\frac{r}{s} \times \sqrt{x} = \sqrt{\frac{rt}{ss}x + p}$, and the Parts being ſquared, $\frac{rr}{ss}x = \frac{rt}{ss}x + p$. Subtract from both Sides $\frac{rt}{ss}x$, multiply all into ss, and divide by $rr - rt$, and there will ariſe $x = \frac{ssp}{rr - rt}$. Which Equation would have come out more ſimple, if I had taken $\frac{p}{s}$ for $\frac{a^3 + b^3}{b^3}$, for there would have come out $\frac{ss}{p - t} = x$. Whence making that $p - t$ ſhall be to s as s to x, you will have x, or ed; to which if you add ec, you will have dc, and the Point c, in which the Globes ſhall fall upon one another. Q. E. F.

PROBLEM LV.

Three Staves being erected, or set up an End, in some certain Part of the Earth perpendicular to the Plane of the Horizon, in the Points A, B, *and* C, *whereof that which is in* A *is six Foot long, that in* B *eighteen, and that in* C *eight, the Line* A B *being thirty Foot long; it happens on a certain Day in the Year that the End of the Shadow of the Staff* A *passes through the Points* B *and* C, *and of the Staff* B *through* A *and* C, *and of the Staff* C *through the Point* A. *To find the Sun's Declination, and the Elevation of the Pole, or the Day and Place where this shall happen.* [*See* Figure 61.]

Because the Shadow of each Staff describes a Conick Section, or the Section of a luminous Cone, whose Vertex is the Top of the Staff; I will feign B C D E F to be such a Curve, (whether it be an Hyperbola, Parabola, or Ellipse) as the Shadow of the Staff A describes that Day, by putting A D, A E, A F, to have been its Shadows, when B C, B A, C A, were respectively the Shadows of the Staves B and C. And besides I will suppose P A Q to be the Meridional Line, or the Axis of this Curve, to which the Perpendiculars B M, C H, D K, E N, and F L, being let fall, are Ordinates. And I will denote these Ordinates indefinitely by the Letter y, and the intercepted Parts of the Axis A M, A H, A K, A N, and A L by the Letter x. I will suppose, lastly, the Equation $aa \perp bx \perp cxx = yy$, to express the Relation of x and y, (*i. e.* the Nature of the Curve) assuming aa, b and c, as known Quantities, as they will be found to be from the Analysis. Where I made the unknown Quantities of two Dimensions only because the Equation is to express a Conick Section: and I omitted the odd Dimensions of y, because it is an Ordinate to the Axis. And I denoted the Signs of b and c, as being indeterminate by the Note $\perp$, which I use indifferently for $+$ or $-$, and its opposite $\top$ for the contrary. But I made the Sign of the Square aa Affirmative, because the concave Part of the Curve necessarily contains the Staff A, projecting its Shadows to the opposite Parts (C and F, D and E); and therefore if at the Point A you erect the Perpendicular

pendicular $A\beta$, this will some where meet the Curve, suppose in β, that is, the Ordinate y, where x is nothing, will still be real. From thence it follows that its Square, which in that Case is aa, will be Affirmative.

It is manifest therefore, that this fictitious Equation $aa \perp bx \perp cxx = yy$, as it is not filled with superfluous Terms, so neither is it more restrained than what is capable of satisfying all the Conditions of this Problem, and will denote the Hyperbola, Ellipse, or Parabola, according as the Values of aa, b, c, shall be determined, or perhaps found to be nothing. But what may be their Values, and with what Signs b and c are to be affected, and thence what Sort of a Curve this may be, will be manifest from the following Analysis.

The former Part of the Analysis.

Since the Shadows are as the Altitudes of the Staves, you will have $BC : AD :: AB : AE$ $(:: 18 : 6) :: 3 : 1$. Also $CA : AF$ $(:: 8 : 6) :: 4 : 3$. Wherefore naming $AM = r$, $MB = s$, $AH = t$, and $HC = \perp v$. From the Similitude of the Triangles AMB, ANE, and AHC, ALF, AN will be $= -\frac{r}{3}$. $NE = -\frac{s}{3}$. $AL = -\frac{3t}{4}$, and $LF = \top\frac{3v}{4}$; whose Signs I put contrary to the Signs of AM, MB, AH, HC, because they tend contrary Ways with respect to the Point A from which they are drawn, or to the Axis PQ on which they stand. Now these being respectively written for x and y in the fictitious Equation $aa \perp bx \perp cxx = yy$.

r and s will give $aa \perp br \perp crr = ss$.

$-\frac{r}{3}$ and $-\frac{s}{3}$ will give $aa \top \frac{br}{3} \perp \frac{1}{9}crr = \frac{1}{9}ss$.

t and $\perp v$ will give $aa \perp bt \perp ctt = vv$.

$-\frac{3}{4}t$ and $\top\frac{3}{4}v$ will give $aa \top \frac{3}{4}bt \perp \frac{9}{16}ctt = \frac{9}{16}vv$.

Now, by exterminating ss from the first and second Equations, in order to obtain r, there comes out $\frac{2aa}{\perp b} = r$. Whence it is manifest, that $\perp b$ is Affirmative. Also by exterminating vv from the third and fourth, to obtain t, there comes

comes out $\frac{aa}{3b}=t$. And having writ $\frac{2aa}{b}$ for r in the first, and $\frac{aa}{3b}$ for t in the third, there arise $3aa \perp \frac{4a^4c}{bb}=ss$, and $\frac{4}{3}aa \perp \frac{a^4c}{9bb}=vv$.

Moreover, having let fall Bλ perpendicular upon CH, BC will be : AD (: : 3 : 1) : : Bλ : AK : : Cλ : DK. Wherefore, since Bλ is ($=$ AM $-$ AH $= r - t$) $= \frac{5aa}{3b}$, AK will be $= \frac{5aa}{9b}$, or rather $= -\frac{5aa}{9b}$. Also since Cλ is ($=$ CH $\perp$ BM $= v \perp s$) $= \sqrt{\frac{4aa}{3} \perp \frac{a^4c}{9bb}} \perp \sqrt{3aa \perp \frac{4a^4c}{bb}}$, it will be DK ($=\frac{1}{3}$Cλ) $= \sqrt{\frac{4aa}{27} \perp \frac{a^4c}{81bb}} \perp \sqrt{\frac{1}{3}aa \perp \frac{4a^4c}{9bb}}$. Which being respectively written in the Equation $aa + bx \perp cxx = yy$, for AK and DK, or x and y, there comes out $\frac{4aa}{9} \perp \frac{25a^4c}{81bb} = \frac{13}{27}aa \perp \frac{37a^4c}{81bb} \perp 2\sqrt{\frac{4aa}{27} \perp \frac{a^4c}{81bb}} \times \sqrt{\frac{aa}{3} \perp \frac{4a^4c}{9bb}}$. And by Reduction $-bb \top 4aac = \perp 2\sqrt{36b \perp 51aabbc + 4a^4cc}$; and the Parts being squared, and again reduced, there comes out $0 = 143b^4 \perp 196aabbc$, or $\frac{-143bb}{196aa} = \perp c$. Whence it is manifest, that $\perp c$ is Negative, and consequently the fictitious Equation $aa \perp bx \perp cxx = yy$ will be of this Form, $aa + bx - cxx = yy$. And therefore the Curve, which it denotes, is an Ellipsis; whose Center and two Axes are thus found.

Making $y=0$, as happens in the Vertex's of the Figure P and Q, you will have $aa + bx = cxx$, and having extracted

tracted the Root $x = \frac{b}{2c} \pm \sqrt{\frac{bb}{4cc} + \frac{aa}{c}} = \left\{ \begin{matrix} AQ \\ AP \end{matrix} \right\}$.

And consequently, taking $AV = \frac{b}{2c}$, V will be the Center of the Ellipse, and VQ, or VP $\left(\sqrt{\frac{bb}{4cc} + \frac{aa}{c}} \right)$ the greatest Semi-Axis. If, moreover, the Value of AV, or $\frac{b}{2c}$, be put for x in the Equation $aa + bx - cxx = yy$, there will come out $aa + \frac{bb}{4c} = yy$. Wherefore $aa + \frac{bb}{4c}$ will be $= VZq$, that is, to the Square of the least Semi-Axis. Lastly, in the Values of AV, VQ, and VZ already found, writing $\frac{143\,bb}{196\,aa}$ for c, there come out $\frac{98\,aa}{143\,b} = AV$, $\frac{112\,aa\sqrt{3}}{143\,b} = VQ$, and $\frac{8a\sqrt{3}}{\sqrt{143}} = VZ$.

The other Part of the Analysis. [*See* Figure 62.]

Suppose now the Staff AR standing on the Point A, and RPQ will be the Meridional Plane, and RPZQ the luminous Cone whose Vertex is R. Let moreover TXZ be a Plane cutting the Horizon in VZ, and the Meridional Plane in TVX, which Section let be perpendicular to the Axis of the World, or of the Cone, and the Plane TXZ will be perpendicular to the same Axis, and will cut the Cone in the Periphery of the Circle TZX, which will be every where at an equal Distance, as RX, RZ, RT, from its Vertex. Wherefore, if PS be drawn parallel to TX, you will have $RS = RP$, by reason of the equal Quantities RX, RT; and also $SX = XQ$, by reason of the equal Quantities PV, VQ; whence RX or $RZ = \left(\frac{RS + RQ}{2}\right)$ is $= \frac{RP + RQ}{2}$. Lastly, draw RV, and since VZ perpendicularly stands on the Plane RPQ, (as being the Section of the Planes perpendicularly standing on the same Plane) the Triangle RVZ will be right-angled at V.

Now

Now making RA $= d$, AV $= e$, VP or VQ $= f$, and VZ $= g$, you will have AP $= f - e$, and RP $= \sqrt{ff - 2ef + ee + dd}$. Alſo AQ $= f + e$, and RQ $= \sqrt{ff + 2ef + ee + dd}$; and conſequently RZ $\left(= \frac{RP + RQ}{2}\right) = \frac{\sqrt{ff - 2ef + ee + dd} + \sqrt{ff + 2ef + ee + dd}}{2}$.

Whoſe Square $\frac{dd + ee + ff}{2} + \frac{1}{2}\sqrt{f^4 - 2eeff + e^4 + 2ddff + 2ddee + d^4}$ is equal (RVq + VZq = RAq + AVq + VZq =) $dd + ee + gg$. Now having reduced, it is $\sqrt{f^4 - 2eeff + e^4 + 2ddff + 2ddee + d^4} = dd + ee - ff + 2gg$, and the Parts being ſquared and reduced into Order, $ddff = ddgg + eegg - ffgg + g^4$, or $\frac{ddff}{gg} = dd + ee - ff + gg$. Laſtly, 6, $\frac{98aa}{143b}$, $\frac{112aa\sqrt{3}}{143b}$, and $\frac{8a\sqrt{3}}{\sqrt{143}}$ (the Values of AR, AV, VQ, and VZ) being reſtored for d, e, f, and g, there ariſes $36 - \frac{196a^4}{143bb} + \frac{192aa}{143} = \frac{36 \times 14 \times 14aa}{143bb}$, and thence by Reduction $\frac{49a^4 + 36 \times 49aa}{48aa + 1287} = bb$.

In the firſt Scheme AMq + MBq is = ABq, that is, $rr + ss = 33 \times 33$. But r was $= \frac{2aa}{b}$, and $ss = 3aa - \frac{4a^4c}{bb}$, whence $rr = \frac{4a^4}{bb}$, and (ſubſtituting $\frac{143bb}{196aa}$ for c) $ss = \frac{4aa}{49}$. Wherefore $\frac{4a^4}{bb} + \frac{4aa}{49} = 33 \times 33$, and thence by Reduction there again reſults $\frac{4 \times 49a^4}{53361 - 4aa} = bb$. Putting therefore an Equality between the two Values of bb, and dividing each Part of the Equation by 49, you will have

$a^4 +$

$\frac{a^4 + 36aa}{48aa + 1287} = \frac{4a^4}{53361 - 4aa}$; whose Parts being multiplied cross-ways, ordered and divided by 49, there comes out $4a^4 = 981\,aa + 39204$, whose Root aa is $\frac{981 + \sqrt{1589625}}{8} = 280,2254144$.

Above was found $\frac{4 \times 49\,a^4}{53361 - 4aa} = bb$, or $\frac{14aa}{\sqrt{53361 - 4aa}} = b$. Whence A V $\left(\frac{98aa}{143b}\right)$ is $\frac{7\sqrt{53361 - 4aa}}{143}$, and V P or V Q $\left(\frac{112aa\sqrt{3}}{143b}\right)$ is $\frac{8}{143}\sqrt{160083 - 12aa}$. That is, by substituting 280,2254144 for aa, and reducing the Terms into Decimals, A V = 11,188297, and V P or V Q = 22,147085; and consequently A P (P V − A V) = 10,958788, and A Q (A V + V Q) 33,335382.

Lastly, if $\frac{1}{6}$ A R or 1 be made Radius, $\frac{1}{6}$ A Q or 5,555897 will be the Tangent of the Angle A R Q of 79 *gr*. 47′. 48″. and $\frac{1}{6}$ A P or 1,826465 the Tangent of the Angle A R P of 61 *gr*. 17′. 57″. half the Sum of which Angles 70 *gr*. 32′. 52″. is the Complement of the Sun's Declination; and the Semi-difference 9 *gr*. 14′. 56″. the Complement of the Latitude of the Place. Therefore, the Sun's Declination was 19 *gr*. 27′. 10″. and the Latitude of the Place 80 *gr*. 45′. 4″. which were to be found.

PROBLEM LVI.

From the Observation of four Places of a Comet, moving with an uniform right-lined Motion through the Heaven, to determine its Distance from the Earth, and Direction and Velocity of its Motion, according to the Copernican *Hypothesis.* [*See* Figure 73.]

If from the Center of the Comet in the four Places observed, you let fall so many Perpendiculars to the Plane of the Ecliptick; and A, B, C, D, be the Points in that Plane on which the Perpendiculars fall; through those Points draw the right Line A D, and this will be cut by the Perpendiculars in the same Ratio with the Line which the Comet describes

scribes by its Motion; that is, so that A B shall be to A C as the Time between the first and second Observation to the Time between the first and third; and A B to A D as the Time between the first and second to the Time between the first and fourth. From the Observations therefore there are given the Proportions of the Lines A B, A C, A D, to one another.

Moreover, let the Sun S be in the same Plane of the Ecliptick, and E H an Arch of the Ecliptical Line in which the Earth moves; E, F, G, H, four Places of the Earth at the Times of the Observations, E the first Place, F the second, G the third, H the fourth. Join A E, B F, C G, D H, and let them be produced until the three latter cut the former in I, K, and L, *viz.* B F in I, C G in K, D H in L. And the Angles A I B, A K C, A L D will be the Differences of the observed Longitudes of the Comet; A I B the Difference of the Longitudes of the first and second Place of the Comet; A K C the Difference of the Longitudes of the first and third Place, and A L D the Difference of the Longitudes of the first and fourth Place. There are given therefore from the Observations the Angles A I B, A K C, A L D.

Join S E, S F, E F; and by reason of the given Points S, E, F, and the given Angle E S F, there will be given the Angle S E F. There is given also the Angle S E A, as being the Difference of Longitude of the Comet and Sun in the Time of the first Observation. Wherefore, if you add its Complement to two right Angles, *viz.* the Angle S E I to the Angle S E F, there will be given the Angle I E F. Therefore there are given the Angles of the Triangle I E F, together with the Side E F, and consequently there is given the Side I F. And by a like Argument there are given K E and L E. There are given therefore in Position the four Lines A I, B I, C K, D L, and consequently the Problem comes to this, that four Lines being given in Position, we may find a fifth, which shall be cut by these four in a given Ratio.

Having let fall to A I the Perpendiculars B M, C N, D O, by reason of the given Angle A I B there is given the Ratio of B M to M I. But B M to C N is in the given Ratio of B A and C A, and by reason of the given Angle C K N there is given the Ratio of C N to K N. Wherefore, there is also given the Ratio of B M to K N; and thence also the Ratio of B M to M I — K N, that is, to M N + I K. Take P to I K as is A B to B C, and since M A is to M N in in the ſame Ratio, P + M A will be to I K + M N in the ſame Ratio, that is, in a given Ratio. Wherefore, there is given the Ratio of B M to P + M A. And by a like Argument, if Q be taken to I L in the Ratio of A B to B D, there will be given the Ratio of B M to Q + M A. And therefore the Ratio of B M to the Difference of P + M A and Q + M A will be alſo given. But that Difference, *viz.* P — Q, or Q — P is given, and therefore there will be given B M. But B M being given, there are alſo given P + M A and M I, and thence, M A, M E, A E, and the Angle E A B.

These being found, erect at A a Line perpendicular to the Plane of the Ecliptick, which ſhall be to the Line E A as the Tangent of the Comet's Latitude in the firſt Obſervation to Radius, and the End of that Perpendicular will be the Place of the Comet's Center in the firſt Obſervation. Whence the Diſtance of the Comet from the Earth is given in the Time of that Obſervation. And after the ſame Manner, if from the Point B you erect a Perpendicular which ſhall be to the Line B F as the Tangent of the Comet's Latitude in the ſecond Obſervation to Radius, you will have the Place of the Comet's Center in that ſecond Obſervation. And a Line drawn from the firſt Place to the ſecond, is that in which the Comet moves through the Heaven.

PROBLEM LVII.

If the given Angle C A D *move about the angular Point* A *given in Poſition, and the given Angle* C B D *about the angular Point* B *given alſo in Poſition, on this Condition, that the Legs* A D, B D, *ſhall always cut one another in the right Line* E F *given likewiſe in Poſition; to find the Curve, which the Interſection* C *of the other Legs* A C, B C, *deſcribes.* [*See* Figure 74.]

Produce C A to *d*, ſo that A *d* ſhall be = A D, and produce C B to *δ*, ſo that B *δ* ſhall be = to B D. Make the Angle A *d e* equal to the Angle A D E, and the Angle B *δ f* equal to the Angle B D F, and produce A B on both Sides until it meet *d e* and *δ f* in *e* and *f*. Produce alſo *e d* to G, that *d* G ſhall be = *δ f*, and from the Point C to the

the Line A B draw C H parallel to ed, and C K parallel to $f\delta$. And conceiving the Lines eG, $f\delta$ to remain immoveable while the Angles C A D, C B D, move by the aforesaid Law about the Poles A and B, G d will always be equal to $f\delta$, and the Triangle C H K will be given in Specie. Make therefore $Ae = a$, $eG = b$, $Bf = c$, $AB = m$, $BK = x$, and $CK = y$. And B K will be : C K :: $Bf : f\delta$. Therefore $f\delta = \frac{cy}{x} = Gd$. Take this from G e, and there will remain $ed = b - \frac{cy}{x}$. Since the Triangle C K H is given in Specie, make $CK : CH :: d : e$, and $CH : HK :: e : f$, and C H will be $= \frac{ey}{d}$, and $HK = \frac{fy}{d}$. And consequently $AH = m - x - \frac{fy}{d}$. But $AH : HC :: Ae : ed$, that is, $m - x - \frac{f}{d}y : \frac{ey}{d} :: a : b - \frac{cy}{x}$. Therefore, by multiplying the Means and Extreams together, there will be made $mb - \frac{mcy}{x} - bx + cy - \frac{bf}{d}y + \frac{cfyy}{dx} = \frac{aey}{d}$. Multiply all the Terms by dx, and reduce them into Order, and there will come out $fcyy \begin{matrix} +dc \\ -ae \\ -fb \end{matrix} xy - dcmy - bdxx + bdmx = 0$. Where, since the unknown Quantities x and y ascend only to two Dimensions, it is evident, that the Curve Line that the Point C describes is a Conick Section. Make $\frac{ae + fb - dc}{c} = 2p$, and there will come out $yy = \frac{2p}{f}xy + \frac{dm}{f}y + \frac{bd}{fc}xx - \frac{bdm}{fc}x$. And the Square Root being extracted, $y = \frac{p}{f}x + \frac{dm}{2f} \pm$

Tab: V: p: 164.

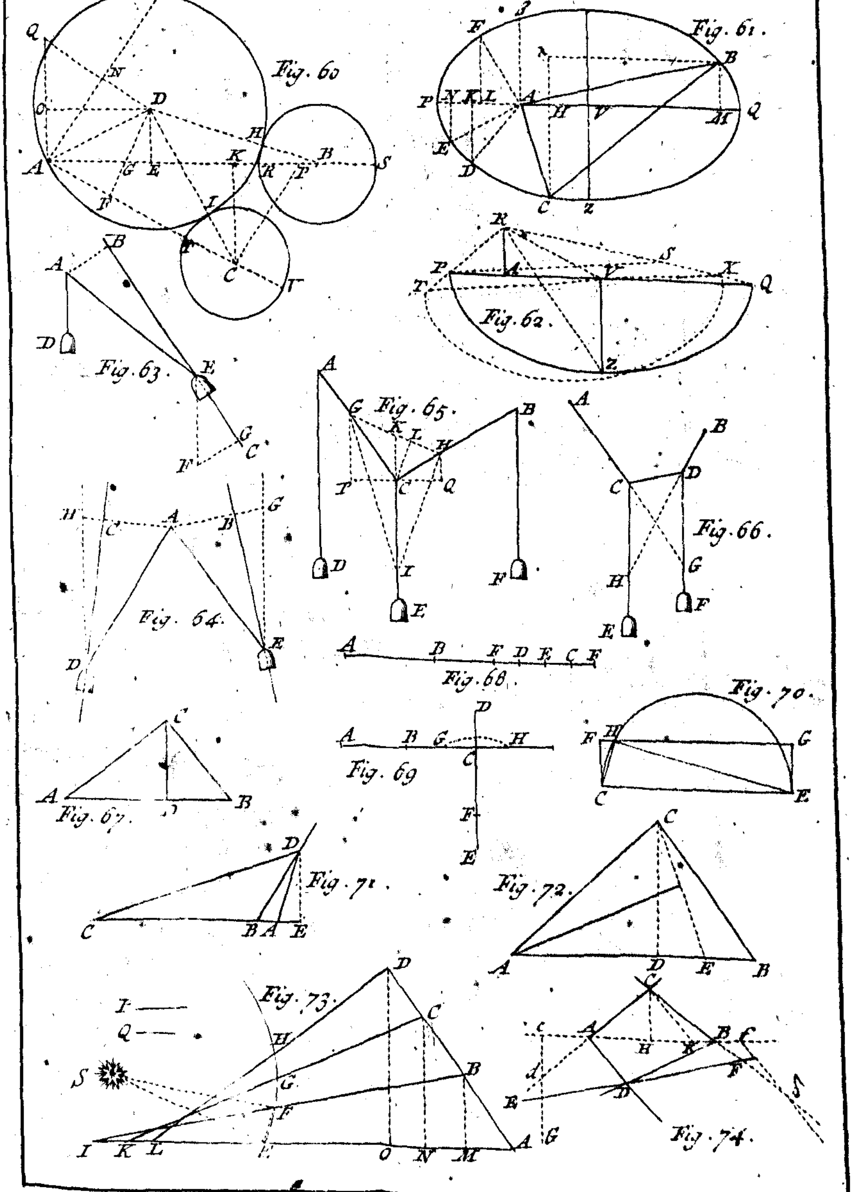


Fig. 60
Fig. 61
Fig. 62
Fig. 63
Fig. 64
Fig. 65
Fig. 66
Fig. 67
Fig. 68
Fig. 69
Fig. 70
Fig. 71
Fig. 72
Fig. 73
Fig. 74

$$\sqrt{\frac{pp}{ff}xx + \frac{bd}{fc}xx + \frac{pdm}{ff}x - \frac{bdm}{fc}x + \frac{ddmm}{4ff}}.$$

Whence we infer, that the Curve is an Hyperbola, if $\frac{bd}{fc}$ be Affirmative, or Negative and leſs than $\frac{pp}{ff}$; and a Parabola, if $\frac{bd}{fc}$ be Negative and equal to $\frac{pp}{ff}$; an Ellipſe or a Circle, if $\frac{bd}{fc}$ be both Negative and greater than $\frac{pp}{ff}$.
Q. E. I.

PROBLEM LVIII.

To deſcribe a Parabola which ſhall paſs through four Points given. [*See* Figure 75.]

Let thoſe given Points be A, B, C, D. Join AB, and biſect it in E. And through E draw VE a right Line, which conceive to be the Diameter of a Parabola, the Point V being its Vertex. Join AC, and draw DG parallel to AB, and meeting AC in G. Make $AB = a$, $AC = b$, $AG = c$, $GD = d$. Upon AC take AP of any Length, and from P draw PQ parallel to AB, and conceiving Q to be a Point of the Parabola; make $AP = x$, $PQ = y$. And take any Equation expreſſive of a Parabola, which may determine the Relation between AP and PQ. As that y is $= e + fx \pm \sqrt{gg + hx}$.

Now if AP or x be put $= 0$, the Point P falling upon A, PQ or y will be $= 0$, as alſo $= -$ AB. And by writing in the aſſumed Equation 0 for x, you will have $y = e \pm \sqrt{gg}$, that is $= e \pm g$. The greater of which Values of y, $e + g$ is $= 0$, the leſſer $e - g = -$ AB, or to $-a$. Therefore $e = -g$, and $e - g$, that is, $-2g = -a$, or $g = \frac{1}{2}a$. And ſo in room of the aſſumed Equation you will have this $y = -\frac{1}{2}a + fx \pm \sqrt{\frac{1}{4}aa + hx}$.

Moreover, if AP or x be made $=$ AC, ſo that the Point P falls upon C, you will have again $PQ = 0$. For x therefore in the laſt Equation write AC or b, and for y write 0; and you will have $0 = -\frac{1}{2}a + fb + \sqrt{\frac{1}{4}aa + hb}$, or $\frac{1}{2}a - fb$

$-fb = \sqrt{\tfrac{1}{4}aa + bb}$; and the Parts being squared $-afb + ffbb = bb$, or $ffb - fa = b$. And so, in room of the assumed Equation, there will be had this, $y = -\tfrac{1}{2}a + fx \pm \sqrt{\tfrac{1}{4}aa + ffbx - fax}$.

Moreover, if AP or x be made $=$ AG or c, PQ or y will be $= -$ GD or $-d$. Wherefore, for x and y in the last Equation write c and $-d$, and you will have $-d = -\tfrac{1}{2}a + fc - \sqrt{\tfrac{1}{4}aa + ffbc - fac}$, or $\tfrac{1}{2}a - d - fc = \sqrt{\tfrac{1}{4}aa + ffbc - fac}$. And the Parts being squared, $-ad - fac + dd + 2dcf + ccff = ffbc - fac$. And the Equation being ordered and reduced, $ff = \frac{2d}{b-c}f + \frac{dd-ad}{bc-cc}$. For $b-c$, that is, for GC write k, and that Equation will become $ff = \frac{2d}{k}f + \frac{dd-ad}{kc}$. And the Root being extracted, $f = \frac{d}{k} \pm \sqrt{\frac{ddc + ddk - adk}{kkc}}$. But f being found, the Parabolick Equation, *viz.* $y = -\tfrac{1}{2}a + fx \pm \sqrt{\tfrac{1}{4}aa + ffbx - fax}$ will be fully determined; by whose Construction therefore the Parabola will also be determined. The Construction is thus: Draw CH parallel to BD meeting DG in H. Between DG and DH take a mean Proportional DK, and draw EI parallel to CK, bisecting AB in E, and meeting DG in I. Then produce IE to V, so that EV shall be to EI :: EBq : DIq — EBq, and V will be the Vertex, VE the Diameter, and $\frac{BEq}{VE}$ the *Latus Rectum* of the Parabola sought.

Problem LIX.

To describe a Conick Section through five Points given. [*See* Figure 76.]

Let those Points be A, B, C, D, E. Join AC, BE, cutting one another in H. Draw DI parallel to BE, and meeting AC in I. As also EK parallel to AC, and meeting DI produced in K. Produce ID to F, and EK to G; so that AHC shall be : BHE :: AIC : FID :: EKG : FKD, and the Points F and G will be in a Conick Section, as is known. But

But you ought to observe this, if the Point H falls between all the Points A, C, and B E, or without them all, the Point I must either fall between all the Points A, C, and F, D, or without them all; and the Point K between all the Points D, F, and E, G, or without them all. But if the Point H falls between the two Points A, C, and without the other two B, E, or between those two B, E, and without the other two A, C, the Point I ought to fall between two of the Points A, C and F, D, and without the other two of them; and in like Manner, the Point K ought to fall between two of the Points D, F, and E, G, and without Side of the two other of them; which will be done by taking I F, K G, on this or that Side of the Points I, K, according to the Exigency of the Problem. Having found the Points F and G, bisect A C and E G in N and O; also B E, F D in L and M. Join N O, L M, cutting one another in R; and L M and N O will be the Diameters of the Conick Section, R its Center, and B L, F M, Ordinates to the Diameter L M. Produce L M on both Sides, if there be Occasion, to P and Q, so that B L q shall be to F M q : : P L Q : P M Q, and P and Q will be the Vertex's of the Conick Section, and P Q the *Latus Transversum.* Make P L Q : L B q : : P Q : T, and T will be the *Latus Rectum.* Which being known, the Figure is known.

It remains only that we may shew how L M is to be produced each Way to P and Q, so that B L q may be : F M q : : P L Q : P M Q, *viz.* P L Q, or P L × L Q, is $\overline{PR - LR} \times \overline{PR + LR}$; for P L is P R — L R, and L Q is R Q + L R, or P R + L R. Moreover, $\overline{PR - LR} \times \overline{PR + LR}$, by multiplying, becomes P R q — L R q. And after the same Manner, P M Q is $\overline{PR + RM} \times \overline{PR - RM}$, or P R q — R M q. Therefore B L q : F M q : : P R q — L R q : P R q — R M q; and by dividing, B L q — F M q : F M q : : R M q — L R q : P R q — R M q. Wherefore since there are given B L q — F M q, F M q and R M q — L R q, there will be given P R q — R M q. Add the given Quantity R M q, and there will be given the Sum P R q, and consequently its Root P R, to which Q R is equal.

PRO-

PROBLEM LX.

To describe a Conick Section which shall pass through four given Points, and in one of those Points shall touch a right Line given in Position. [*See* Figure 77.]

Let the four given Points be A, B, C, D, and the right Line given in Position be AE, which let the Conick Section touch in the Point A. Join any two Points D, C, and let DC produced, if there be Occasion for it, meet the Tangent in E. Through the fourth Point B draw BF parallel to DC, which may meet the same Tangent in F. Also draw DI parallel to the Tangent, and which may meet BF in I. Upon FB, DI, produced if there be Occasion, take FG, HI, of such Length as that it may be $AEq : CED :: AFq : BFG :: DIH : BIG$. And the Points G and H will be in a Conick Section as is known; provided you take FG, IH, on the proper Sides of the Points F and I, according to the Rule delivered in the foregoing Problem. Bisect BG, DC, DH, in K, L, and M. Join KL, AM, cutting one another in O, and O will be the Center, A the Vertex, and HM an Ordinate to the Semi-Diameter AO; which being known, the Figure is known.

PROBLEM LXI.

To describe a Conick Section which shall pass through three given Points, and touch right Lines given in Position in two of those Points. [*See* Figure 78.]

Let those given Points be A, B, C, the Tangents AD, BD, to the Points A and B, and let D be the common Intersection of those Tangents. Bisect AB in E. Draw DE, and produce it till in F it meets CF drawn parallel to AB; and DF will be the Diameter, and AE and CF the Ordinates to that Diameter. Produce DF to O, and on DO take OV a mean Proportional between DO and EO, on this Condition, that also $AEq : CFq :: VE \times \overline{VO+OE} : VF \times \overline{VO+OF}$; and V will be the Vertex, and O the Center of the Figure. Which being known, the Figure will also be known. But VE is $= VO - OE$, and consequently $VE \times \overline{VO+OE} = \overline{VO-OE} \times \overline{VO+OE} = VOq - OEq$. Besides, Because VO is a mean Proportional

onal between DO and EO, VOq will be $= DOE$, and consequently $VOq - OEq = DOE - OEq = DEO$. And by a like Argument you will have $VF \times \overline{VO + OF} = VOq - OFq = DOE - OFq$. Therefore $AEq : CFq :: DEO : DOE - OFq$. OFq is $= EOq - 2FEO + FEq$. And consequently $DOE - OFq = DOE - OEq + 2FEO - FEq = DEO + 2FEO - FEq$. And $AEq : CFq :: DEO : DEO + 2FEO - FEq :: DE : DE + 2FE - \frac{FEq}{EO}$. Therefore there is given $DE + 2FE - \frac{FEq}{EO}$. Take away from this given Quantity $DE + 2FE$, and there will remain $\frac{FEq}{EO}$ given. Call that N; and $\frac{FEq}{N}$ will be $= EO$, and consequently EO will be given. But EO being given, there is also given VO, the mean Proportional between DO and EO.

After this Way, by some of *Apollonius*'s Theorems, these Problems are expeditiously enough solved; which yet may be solved by Algebra alone without those Theorems. As if the first of the last three Problems be proposed: [See *Figure* 78.] Let the five given Points be A, B, C, D, E, through which the Conick Section is to pass. Join any two of them, A, C, and any other two, B, E, by right Lines intersecting one another in H. Draw DI parallel to BE meeting AC in I; as also any other right Line KL meeting AC in K, and the Conick Section in L. And imagine the Conick Section to be given, so that the Point K being known, there will at the same Time be known the Point L; and making $AK = x$, and $KL = y$, to express the Relation between x and y, assume any Equation which generally expresses the Conick Sections; suppose this, $a + bx + cxx + dy + exy + yy = 0$. Wherein a, b, c, d, e, denote determinate Quantities with their Signs, but x and y indeterminate Quantities. Now if we can find the determinate Quantities a, b, c, d, e, the Conick Section will be known. Let us therefore feign the Point L to fall successively upon the Points A, C, B, E, D, and let us see what will follow thence. If therefore the Point L falls upon the Point A, in that Case AK and KL, that is, x and y, will be 0. Then all the Terms of the Equation besides a will vanish, and there will remain $a = 0$. Wherefore a is to be blotted out in that Equation, and the other Terms $bx + cxx + dy + exy + yy$ will be $= 0$.

Moreover if L falls upon C, AK, or x, will be $=$ AC, and LK or $y=0$. Put therefore AC$=f$, and by substituting f for x and o for y, the Equation for the Curve $bx+cxx+dy+exy+yy=0$, will become $bf+cff=0$, or $b=-cf$. And having writ in that Equation $-cf$ for b, it will become $-cfx+cxx+dy+exy+yy=0$. Farther, if the Point L falls upon the Point B, AK or x will be $=$ AH, and KL or $y=$ BH. Put therefore AH$=g$, and BH$=h$, and then write g for x and h for y, and the Equation $-cfx+cxx$, &c. will become $-cfg+cgg+dh+egh+hh=0$. But if the Point L falls upon E, AK will be $=$ AH, or $x=g$, and KL or $y=$ HE. For HE therefore write $-k$, with a Negative Sign, because HE lies on the contrary Side of the Line AC, and by substituting g for x and $-k$ for y, the Equation $-cfx+cxx$, &c. will become $-cfg+cgg-dk-egk+kk=0$. Take away this from the former Equation $-cfg+cgg+dh+egh+hh$, and there will remain $dh+egh+hh+dk+egk-kk=0$. Divide this by $h+k$, and there will come out $d+eg+h-k=0$. Take away this multiplied by h from $-cfg+cgg+dh+egh+hh=0$, and there will remain $-cfg+cgg+hk=0$, or $\frac{hk}{-gg+fg}=c$. Lastly, if the Point L falls upon the Point D, AK or x will be $=$ AI, and KL or y will be $=$ ID. Wherefore, for AI write m, and for ID, n, and likewise for x and y substitute m and n, and the Equation $-cfx+cxx$, &c will become $-cfm+cmm+dn+emn+nn=0$. Divide this by n, and there will come out $\frac{-cfm+cmm}{n}+d+em+n=0$. Take away $d+eg+h-k=0$, and there will remain $\frac{-cfm+cmm}{n}+em-eg+n-h+k=0$, or $\frac{cmm-cfm}{n}+n-h+k=eg-em$. But now by reason of the given Points A, B, C, D, E, there are given AC, AH, AI, BH, EH, DI, that is, f, g, m, h, k, n. And consequently by the Equation $\frac{hk}{fg-gg}=c$ there is given c. But c being given by the Equation $\frac{cmm-cfm}{n}+n-h+k=eg-em$ there

there is given $eg - em$. Divide this given Quantity by the given one $g - m$, and there will come out the given e. Which being found, the Equation $d + eg + h - k = 0$, or $d = k - h - eg$, will give d. And these being known, there will at the same Time be determined the Equation expressive of the Conick Section sought, *viz.* $cfx = cxx + dy + exy + yy$. And from that Equation, by the Method of *Des Cartes*, the Conick Section will be determined.

Now if the four Points A, B, C, E, and the Position of the right Line A F, which touches the Conick Section in one of those Points A were given, the Conick Section may be thus more easily determined. Having found, as above, the Equations $cfx = cxx + dy + exy + yy$, $d = k - h - eg$, and $c = \frac{hk}{fg - gg}$, conceive the Tangent A F to meet the right Line E H in F, and then the Point L to be moved along the Perimeter of the Figure C D E till it fall upon the Point A; and the ultimate Ratio of L K to A K will be the Ratio of F H to A H, as will be evident to any one that contemplates the Figure. Make F H $= p$, and in this Case where L K, A K, are in a vanishing State, you will have $p : g :: y : x$, or $\frac{gy}{p} = x$. Wherefore for x, in the Equation $cfx = cxx + dy + exy + yy$, write $\frac{gy}{p}$, and there will arise $\frac{cfgy}{p} = \frac{cggyy}{pp} + dy + \frac{egyy}{p} + yy$. Divide all by y, and there will come out $\frac{cfg}{p} = \frac{cggy}{pp} + d + \frac{cgy}{p} + y$. Now because the Point L is supposed to fall upon the Point A, and consequently K L, or y, to be infinitely small or nothing, blot out the Terms which are multiplied by y, and there will remain $\frac{cfg}{p} = d$. Wherefore make $\frac{hk}{fg - gg} = c$, then $\frac{cfg}{p} = d$. Lastly, $\frac{k - h - d}{g} = e$, and having found c, d, and e, the Equation $cfx = cxx + dy + exy + yy$ will determine the Conick Section.

 If,

If, lastly, there are only given the three Points A, B, C, together with the Position of the two right Lines A T, C T, which touch the Conick Section in two of those Points, A and C, there will be obtained, as above, this Equation expressive of a Conick Section, $cfx = cxx + dy + exy + yy$. [See *Figure* 80.] Then if you suppose the Ordinate K L to be parallel to the Tangent A T, and it be conceived to be produced, till it again meets the Conick Section in M, and that Line L M to approach to the Tangent A T till it coincides with it at A; the ultimate Ratio of the Lines K L and K M to one another, will be a Ratio of Equality, as will appear to any one that contemplates the Figure. Wherefore in that Case K L and K M being equal to each other, that is, the two Values of y, (*viz.* the Affirmative one K L, and the Negative one K M) being equal, those Terms of the Equation $cfx = cxx + dy + exy + yy$ in which y is of an odd Dimension, that is, the Terms $dy + exy$ in respect of the Term yy, wherein y is of an even Dimension, will vanish. For otherwise the two Values of y, *viz.* the Affirmative and the Negative, cannot be equal; and in that Case A K is infinitely less than L K, that is x than y, and consequently the Term exy than the Term yy. And consequently being infinitely less, may be reckoned for nothing. But the Term dy, in respect of the Term yy, will not vanish as it ought to do, but will grow so much the greater, unless d be supposed to be nothing. Therefore the Term dy is to be blotted out, and so there will remain $cfx = cxx + exy + yy$, an Equation expressive of a Conick Section. Conceive now the Tangents A T, C T, to meet one another in T, and the Point L to come to approach to the Point C, till it coincides with it. And the ultimate Ratio of K L to K C will be that of A T to A C. K L was y; A K, x; and A C, f; and consequently K C, $f - x$; make A T $= g$, and the ultimate Ratio of y to $f - x$, will be the same as of g to f. The Equation $cfx = cxx + exy + yy$, subtracting on both Sides cxx, becomes $cfx - cxx = exy + yy$, that is, $\overline{f - x}$ into $cx = y$ into $ex + y$. Therefore it is $y : f - x :: cx : ex + y$, and consequently $g : f :: cx : ex + y$. But the Point L falling upon C, y becomes nothing. Therefore $g : f :: cx : ex$. Divide the latter Ratio by x, and it will become $g : f :: c : e$, and $\frac{cf}{g} = e$. Wherefore, if in the Equation $cfx = cxx + exy$

$+exy+yy$; you write $\frac{cf}{g}$ for e, it will become $cfx=cxx$ $+\frac{cf}{g}xy+yy$, an Equation expreſſive of a Conick Section. Laſtly, draw B H parallel to K L, or A T, from the given Point B, through which the Conick Section ought to paſs, and which ſhall meet A C in H, and conceiving K L to come towards B H, until it coincides with it, in that Caſe A H will be $=x$, and B H $=y$. Call therefore the given quantity A H $=m$, and the given B H $=n$, and then for x and y, in the Equation $cfx=cxx+\frac{cf}{g}xy+yy$, write m and n, and there will ariſe $cfm=cmm+\frac{cf}{g}mn+nn$. Take away on both Sides $cmm+\frac{cf}{g}mn$, and there will come out $cfm-cmm-\frac{cf}{g}mn=nn$. Put $f-m-\frac{fn}{g}=s$, and csm will be $=nn$. Divide each Part of the Equation by sm, and there will ariſe $c=\frac{nn}{sm}$. But having found c, the Equation for the Conick Section is determined cfx $=cxx+\frac{cf}{g}xy+yy$. And then, by the Method of *Des Cartes*, the Conick Section is given, and may be deſcribed.

Hitherto I have been ſolving ſeveral Problems. For in learning the Sciences, Examples are of more Uſe than Precepts. Wherefore I have been the larger on this Head. And ſome which occurred as I was putting down the reſt, I have given their Solutions without uſing Algebra, that I might ſhew that in ſome Problems that at firſt Sight appear difficult, there is not always Occaſion for Algebra. But now it is Time to ſhew the Solution of Equations. For after a Problem is brought to an Equation, you muſt extract the Roots of that Equation, which are the Quantities that ſatisfy the Problem.

 How

How EQUATIONS are to be solved.

AFTER therefore in the Solution of a Question you are come to an Equation, and that Equation is duly reduced and ordered; when the Quantities which are denoted by Species, and which are supposed given, are really given in Numbers, those Numbers are to be substituted in their room in the Equation, and you will have a Numeral Equation, whose Root being extracted will satisfy the Question. As if in the Division of an Angle into five equal Parts, by putting r for the Radius of the Circle, q for the Chord of the Complement of the proposed Angle to two right ones, and x for the Chord of the Complement of the fifth Part of that Angle, I had come to this Equation, $x^5 - 5rrx^3 + 5r^4x - r^4q = 0$. Where in any particular Case the Radius r is given in Numbers, and the Line q subtending the Complement of the given Angle; as if the Radius were 10, and the Chord 3; I substitute those Numbers in the Equation for r and q, and there comes out the Numeral Equation $x^5 - 500x^3 + 50000x - 30000 = 0$, whereof the Root being extracted will be x, or the Line subtending the Complement of the fifth Part of that given Angle.

Of the Nature of the Roots of an EQUATION.

But the Root is a Number which being substituted in the Equation for the Letter or Species signifying the Root, will make all the Terms vanish.

Thus Unity is the Root of the Equation $x^4 - x^3 - 19xx + 49x - 30 = 0$, because being writ for x it produces $1 - 1 - 19 + 49 - 30$, that is, nothing. But there may be more Roots of the same Equation. As if in this same Equation $x^4 - x^3 - 19xx + 49x - 30 = 0$, for x you write the Number 2, and for the Powers of x the like Powers of the Number 2, there will be produced $16 - 8 - 76 + 98 - 30$, that is nothing. And so if for x you write the Number 3, or the Negative Number -5, in both Cases there will be produced nothing, the Affirmative and Negative Terms in these four Cases destroying one another. Therefore since any of the Numbers written in the Equation fulfils the Condition of x, by making all the Terms of the Equation together equal to nothing, any of them will be the Root of the Equation.

And

And that you may not wonder that the same Equation may have *several Roots*, you must know *that there may be more Solutions than one of the same Problem.*

As if there was sought the *Intersection* of two given Circles; there are *two* Intersections, and consequently the Question admits *two Answers*; and therefore the Equation determining the Intersection will have *two Roots*, whereby it determines both the Intersections, *provided there be nothing in the Data whereby the Answer is determined to only one Intersection.* [See *Figure* 87.]

And thus, if of the Arch A P B its fifth Part A P were to be found, though perhaps you might apply your Thoughts only to the Arch A P B, yet the Equation, whereby the Question will be solved, will determine the fifth Part of all the Arches which are terminated at the Points A and B; *viz.* the fifth Part of the Arches A S B, A P B S A P B, A S B P A S B, and A P B S A P B S A P B, as well as the fifth Part of the Arch A P B; which fifth Parts, if you divide the whole Circumference into five equal Parts P Q, Q R, R S, S T, T P, will be A T, A Q, A T S, A Q R. Wherefore, by seeking the fifth Parts of the Arches which the right Line A B subtends, to determine all the Cases the whole Circumference ought to be divided in the five Points P, Q, R, S, T, therefore the Equation that will determine all the Cases will have five Roots. For the fifth Parts of all these Arches depend on the same Data, and are found by the same kind of Calculus; so that you will always fall upon the same Equation, whether you seek the fifth Part of the Arch A P B, or the fifth Part of the Arch A S B, or the fifth Part of any other of the Arches. Whence, if the Equation by which the fifth Part of the Arch A P B is determined, should not have more than one Root, while by seeking the fifth Part of the Arch A S B we fall upon that same Equation; it would follow, that this greater Arch would have the same fifth Part with the former, which is less, because its Subtense or Chord is expressed by the same Root of the Equation. *In every Problem therefore it is necessary, that the Equation which answers should have as many Roots as there are different Cases of the Quantity sought depending on the same Data, and to be determined by the same Method of Reasoning.*

But an Equation may have as many Roots as it has Dimensions, and not more.

Thus the Equation $x^4 - x^3 - 19xx + 49x - 30$, $= 0$, has four Roots, $1, 2, 3, -5$; but not more. For any

any of these Numbers writ in the Equation for x will cause all the Terms to destroy one another as we have said; but besides these, there is no Number by whose Substitution this will happen.

But the Number and Nature of the Roots will be best understood from the Generation of the Equation.

As if we would know how an Equation is generated, whose Roots are 1, 2, 3, and -5; we are to suppose x to signify ambiguously those Numbers, or x to be $=1$, $x=2$, $x=3$, and $x=-5$, or which is the same Thing, $x-1=0$, $x-2=0$, $x-3=0$, and $x+5=0$; and multiplying these together, there will come out by the Multiplication of $x-1$ by $x-2$ this Equation $xx-3x+2=0$, which is of two Dimensions, and has two Roots 1 and 2. And by the Multiplication of this by $x-3$, there will come out $x^3-6xx+11x-6=0$, an Equation of three Dimensions and as many Roots; which again multiplied by $x+5$ becomes $x^4-x^3-19xx+49x-30=0$, as above. Since therefore this Equation is generated by four Factors, $x-1$, $x-2$, $x-3$, and $x+5$, continually multiplied by one another, where any of the Factors is nothing, that which is made by all will be nothing; but where none of them is nothing, that which is contained under them all cannot be nothing. That is, $x^4-x^3-19xx+49x-30$ cannot be $=0$, as ought to be, except in these four Cases, where $x-1$ is $=0$, or $x-2=0$, or $x-3=0$, or, lastly, $x+5=0$, therefore only the Numbers 1, 2, 3, and -5 can exhibit x, or be the Roots of the Equation. And you are to reason alike of all Equations. For we may imagine all to be generated by such a Multiplication, although it is usually very difficult to separate the Factors from one another, and is the same Thing as to resolve the Equation and extract its Roots. For the Roots being had, the Factors are had also.

But the Roots are of two Sorts, Affirmative, *as in the Example brought,* 1, 2, *and* 3, *and* Negative, *as* -5. *And of these some are often* impossible.

Thus, the two Roots of the Equation $xx-2ax+bb=0$, which are $a+\sqrt{aa-b}$ and $a-\sqrt{aa-bb}$, are real when aa is greater than bb; but when aa is less than bb, they become impossible, because then $aa-bb$ will be a Negative Quantity, and the Square Root of a Negative Quantity is impossible. For every possible Root, whether it be be Affirmative or Negative, if it be multiplied by it felf, produces an Affirmative Square; therefore that will be an impoffible one which is to produce a Negative Square. By the fame Argument you may conclude, that the Equation $x^3 - 4xx + 7x - 6 = 0$, has one real Root, which is 2, and two impoffible ones $1 + \sqrt{-2}$ and $1 - \sqrt{-2}$. For any of thefe, 2, $1 + \sqrt{-2}$, $1 - \sqrt{-2}$ being writ in the Equation for x, will make all its Terms deftroy one another; but $1 + \sqrt{-2}$, and $1 - \sqrt{-2}$, are impoffible Numbers, becaufe they fuppofe the Extraction of the Square Root out of the Negative Number -2.

But it is juft, that the Roots of Equations fhould be often impoffible, left they fhould exhibit the Cafes of Problems that are often impoffible as if they were poffible.

As if you were to determine the Interfection of a right Line and a Circle, and you fhould put two Letters for the Radius of the Circle and the Diftance of the right Line from its Center; and when you have the Equation defining the Interfection, if for the Letter denoting the Diftance of the right Line from the Center, you put a Number lefs than the Radius, the Interfection will be poffible; but if it be greater, impoffible; and the two Roots of the Equation, which determine the two Interfections, ought to be either poffible or impoffible, that they may truly exprefs the Matter. [See *Figure* 88.] And thus, if the Circle C D E F, and the Ellipfis A C B F, cut one another in the Points C, D, E, F, and to any right Line given in Pofition, as A B, you let fall the Perpendiculars C G, D H, E I, F K, and by feeking the Length of any one of the Perpendiculars, you come at length to an Equation; that Equation, when the Circle cuts the Ellipfis in four Points, will have four real Roots, which will be thofe four Perpendiculars. But if the Radius of the Circle, its Center remaining, be diminifhed until the Points E and F meeting, the Circle at length touches the Ellipfe, thofe two of the Roots which exprefs the Perpendiculars E I and F K now coinciding, will become equal. And if the Circle be yet diminifhed, fo that it does not touch the Ellipfe in the Point E F, but only cuts it in the other two Points C, D, then out of the four Roots thofe two which expreffed the Perpendiculars E I, F K, which are now become impoffible, will become, together with thofe Perpendiculars, alfo impoffible. And after this Way in all Equations, by augmenting or diminifhing their Terms, of the unequal Roots, two will become firft equal and then impoffible.

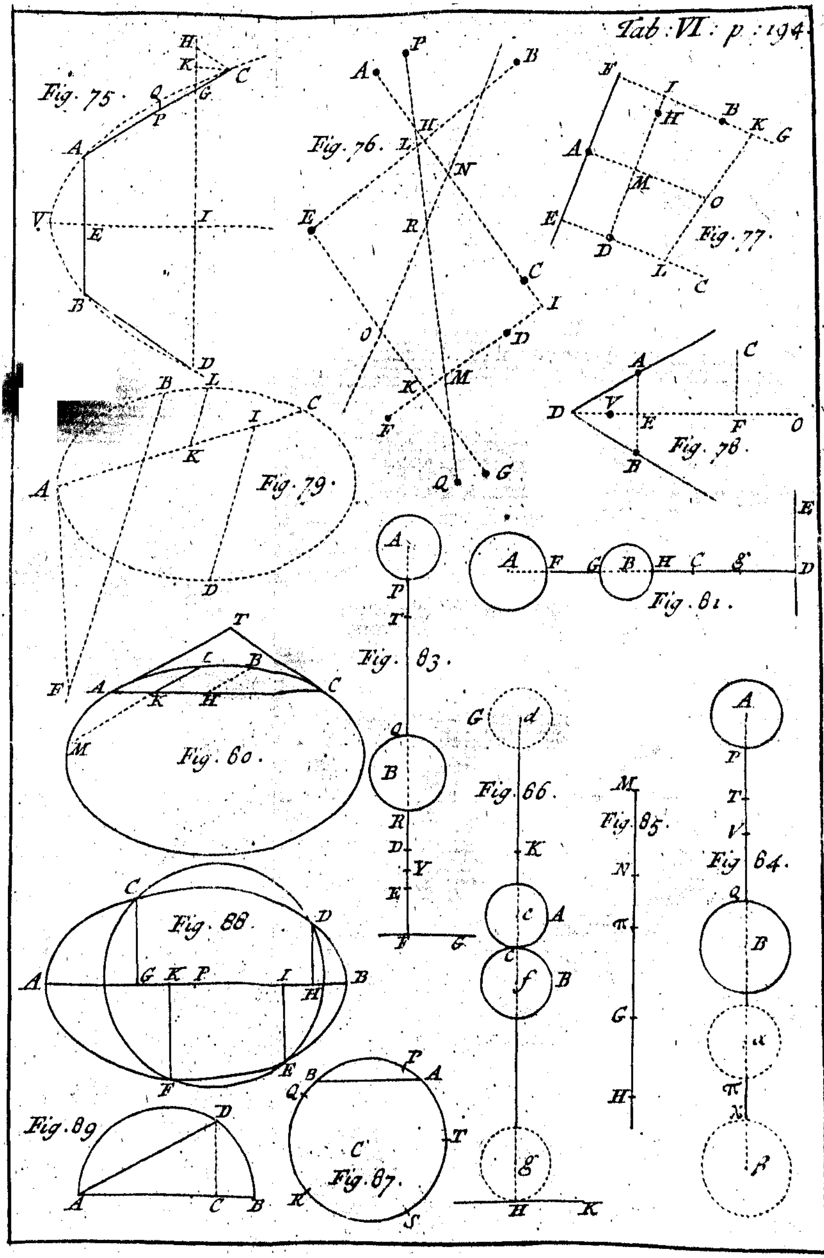
possible. And thence it is, that the Number of the impossible Roots is always even.

But sometimes the Roots of Equations are possible, when the Schemes exhibit them as impossible. But this happens by reason of some Limitation in the Scheme, which does not belong to the Equation. [*See* Figure 89.]

As if in the Semi-Circle A D B, having given the Diameter A B, and the Chord A D, and having let fall the Perpendicular D C, I was to find the Segment of the Diameter A C, you will have $\frac{ADq}{AB} = AC$. And, by this Equation, A C is exhibited a real Quantity, where the inscribed Line A D is greater than the Diameter A B; but by the Scheme, A C then becomes impossible, *viz.* in the Scheme the Line A D is supposed to be inscribed in the Circle, and therefore cannot be greater than the Diameter of the Circle; but in the Equation there is nothing that depends upon that Condition. From this Condition alone of the Lines the Equation comes out, that A B, A D, and A C are continually proportional. And because the Equation does not contain all the Conditions of the Scheme, it is not necessary that it should be bound to the Limits of all Conditions. Whatever is more in the Scheme than in the Equation may constrain that to Limits, but not this. For which reason, when Equations are of odd Dimensions, and consequently cannot have all their Roots impossible, the Schemes often set Limits to the Quantities on which all the Roots depend, which Limits it is impossible they can exceed, keeping the same Conditions of the Schemes.

Of those Roots that are real ones, the Affirmative and Negative ones lie on contrary Sides, or tend contrary Ways.

Thus, in the last Scheme but one, by seeking the Perpendicular C G, you will light upon an Equation that has two Affirmative Roots C G and D H, tending from the Points C and D the same Way; and two Negative ones, E I and F K, tending from the Points E and F the opposite Way. Or if in the Line A B to which the Perpendiculars are let fall, there be given any Point P, and the Part of it P G extending from that given Point to some of the Perpendiculars, as C G, be sought, we shall light on an Equation of four Roots, P G, P H, P I, and P K, whereof the Quantity sought P G, and those that tend from the Point P the same Way with P G, (as


Tab: VI: p: 194.
Fig. 75.
Fig. 76.
Fig. 77.
Fig. 78.
Fig. 79.
Fig. 80.
Fig. 81.
Fig. 83.
Fig. 84.
Fig. 85.
Fig. 86.
Fig. 87.
Fig. 88.
Fig. 89

P K) will be Affirmative, but those which tend the contrary Way (as P H, P I) Negative.

Where there are none of the Roots of the Equation impossible, the Number *of the Affirmative and Negative* Roots *may be known from the* Signs *of the Terms of the Equation. For there are* so many *Affirmative Roots as there are* Changes *of the Signs in a continual Series from* + *to* —, *and from* — *to* +; *the rest are* Negative.

As in the Equation $x^4 - x^3 - 19xx + 49x - 30 = 0$, where the Signs of the Terms follow one another in this Order, $+ - - + -$, the Variations of the second — from the first +, of the fourth + from the third —, and of the fifth — from the fourth +, shew, that there are three Affirmative Roots, and consequently, that the fourth is a Negative one. But where some of the Roots are *impossible*, the Rule is of no Force, unless as far as those impossible Roots, which are neither Negative nor Affirmative, may be taken for *ambiguous ones.* Thus in the Equation $x^3 + pxx + 3ppx - q = 0$, the Signs shew that there is one Affirmative Root and two Negative ones. Suppose $x = 2p$, or $x - 2p = 0$, and multiply the former Equation by this, $x - 2p = 0$, that one Affirmative Root more may be added to the former, and you will have this Equation, $x^4 - px^3 + ppxx \begin{matrix} -bp^3 \\ -q \end{matrix} x + 2pq = 0$, which ought to have two Affirmative and two Negative Roots; yet it has, if you regard the Change of the Signs, four Affirmative ones. There are therefore *two impossible ones*, which for their Ambiguity in the former Case seem to be Negative ones, in the latter, Affirmative ones.

But you may know almost by this Rule how many Roots are impossible.

Make a Series of Fractions, whose Denominators are Numbers in this Progression 1, 2, 3, 4, 5, &c. *going on to the Number which shall be the same as that of the Dimensions of the Equation; and the Numerators the same Series of Numbers in a contrary Order. Divide each of the latter Fractions by each of the former. Place the Fractions that come out over the middle Terms of the Equation. And under any of the middle Terms, if its Square, multiplied into the Fraction standing over its Head, is greater than the Rectangle of the Terms on both Sides, place the Sign* +; *but if it be less, the Sign* —. *But under the first and last Term place the Sign* +. *And there will be as many impossible Roots as there are*

are Changes in the Series of the under-written Signs from + *to* —, *and* — *to* +.

As if you have the Equation $x^3 + pxx + 3ppx - q = 0$; I divide the second of the Fractions of this Series $\frac{3}{1}. \frac{2}{2}. \frac{1}{3}$, *viz.* $\frac{2}{2}$ by the first $\frac{3}{1}$, and the third $\frac{1}{3}$ by the second $\frac{2}{2}$, and I place the Fractions that come out, *viz.* $\frac{1}{3}$ and $\frac{1}{3}$ over the middle Terms of the Equation, as follows;

$$\begin{array}{ccccccccc} & & \frac{1}{3} & & & \frac{1}{3} & & & \\ x^3 & + & p\,xx & + & 3p & p & x & - & q = 0. \\ + & & - & & & + & & & + \end{array}$$

Then, because the Square of the second Term pxx multiplied into the Fraction over its Head $\frac{1}{3}$, *viz.* $\frac{ppx^4}{3}$ is less than $3ppx^4$, the Rectangle of the first Term x^3 and third $3ppx$, I place the Sign — under the Term pxx. But because $9p^4xx$ (the Square of the third Term $3ppx$) multiplied into the Fraction over its Head $\frac{1}{3}$, is greater than nothing, and therefore much greater than the Negative Rectangle of the second Term pxx, and the fourth $-q$, I place the Sign + under that third Term. Then, under the first Term x^3 and the last $-q$, I place the Sign +. And the two Changes of the underwritten Signs; which are in this Series + — + +, the one from + into —, and the other from — into +, shew that there are two impossible Roots. And thus the Equation $x^3 - 4xx + 4x - 6 = 0$ has two impossible Roots,

$$\begin{array}{cccccc} & \frac{1}{3} & & \frac{1}{3} & & \\ x^3 & -4\,xx & + & 4\,x & - & 6 = 0. \\ + & + & & - & & + \end{array}$$

Also the Equation $x^4 - 6xx - 3x - 2 = 0$ has two.

$$\begin{array}{cccccc} & \frac{3}{8} & \frac{4}{9} & \frac{3}{8} & & \\ x^4 & * & -\,6\,xx & -\,3\,x & - & 2 = 0. \\ + & + & + & - & & + \end{array}$$

For this Series of Fractions $\frac{4}{1}. \frac{3}{2}. \frac{2}{3}. \frac{1}{4}$, by dividing the second by the first, and the third by the second, and the fourth by the third, gives this Series $\frac{3}{8}. \frac{4}{9}. \frac{3}{8}$, to be placed over the middle Terms of the Equation. Then the Square of the second Term, which is here nothing, multiplied into the Fraction over Head, *viz.* $\frac{3}{8}$, produces nothing, which is yet greater than the Negative Rectanlge $-6x^6$ contained under the Terms on each side x^4 and $-6xx$. Wherefore, under the Term that is wanting I write +. In the rest I go on as in the former Example; and there comes out this Series of the underwritten Signs + + + — +, where two Changes shew there are two impossible Roots. And

And after the ſame Way in the Equation $x^5 - 4x^4 + 4x^3 - 2xx - 5x - 4 = 0$, are diſcovered two impoſſible Roots, as follows;

$$\begin{array}{cccccc} & \frac{2}{5} & \frac{1}{2} & \frac{1}{2} & \frac{2}{5} & \\ x^5 & -4x^4 & +4x^3 & -2xx & -5x & -4=0. \\ + & + & - & + & + & + \end{array}$$

Where two or more Terms are wanting together, under the firſt of the deficient Terms you muſt write the Sign $-$, under the ſecond Sign $+$, under the third the Sign $-$, and ſo on, always varying the Signs, except that under the laſt of ſuch deficient Terms you muſt always place $+$, when the Terms next on both Sides the deficient Terms have contrary Signs. As in the Equations $\begin{array}{cccccc} x^5 & +ax^4 & * & * & * & +a^5=0, \\ + & + & - & + & - & + \end{array}$ and $\begin{array}{cccccc} x^5 & +ax^4 & * & * & * & -a^5=0 \\ + & + & - & + & + & + \end{array}$; the firſt whereof has four, and the latter two impoſſible Roots. Thus alſo the Equation,

$$\begin{array}{cccccccc} & \frac{3}{7} & \frac{5}{9} & \frac{3}{5} & \frac{3}{5} & \frac{5}{9} & \frac{3}{7} & \\ x^7 & -2x^6 & +3x^5 & -2x^4 & +x^3 & * & * & -3=0 \\ + & - & + & - & + & - & + & + \end{array}$$

has ſix impoſſible Roots.

Hence alſo may be known whether the impoſſible Roots are among the Affirmative or Negative ones. For the Signs of the Terms over Head of the ſubſcribed changing Terms ſhew, that there are as many impoſſible Affirmative Roots as there are Variations of them, and as many Negative ones as there are Succeſſions without Variations. Thus, in the Equation $\begin{array}{cccccc} x^5 & -4x^4 & +4x^3 & -2xx & -5x & -4=0 \\ + & + & - & + & + & + \end{array}$, becauſe by the Signs that are writ underneath that are changeable, *viz.* $+ - +$, by which it is ſhewn there are two impoſſible Roots, the Terms over Head $-4x^4 + 4x^3 - 2xx$ have the Signs $- + -$, which by two Variations ſhew there are two Affirmative Roots; therefore there will be two impoſſible Roots among the Affirmative ones. Since therefore the Signs of all the Terms of the Equation $+ - + - - -$ by three Variations ſhew that there are three Affirmative Roots, and that the other two are Negative, and that among the Affirmative ones there are two impoſſible ones; it follows that the Equation has one true affirmative Root, two negative ones, and two impoſſible ones.

ones. But if the Equation had been $\underset{+}{x^5} - \underset{+}{4x^4} - \underset{-}{4x^3} - \underset{-}{2xx} - \underset{+}{5x} - \underset{+}{4} = 0$, then the Terms over Head of the subscribed former varying Terms $+-$, *viz.* $-4x^4 - 4x^3$, by their Signs that do not change $-$ and $-$, shew, that one of the Negative Roots is impossible; and the Terms over the latter underwritten varying Terms $-+$, *viz.* $-2xx - 5x$, by their Terms not varying, $-$ and $-$, shew that another of the Negative Roots is impossible. Wherefore, since the Signs of the Equation $+-----$ by one Variation shew there is one Affirmative Root, and that the other four are Negative; it follows, there is one Affirmative, two Negative, and two Impossible ones. And this is so where there are not more impossible Roots than what are discovered by the Rule preceding. For there may be more, although it seldom happens.

Of the TRANSMUTATIONS *of* EQUATIONS.

Moreover all the Affirmative *Roots of any Equation may be changed into* Negative *ones, and the* Negative *into* Affirmative *ones, and that only by changing the Signs of the alternate Terms.*

Thus in the Equation $x^5 - 4x^4 + 4x^3 - 2xx - 5x - 4 = 0$, the three Affirmative Roots will be changed into Negative ones, and the two Negative ones into Affirmatives, by changing only the Signs of the second, fourth, and sixth Terms, as is done here, $x^5 + 4x^4 + 4x^3 + 2xx - 5x + 4 = 0$. This Equation has the same Roots with the former, unless that in this, those Roots are Affirmative that were there Negative, and Negative here that there were Affirmative; and the two impossible Roots, which lay hid there among the Affirmative ones, lie hid here among the Negative ones; so that these being deducted, there remains only one Root truly Negative.

There are also other Transmutations of Equations which are of Use in divers Cases. *For we may suppose the Root of an Equation to be composed any how out of a known and unknown Quantity, and then substitute what we suppose equivalent to it.* As if we suppose the Root to be equal to the Sum or Difference of any known and unknown Quantity. For after

after this Rate we may augment or diminish the Roots of the Equation by that known Quantity, or subtract them from it; and thereby cause that some of them that were before Negative shall now become Affirmative, or some of the Affirmative ones become Negative, or also that all shall become Affirmative or all Negative. Thus in the Equation $x^4 - x^3 - 19xx + 49x - 30 = 0$, if I have a mind to augment the Roots by Unity, I suppose $x + 1 = y$, or $x = y - 1$; and then for x I write $y - 1$ in the Equation, and for the Square, Cube, or Biquadrate of x, I write the like Power of $y - 1$, as follows.

$$\begin{array}{r|rrrrr}
x^4 & y^4 - 4y^3 & + 6yy & - 4y & + 1 & \\
-x^3 & - y^3 & + 3yy & - 3y & + 1 & \\
-19xx & & -19yy & +38y & -19 & \\
+49x & & & +49y & -49 & \\
-30 & & & & -30 & \\
\hline
\text{Sum} & y^4 - 5y^3 & -10yy & +80y & -96 & = 0.
\end{array}$$

And the Roots of the Equation that is produced, (*viz.*) $y^4 - 5y^3 - 10yy + 80y - 96 = 0$, will be 2, 3, 4, -4, which before were 1, 2, 3, -5, *i.e.* bigger by Unity. Now, if for x I had writ $y + 1\frac{1}{2}$, there would have come out the Equation $y^4 + 5y^3 - 10yy - \frac{5}{4}y + \frac{30}{16} = 0$, whereof there be two Affirmative Roots, $\frac{1}{2}$ and $1\frac{1}{2}$, and two Negative ones, $-\frac{1}{2}$ and $-6\frac{1}{2}$. But by writing $y - 6$ for x, there would have come out an Equation whose Roots would have been 7, 8, 9, 1, *viz.* all Affirmative; and writing for the same [x] $y + 4$, there would have come out those Roots diminished by 4, *viz.* -3 -2 -1 -9, all of them Negative.

After this Manner, by augmenting or diminishing the Roots, if any of them are impossible, they will sometimes be more easily detected than before. Thus in the Equation $x^3 - 3aax - 3a^3 = 0$, there are no Roots that appear impossible by the preceding Rule; but if you augment the Roots by the Quantity a, writing $y - a$ for x, you may now by that Rule discover two impossible Roots in the Equation resulting, $y^3 - 3ayy - a^3 = 0$.

By the same Operation you may also take away the second Terms of Equations. This will be done, if you subduct the known Quantity of the second Term of the Equation proposed, divided by the Number of Dimensions of the highest Term

Term of the Equation, from the Quantity which you aſſume to ſignify the Root of the new Equation, and ſubſtitute the Remainder for the Root of the Equation propoſed. As if there was propoſed the Equation $x^3 - 4xx + 4x - 6 = 0$, I ſubtract the known Quantity of the ſecond Term, which is -4, divided by the Number of the Dimenſions of the Equation, *viz.* 3, from the Species or Letter which is aſſumed to ſignify the new Root, ſuppoſe from y, and the Remainder $y + \frac{4}{3}$ I ſubſtitute for x, and there comes out

$$\begin{array}{rrrr} y^3 & + 4yy & + \frac{16}{3}y & + \frac{64}{27} \\ & - 4yy & - \frac{32}{3}y & - \frac{64}{9} \\ & & + 4y & + \frac{16}{3} \\ & & & - 6 \\ \hline y^3 & * & - \frac{4}{3}y & - \frac{146}{27} = 0. \end{array}$$

By the ſame Method, the third Term of an Equation may be alſo taken away. Let there be propoſed the Equation $x^4 - 3x^3 + 3xx - 5x - 2 = 0$, and make $x = y - e$, and ſubſtituting $y - e$ in the room of x, there will ariſe this Equation;

$$y^4 \begin{array}{l} -4e \\ -3 \end{array} y^3 \begin{array}{l} +6ee \\ +9e \\ +3 \end{array} yy \begin{array}{l} -4e^3 \\ -9ee \\ -6e \\ -5 \end{array} y \left.\begin{array}{l} +e^4 \\ +3e^3 \\ +3ee \\ +5e \\ -2 \end{array}\right\} = 0.$$

The third Term of this Equation is $6ee + 9e + 3$ multiplied by yy. Where, if $6ee + 9e + 3$ were nothing, you would have what you deſired. Let us ſuppoſe it therefore to be nothing, that we may thence find what Number ought to be ſubſtituted in this Caſe for e, and we ſhall have the Quadratick Equation $6ee + 9e + 3 = 0$; which divided by 6 will become $ee + \frac{3}{2}e + \frac{1}{2} = 0$, or $ee = -\frac{3}{2}e - \frac{1}{2}$, and extracting the Root $e = -\frac{3}{4} \pm \sqrt{\frac{9}{16} - \frac{1}{2}}$, or $= -\frac{3}{4} \pm \sqrt{\frac{1}{16}}$, that is, $= -\frac{3}{4} \pm \frac{1}{4}$, and conſequently either $= -\frac{1}{2}$ or $= -1$. Whence $y - e$ will be either $y + \frac{1}{2}$, or $y + 1$. Wherefore, ſince $y - e$ was writ for x; in the room of $y - e$ there ought to be writ $y + \frac{1}{2}$, or $y + 1$ for x, that the third Term of the Equation that reſults may be taken away. And that will happen in both Caſes. For if for x you write $y + \frac{1}{2}$, there will ariſe this Equation, $y^4 - y^3 - \frac{13}{4}y - \frac{65}{16} = 0$; but if you write $y + 1$, there will ariſe this Equation, $y^4 + y^3 - 4y - 6 = 0$. *More-*

Moreover, the Roots of Equations may be multiplied or divided by given Numbers; and after this Rate, the Terms of Equations be diminished, and Fractions and Radical Quantities sometimes be taken away.

As if the Equation were $y^3 - \frac{4}{3}y - \frac{146}{27} = 0$; in order to take away the Fractions, I suppose y to be $= \frac{1}{3}z$, and then by substituting $\frac{1}{3}z$ for y, there comes out this new Equation, $\frac{z^3}{27} - \frac{12z}{27} - \frac{146}{27} = 0$, and having rejected the common Denominator of the Terms, $z^3 - 12z - 146 = 0$, the Roots of which Equation are thrice greater than before. And again to diminish the Terms of this Equation, if you write $2v$ for z, there will come out $8v^3 - 24v - 146 = 0$, and dividing all by 8, you will have $v^3 - 3v - 18\frac{1}{4} = 0$; the Roots of which Equation are half of the Roots of the former. And here, if at last you find v make $2v = z$, $\frac{1}{3}z = y$, and $y + \frac{4}{3} = x$, and you will have x the Root of the Equation $x^3 - 4xx + 4x - 6 = 0$, as first proposed.

And thus, in the Equation $x^3 - 2x + \sqrt{3} = 0$, to take away the Radical Quantity $\sqrt{3}$; for x I write $y\sqrt{3}$, and there comes out the Equation $3y^3\sqrt{3} - 2y\sqrt{3} + \sqrt{3} = 0$, which, dividing all the Terms by $\sqrt{3}$, becomes $3y^3 - 2y + 1 = 0$.

Again, the Roots of an Equation may be changed into their Reciprocals, and after this Way the Equation may be sometimes reduced to a more commodious Form.

Thus, the last Equation $3y^3 - 2y + 1 = 0$, by writing $\frac{1}{z}$ for y, becomes $\frac{3}{z^3} - \frac{2}{z} + 1 = 0$, and all the Terms being multiplied by z^3, and the Order of the Terms changed, $z^3 - 2zz + 3 = 0$. The last Term but one of an Equation may also by this Method be taken away, provided the second was taken away before, as you see done in the precedent Example. Or if you would take away the last but two, it may be done, provided you have taken away the third before. Moreover, the least Root may be thus converted into the greatest, and the greatest into the least, which may be of some Use in what follows. Thus i the Equation $x^4 - x^3 - 19xx + 49x - 30 = 0$, whose Roots are 3, 2, 1, −5, if you write $\frac{1}{y}$ for x, there will come out the

the Equation $\frac{1}{y^4} - \frac{1}{y^3} - \frac{19}{yy} + \frac{49}{y} - 30 = 0$, which, multiplying all the Terms by y^4, and dividing them by 30, the Signs being changed, becomes $y^4 - \frac{49}{30}y^3 + \frac{19}{30}y^2 + \frac{1}{30}y - \frac{1}{30} = 0$, the Roots whereof are $\frac{1}{3}$, $\frac{1}{2}$, 1, $-\frac{1}{5}$; the greateſt of the Affirmative Roots 3 being now changed into the leaſt $\frac{1}{3}$, and the leaſt 1 being now made greateſt, and the Negative Root -5, which of all was the moſt remote from 0, now coming neareſt to it.

There are alſo other Tranſmutations of Equations, but which may all be performed after that Way of tranſmutating we have ſhewn, when we took away the third Term of the Equation.

From the Generation of Equations it is evident, that the known Quantity of the ſecond *Term of the Equation, if its Sign be changed, is equal to the Aggregate of all the Roots under their proper Signs; and that of the* third *Term equal to the Aggregate of the Rectangles* of each two of the Roots; that of the* fourth, *if its Sign be changed, is equal to the Aggregate of the Contents under each three of the Roots; that of the* fifth *is equal to the Aggregate of the Contents under each four, and ſo on* ad infinitum.

Let us aſſume $x = a$, $x = b$, $x = -c$, $x = d$, &c. or $x - a = 0$, $x - b = 0$, $x + c = 0$, $x - d = 0$, and by the continual Multiplication of theſe we may generate Equations, as above. Now, by multiplying $x - a$ by $x - b$ there will be produced the Equation $xx \begin{matrix} -a \\ -b \end{matrix} x + ab = 0$; where the known Quantity of the ſecond Term, if its Signs are changed, *viz.* $a + b$, is the Sum of the two Roots a and b, and the known Quantity of the third Term is the only Rectangle contained under both. Again, by multiplying this Equation by $x + c$, there will be produced the Cubick Equation $x^3 \begin{matrix} -a \\ -b \\ +c \end{matrix} xx \begin{matrix} +ab \\ -ac \\ -bc \end{matrix} x + abc = 0$, where the known Quantity of the ſecond Term having its Signs changed, *viz.* $a + b - c$, is the Sum of the Roots a, and b, and $-c$; the known Quantity of the third Term $ab - ac - bc$ is the Sum of the Rectangles under each two of the Roots a and b, a and

and $-c$, b and $-c$; and the known Quantity of the fourth Term under its Sign changed, $-abc$, is the only Content generated by the continual Multiplication of all the Roots, a by b into $-c$. Moreover, by multiplying that Cubick Equation by $x-d$, there will be produced this Biquadratick one;

$$x^4 \begin{matrix} -a \\ -b \\ +c \\ -d \end{matrix} x^3 \begin{matrix} +ab \\ -ac \\ -bc \\ +ad \\ +bd \\ -cd \end{matrix} xx \begin{matrix} +abc \\ -abd \\ +bcd \\ +acd \end{matrix} x - abcd = 0.$$

Where the known Quantity of the second Term under its Signs changed, *viz.* $a+b-c+d$, is the Sum of all the Roots; that of the third, $ab-ac-bc+ad+bd-cd$, is the Sum of the Rectangles under every two Roots; that of the fourth, its Signs being changed, $-abc+abd-bcd-acd$, is the Sum of the Contents under each Ternary; that of the fifth, $-abcd$, is the only Content under them all. And hence we first infer, that of any Equation that involves neither Surds nor Fractions all the rational Roots, and the Rectangles of any two of the Roots, or the Contents of any three or more of them, or some of the Integral Divisors of the last Term; and therefore when it is evident that no Divisor of the last Term is either a Root of the Equation, or Rectangle, or Content of two or more Roots, it will also be evident that there is no Root, or Rectangle, or Content of Roots, except what is Surd.

Let us suppose now, that the known Quantities of the Terms of any Equation under their Signs changed, are p, q, r, s, t, v, &c. *viz.* that of the second p, that of the third q, of the fourth r, of the fifth s, and so on. And the Signs of the Terms being rightly observed, make $p=a$, $pa+2q=b$, $pb+qa+3r=c$, $pc+qb+ra+4s=d$, $pd+qc+rb+sa+5t=e$, $pe+qd+rc+sb+ta+6v=f$, and so on *in infinitum*, observing the Series of the Progression. And a will be the Sum of the Roots, b the Sum of the Squares of each of the Roots, c the Sum of the Cubes, d the Sum of the Biquadrates, e the Sum of the Quadrato-Cubes, f the Sum of the Cubo-Cubes, and so on. As in the Equation $x^4-x^3-19xx+49x-30=0$, where the known Quantity of the second Term is -1, of the third -19, of the fourth $+49$, of the fifth -30; you must make $1=p$,

$1=p$, $19=q$, $-49=r$, $30=s$. And there will thence arise $a=(p=)\ 1$, $b=(pa+2q=1+38=)\ 39$, $c=(pb+qa+3r=39+19-147=)-89$, $d=(pc+qb+ra+4s=-89+741-49+120=)\ 723$. Wherefore the Sum of the Roots will be 1, the Sum of the Squares of the Roots 39, the Sum of the Cubes -89, and the Sum of the Biquadrates 723, *viz.* the Roots of that Equation are 1, 2, 3, and -5, and the Sum of these $1+2+3-5$ is 1; the Sum of the Squares, $1+4+9+25$, is 39; the Sum of the Cubes, $1+8+27-125$, is -89; and the Sum of the Biquadrates, $1+16+81+625$, is 723.

Of the LIMITS *of* EQUATIONS.

AND hence are collected the *Limits* between which the Roots of the Equation shall consist, if none of them is impossible. For when the Squares of all the Roots are Affirmative, the Sum of the Squares will be Affirmative, and therefore greater than the Square of the greatest Root. And by the same Argument, the Sum of the Biquadrates of all the Roots will be greater than the Biquadrate of the greatest Root, and the Sum of the Cubo-Cubes greater than the Cubo-Cube of the greatest Root.

Wherefore, if you desire the Limit which no Roots can pass, seek the Sum of the Squares of the Roots, and extract its Square Root. For this Root will be greater than the greatest Root of the Equation. But you will come nearer the greatest Root if you seek *the Sum of the Biquadrates, and extract its Biquadradick Root; and yet nearer, if you seek the Sum of the Cubo-Cubes, and extract its Cubo-Cubical Root; and so on* in infinitum.

Thus, in the precedent Equation, the Square Root of the Sum of the Squares of the Roots, or $\sqrt{39}$, is $6\frac{1}{2}$ nearly, and $6\frac{1}{2}$ is farther distant from 0 than any of the Roots 1, 2, 3, -5. But the Biquadratick Root of the Sum of the Biquadrates of the Roots, *viz.* $\sqrt[4]{723}$, which is $5\frac{1}{4}$ nearly, comes nearer to the Root that is most remote from nothing, *viz.* -5.

If, between the Sum of the Squares and the Sum of the Biquadrates of the Roots you find a mean Proportional, that will be a little greater than the Sum of the Cubes of the Roots

Roots connected under Affirmative Signs. And hence, the half Sum of this mean Proportional, and of the Sum of the Cubes collected under their proper Signs, found as before, will be greater than the Sum of the Cubes of the Affirmative Roots, and the half Difference greater than the Sum of the Cubes of the Negative Roots.

And consequently, the greatest of the Affirmative Roots will be less than the Cube Root of that half Sum, and the greatest of the Negative Roots less than the Cube Root of that Semi-difference.

Thus, in the precedent Equation, a mean Proportional between the Sum of the Squares of the Roots 39, and the Sum of the Biquadrates 723, is nearly 168. The Sum of the Cubes under their proper Signs was, as above, — 89, the half Sum of this and 168 is $39\frac{1}{2}$, the Semi-difference $128\frac{1}{2}$. The Cube Root of the former, which is about $3\frac{1}{3}$, is greater than the greatest of the Affirmative Roots 3. The Cube Root of the latter, which is $5\frac{7}{38}$ nearly, is greater than the Negative Root — 5. By which Example it may be seen how near you may come this Way to the Root, where there is only one Negative Root or one Affirmative one. *And yet you might come nearer still*, if you found a mean Proportional between the Sum of the Biquadrates of the Roots and the Sum of the Cubo Cubes, and if from the Semi-Sum and Semi-Difference of this, and of the Sum of the Quadrato-Cube of the Roots, you extracted the Quadrato-Cubical Roots. For the Quadrato-Cubical Root of the Semi-Sum would be greater than the greatest Affirmative Root, and the Quadrato-Cubick Root of the Semi-Difference would be greater than the greatest Negative Root, but by a less Excess than before. Since therefore any Root, by augmenting or diminishing all the Roots, may be made the least, and then the least converted into the greatest, and afterwards all besides the greatest be made Negative, it is manifest how any Root desired may be found nearly.

If all the Roots except two are Negative, those two may be both together found this Way.

The Sum of the Cubes of those two Roots being found according to the precedent Method, as also the Sum of the Quadrato-Cubes, and the Sum of the Quadrato-Quadrato-Cubes of all the Roots: between the two latter Sums seek a mean Proportional, and that will be the Difference between the Sum of the Cubo-Cubes of the Affirmative Roots, and the Sum of the Cubo-Cubes of the Negative Roots nearly; and consequently,

quently, the half Sum of this mean Proportional, and of the Sum of the Cubo-Cubes of all the Roots, will be the Sum of the Cubo-Cubes of the Affirmative Roots, and the Semi-Difference will be the Sum of the Cubo-Cubes of the Negative Roots. Having therefore both the Sum of the Cubes, and alſo the Sum of the Cubo-Cubes of the two Affirmative Roots, from the double of the latter Sum ſubtract the Square of the former Sum, and the Square Root of the Remainder will be the Difference of the Cubes of the two Roots. And having both the Sum and Difference of the Cubes, you will have the Cubes themſelves. Extract their Cube Roots, and you will nearly have the two Affirmative Roots of the Equation. And if in higher Powers you ſhould do the like, you will have the Roots yet more nearly. But theſe Limitations, by reaſon of the Difficulty of the Calculus, are of leſs Uſe, and extend only to thoſe Equations that have no imaginary Roots. Wherefore I will now ſhew how to find the Limits another Way, which is more eaſy, and extends to all Equations.

Multiply every Term of the Equation by the Number of its Dimenſions, and divide the Product by the Root of the Equation. Then again multiply every one of the Terms that come out by a Number leſs by Unity than before, and divide the Product by the Root of the Equation. And ſo go on, by always multiplying by Numbers leſs by Unity than before, and dividing the Product by the Root, till at length all the Terms are deſtroyed, whoſe Signs are different from the Sign of the firſt or higheſt Term, except the laſt. And that Number will be greater than any Affirmative Root; which being writ in the Terms that come out for the Root, makes the Aggregate of thoſe which were each Time produced by Multiplication to have always the ſame Sign with the firſt or higheſt Term of the Equation.

As if there was propoſed the Equation $x^5 - 2x^4 - 10x^3 + 30xx + 63x - 120 = 0$. I firſt multiply this thus;

$$\overset{5}{x^5} - \overset{4}{2x^4} - \overset{3}{10x^3} + \overset{2}{30xx} + \overset{1}{63x} - \overset{0}{120}.$$

Then I again multiply the Terms that come out divided by x, thus;

$$\overset{4}{5x^4} - \overset{3}{8x^3} - \overset{2}{30xx} + \overset{1}{60x} + \overset{0}{63},$$

and dividing the Terms that come out again by x, there comes out $20x^3 - 24xx - 60x + 60$; which, to leſſen them, I divide by the

the greatest common Divisor 4, and you have $5x^3 - 6xx - 15x + 15$. These being again multiplied by the Progression 3, 2, 1, 0, and divided by x, become $15xx - 12x - 15$, and again divided by 3 become $5xx - 4x - 5$. And these multiplied by the Progression 2, 1, 0, and divided by $2x$ become $5x - 2$. Now, since the highest Term of the Equation x^5 is Affirmative, I try what Number writ in these Products for x will cause them all to be Affirmative. And by trying 1, you have $5x - 2 = 3$ Affirmative; but $5xx - 4x - 5$, you have -4 Negative. Wherefore the Limit will be greater than 1. I therefore try some greater Number, as 2. And substituting 2 in each for x, they become.

$$5x - 2 = 8$$
$$5xx - 4x - 5 = 7$$
$$5x^3 - 6xx - 15x + 15 = 1$$
$$5x^4 - 8x^3 - 30xx + 60x + 63 = 79$$
$$x^5 - 2x^4 - 10x^3 + 30xx + 63x - 120 = 46.$$

Wherefore, since the Numbers that come out 8. 7. 1. 79. 46. are all Affirmative, the Number 2 will be greater than the greatest of the Affirmative Roots. In like manner, if I would find the Limit of the Negative Roots, I try Negative Numbers. Or that which is all one, I change the Signs of every other Term, and try Affirmative ones. But having changed the Signs of every other Term, the Quantities in which the Numbers are to be substituted, will become

$$5x + 2$$
$$5xx + 4x - 5$$
$$5x^3 + 6xx - 15x - 15$$
$$5x^4 + 8x^3 - 30xx - 60x + 63$$
$$x^5 + 2x^4 - 10x^3 - 30xx + 63x + 120.$$

Out of these I chuse some Quantity wherein the Negative Terms seem most prevalent; suppose $5x^4 + 8x^3 - 30xx - 60x + 63$, and here substituting for x the Numbers 1 and 2, there come out the Negative Numbers -14 and -33. Whence the Limit will be greater than -2. But substituting the Number 3, there comes out the Affirmative Number 234. And in like manner in the other Quantities, by substituting the Number 3 for x, there comes out always an Affirmative Number, which may be seen by bare Inspection. Wherefore the Number -3 is greater than all the Negative Roots. And so you have the Limits 2 and -3, between which are all the Roots.

But

But the Invention of these Limits is of Use both in the Reduction of Equations by Rational Roots, and in the Extraction of Surd Roots out of them; lest we might sometimes go about to look for the Root beyond these Limits. Thus, in the last Equation, if I would find the Rational Roots, if perhaps it has any; from what we have said, it is certain they can be no other than the Divisors of the last Term of the Equation, which here is 120. Then trying all its Divisors, if none of them writ in the Equation for x would make all the Terms vanish, it is certain that the Equation will admit of no Root but what is Surd. But there are many Divisors of the last Term 120, *viz.* 1. — 1. 2. — 2. 3. — 3. 4. — 4. 5. — 5. 6. — 6. 8. — 8. 10. — 10. 12. — 12. 15. — 15. 20. — 20. 24. — 24. 30. — 0. 40. — 40. 60. — 60. 120. and — 120. To try all these Divisors would be tedious. But it being known that the Roots are between 2 and — 3, we are freed from that Labour. For now there will be no need to try the Divisors, unless those only that are within these Limits, *viz.* the Divisors 1, and — 1. and — 2. For if none of these are the Root, it is certain that the Equation has no Root but what is Surd.

The Reduction of EQUATIONS *by Surd Divisors.*

HItherto I have treated of the Reduction of Equations which admit of Rational Divisors. But before we can conclude, that an Equation of four, six, or more Dimensions is irreducible, we must first try whether or not it may be reduced by any Surd Divisor; or, which is the same Thing, you must try whether the Equation can be so divided into two equal Parts, that you can extract the Root out of both. But that may be done by the following Method.

Dispose the Equation according to the Dimensions of some certain Letter, so that all its Terms jointly under their proper Signs, may be equal to nothing, and let the highest Term be affected with an Affirmative Sign. Then, if the Equation be a Quadratick, (*for we may add this Case for the Analogy of the Matter*) *take from both Sides the lowest Term, and add one fourth Part of the Square of the known Quantity of the middle Term.*

As if the Equation be $xx - ax - b = 0$, subtract from both Sides $-b$, and add $\frac{1}{4}aa$, and there will come out $xx -$

$xx - ax + \frac{1}{4}aa = b + \frac{1}{4}aa$, *and extracting on both Sides the Root, you will have* $x - \frac{1}{2}a = \pm\sqrt{b + \frac{1}{4}aa}$, or $x = \frac{1}{2}a \pm \sqrt{b + \frac{1}{4}aa}$.

But if the Equation be of four Dimensions, *suppose* $x^4 + px^3 + qxx + rx + s = 0$, where p, q, r, and s *denote the known Quantities of the Terms of the Equation adfected by their proper Signs, make*

$$q - \frac{1}{4}pp = \alpha.\quad r - \frac{1}{2}\alpha p = \beta.$$
$$s - \frac{1}{4}\alpha\alpha = \zeta.$$

Then put for n some common Integral Divisor of the Terms β *and* 2ζ, *that is not a Square, and which ought to be odd, and divided by* 4 *to leave Unity, if either of the Terms p and r be odd. Put also for* k *some Divisor of the Quantity* $\frac{\beta}{n}$ *if p be even; or half of the odd Divisor, if p be odd; or nothing, if the Dividual* β *be nothing. Take the Quotient from* $\frac{1}{2}pk$, *and call the half of the Remainder l. Then for Q put* $\frac{\alpha + nkk}{2}$, *and try if n divides* $QQ - s$, *and the Root of the Quotient be rational and equal to l; which if it happen, add to each Part of the Equation* $nkkxx + 2nklx + nll$, *and extract the Rooot on both Sides, there coming out* $xx + \frac{1}{2}px + Q = \sqrt{n}$ *into* $\overline{kx + l}$.

For Example, let there be proposed the Equation $x^4 + 12x - 17 = 0$, and because p and q are both here wanting, and r is 12, and s is -17, having substituted these Numbers, you will have $\alpha = 0$, $\beta = 12$, and $\zeta = -17$, and the only common Divisor of β and 2ζ, or 12 and -34, *viz.* 2, will be n. Moreover, $\frac{\beta}{n}$ is 6, and its Divisors 1, 2, 3, and 6, are successively to be tried for k, and -3, $-\frac{3}{2}$, -1, $-\frac{1}{2}$, for l respectively. But $\frac{\alpha + nkk}{2}$, that is, kk is equal to Q. Moreover, $\sqrt{\frac{QQ - s}{n}}$, that is, $\sqrt{\frac{QQ + 17}{2}}$ is $= l$.

 Where

Where the even Numbers 2 and 6 are writ for k, Q becomes 4 and 36, and $QQ - s$ will be an odd Number, and consequently cannot be divided by n or 2. Wherefore those Numbers 2 and 6 are to be rejected. But when 1 and 3 are writ for k, Q becomes 1 and 9, and $QQ - s$ is 18 and 98, which Numbers may be divided by n, and the Roots of the Quotients extracted. For they are ± 3 and ± 7; whereof however only -3 agrees with l. I put therefore $k = 1$, $l = -3$, and $Q = 1$, and I add the Quantity $nkkxx + 2nklx + nll$, that is, $2xx - 12x + 18$ to each Part of the Equation, and there comes out $x^4 + 2xx + 1 = 2xx - 12x + 18$, and extracting on both Sides the Root $xx + 1 = x\sqrt{2} - 3\sqrt{2}$. But if you had rather avoid the Extraction of the Root, make $xx + \frac{1}{2}px + Q = \sqrt{n} \times \overline{kx + l}$, and you will find, as before, $xx + 1 = \pm\sqrt{2} \times \overline{x - 3}$. And if again you extract the Root of this Equation, there will come out $x = \pm\frac{1}{2}\sqrt{2} \pm \sqrt{\frac{-1}{2} \mp 3\sqrt{2}}$, that is, according to the Variations of the Signs, $x = -\frac{1}{2}\sqrt{2} + \sqrt{3\sqrt{2} - \frac{1}{2}}$, and $x = -\frac{1}{2}\sqrt{2} - \sqrt{3\sqrt{2} - \frac{1}{2}}$. Also $x = \frac{1}{2}\sqrt{2} + \sqrt{-3\sqrt{2} - \frac{1}{2}}$, and $x = \frac{1}{2}\sqrt{2} - \sqrt{-3\sqrt{2} - \frac{1}{2}}$. Which are four Roots of the Equation at first proposed, $x^4 + 12x - 17 = 0$. But the two last of them are impossible.

Let us now propose the Equation $x^4 - 6x^3 - 58xx - 114x - 11 = 0$, and by writing -6, -58, -114, and -11, for p, q, r, and s respectively, there will arise $-67 = \alpha$, $-315 = \beta$, and $-1133\frac{1}{4} = \zeta$. The only common Divisor of the Numbers β and 2ζ, or of -315 and $-\frac{4533}{2}$ is 3, and consequently will be here n, and the Divisors of $\frac{\beta}{n}$ or -105, are 3, 5, 7, 15, 21, 35, and 105, which are therefore to be tried for k. Wherefore, I try first 3, and the Quotient -35, which comes out by dividing $\frac{\beta}{n}$ by k, or -105 by 3, I subtract from $\frac{1}{2}pk$, or -3×3, and there remains 26; the half whereof, 13 ought to be l. But

But $\frac{\alpha + nkk}{2}$, or $\frac{-67+27}{2}$, that is, -20, will be Q, and $QQ - s$ will be 411, which may be divided by n, or 3, but the Root of the Quotient 137 cannot be extracted. Wherefore I reject 3, and try 5 for k. The Quotient that now comes out by dividing $\frac{\beta}{n}$ by k, or -105 by 5, is -21, and subtracting this from $\frac{1}{2}pk$, or -3×5, there remains 6, the half whereof 3 will be l. Also Q or $\frac{\alpha + nkk}{2}$, that is, $\frac{-67+75}{2}$, is the Number 4. And $QQ - s$, or $16 + 11$ may be divided by n; and the Root of the Quotient, which is 9, being extracted, *i. e.* 3 agrees with l. Wherefore I conclude, that l is $= 3$, $k = 5$, $Q = 4$, and $n = 3$; and if $nkkxx + 2nklx + nll$, that is, $75xx + 90x + 27$ be added to each Part of the Equation, the Root may be extracted on both Sides, and there will come out $xx + \frac{1}{2}px + Q = \sqrt{n} \times \overline{kx + l}$, or $xx - 3x + 4 = \pm\sqrt{3} \times \overline{5x + 3}$; and the Root being again extracted, $x = \frac{3 \pm 5\sqrt{3}}{2}$ $\pm\sqrt{17 \pm \frac{21 \times \sqrt{3}}{2}}$,

Thus, if there was proposed this Equation $x^4 - 9x^3 + 15xx - 27x + 9 = 0$, by writing -9, $+15$, -27, and $+9$ for p, q, r, and s respectively, there will come out $-5\frac{1}{4} = \alpha$, $-50\frac{5}{8} = \beta$, and $2\frac{7}{64} = \zeta$. The common Divisors of β and 2ζ, or $-\frac{405}{8}$ and $\frac{135}{32}$ are 3, 5, 9, 15, 27, 45, and 135; but 9 is a Square Number, and 3, 15, 27, 135, divided by the Number 4, do not leave Unity, as, by reason of the odd Term p, they ought to do. These therefore being rejected, there remain only 5 and 45 to be tried for n. Let us put therefore, first $n = 5$, and the odd Divisors of $\frac{\beta}{n}$ or $-\frac{81}{8}$ being halved, *viz.* $\frac{1}{2}$, $\frac{3}{2}$, $\frac{9}{2}$, $\frac{27}{2}$, $\frac{81}{2}$, are to be tried for k. If k be made $\frac{1}{2}$, the Quotient $-\frac{81}{4}$, which comes out by dividing $\frac{\beta}{n}$ by k, subtracted from $\frac{1}{2}pk$, or

$-\frac{9}{4}$, leaves 18 for $2l$, and $\frac{\alpha + nkk}{2}$ or -2 is Q, and QQ $-s$, or -5 may be divided indeed by n or 5, but the Root of the Negative Quotient -1 is impossible, which yet ought to be 9. Wherefore I conclude k not to be $\frac{1}{1}$, and then I try if it be $\frac{3}{2}$. The Quotient which arises by dividing $\frac{\beta}{n}$ by k, or $-\frac{81}{8}$ by $\frac{3}{2}$, *viz.* the Quotient $-\frac{27}{4}$ I subtract from $\frac{1}{2}pk$ or $-\frac{27}{4}$, and there remains 0. Whence now l will be nothing. But $\frac{\alpha + nkk}{2}$ or 3 is equal to Q, and QQ $-s$ is nothing; whence again l, which is the Root of QQ $-s$, divided by n, is found to be nothing. Wherefore these Things thus agreeing, I conclude n to be $= 5$, $k = \frac{3}{2}$, $l = 0$, and Q $= 3$, and therefore by adding to each Part of the Equation proposed the Terms $nkkxx + 2nlkx + nll$, that is, $\frac{45}{4}xx$, and by extracting on both Sides the Square Root, there comes out $xx + \frac{1}{2}px + Q = \sqrt{n} \times \overline{kx + l}$, that is, $xx - 4\frac{1}{2}x + 3 = \sqrt{5} \times \frac{3}{2}x$.

By the same Method Literal Equations are also reduced. As if there was $x^4 - 2ax^3 \begin{matrix} +2aa \\ -cc \end{matrix} xx - 2a^3x + a^4 = 0$, by substituting $-2a$, $2aa - cc$, $-2a^3$, and $+a^4$ for p, q, r, and s respectively, you will obtain $aa - cc = \alpha$, $-acc - a^3 = \beta$, and $\frac{1}{4}a^4 + \frac{1}{2}aacc - \frac{1}{4}c^4 = \zeta$. The common Divisor of the Quantities β and 2ζ is $aa + cc$, which therefore will be n; and $\frac{\beta}{n}$ or $-a$, has the Divisors 1 and a. But because n is of two Dimensions, and $k\sqrt{n}$ ought to be of no more than one, therefore k will be of none, and consequently cannot be a. Let therefore k be 1, and $\frac{\beta}{n}$ being divided by k, take the Quotient $-a$ from $\frac{1}{2}pk$ or $-a$ and there will remain nothing for l. Moreover, $\frac{\alpha + nkk}{2}$ or aa is Q, and QQ $-s$ or $a^4 - a^4$ is 0; and thence again there comes out nothing for l. Which shews the Quantities n, k, l, and Q to be rightly found; and adding to each Part of the Equation proposed, the Terms $nkkxx + 2nklx + nll$, that

that is, $aaxx + ccxx$, the Root may be extracted on both Sides; and by that Extraction there will come out $xx + \frac{1}{2}px + Q = \sqrt{n} \times \overline{kx + l}$, that is, $xx - ax + aa = \pm x\sqrt{aa + cc}$. And the Root being again extracted, you will have $x = \frac{1}{2}a \pm \frac{1}{2}\sqrt{aa + cc} \pm \sqrt{\frac{1}{4}cc - \frac{1}{2}aa \pm \frac{1}{2}a\sqrt{aa + cc}}$.

Hitherto I have applied the Rule to the Extraction of *surd Roots*; the same may also be applied to the Extraction of *Rational Roots*, if for the Quantity n you make Use of Unity; and after that Manner we may examine, whether an Equation that wants Fracted or Surd Terms can admit of any Divisor, either Rational or Surd, of two Dimensions. As if the Equation $x^4 - x^3 - 5xx + 12x - 6 = 0$ was proposed, by substituting -1, -5, $+12$, and -6 for p, q, r, and s respectively, you will find $-5\frac{1}{4} = \alpha$, $9\frac{3}{8} = \beta$, and putting $n = 1$. The Divisors of the Quantity $\frac{\beta}{n}$, or $\frac{75}{8}$, are $1, 3, 5, 15, 25, 75$; the Halves whereof (if p be odd) are to be tried for k. And if for k we try $\frac{5}{2}$, you will have $\frac{1}{2}pk - \frac{\beta}{nk} = -5$, and its half $-\frac{5}{2} = l$. Also $\frac{\alpha + nkk}{2} = \frac{1}{2} = Q$, and $\frac{QQ - s}{n} = 6\frac{1}{4}$, the Root whereof agrees with l.

I therefore conclude, that the Quantities n, k, l, Q, are rightly found; and having added to each Part of the Equation the Terms $nkkxx + 2nklx + nll$, that is, $6\frac{1}{4}xx - 12\frac{1}{2}x + 6\frac{1}{4}$, the Root may be extracted on both Sides; and by that Extraction there will come out $xx + \frac{1}{2}px + Q = \pm\sqrt{n} \times \overline{kx + l}$, that is, $xx - \frac{1}{2}x + \frac{1}{2} = \pm 1 \times 2\frac{1}{2}x - 2\frac{1}{2}$, or $xx - 3x + 3 = 0$, and $xx + 2x - 2 = 0$, and so by these two Quadratick Equations the Biquadratick one proposed may be divided. But Rational Divisors of this Sort may more expeditiously be found by another Method delivered above.

If at any Time there are many Divisors of the Quantity $\frac{\beta}{n}$, so that it may be too difficult to try all of them for k, their Number may be soon diminished, by seeking all the Divisors of of the Quantity, $\alpha s - \frac{1}{4} rr$. For the Quantity Q ought to be equal to fome of thefe, or to the half of fome odd one. Thus, in the laft Example, $\alpha s - \frac{1}{4} rr$ is $-\frac{9}{2}$, fome one of whofe Divifors, 1, 3, 9, or of them halved $\frac{1}{2}$, $\frac{3}{2}$, $\frac{9}{2}$, ought to be Q. Wherefore, by trying fingly the halved Divifors of the Quantity $\frac{\beta}{n}$, *viz.* $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, $\frac{15}{2}$, $\frac{25}{2}$, and $\frac{75}{2}$ for k, I reject all that do not make $\frac{1}{2}\alpha + \frac{1}{2}nkk$, or $-\frac{21}{8} + \frac{1}{2}kk$; that is, Q to be one of the Numbers 1, 3, 9, $\frac{1}{2}$, $\frac{3}{2}$, $\frac{9}{2}$. But by writing $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, $\frac{15}{2}$, &c. for k, there come out refpectively $-\frac{5}{2}$, $-\frac{3}{2}$, $+\frac{1}{2}$, $+\frac{51}{2}$, &c. for Q; out of which only $-\frac{3}{2}$ and $\frac{1}{2}$ are found among the aforefaid Numbers 1, 3, 9, $\frac{1}{2}$, $\frac{3}{2}$, $\frac{9}{2}$, and confequently the reft being rejected, either k will be $= \frac{3}{2}$ and $Q = -\frac{3}{2}$, or $k = \frac{5}{2}$ and $Q = \frac{1}{2}$. Which two Cafes let be examined. And fo much of Equations of *four Dimenfions.*

If an Equation of fix Dimenfions is to be reduced, let it be $x^6 + px^5 + qx^4 + rx^3 + sxx + tx + v = 0$, and make

$$q - \tfrac{1}{4}pp = \alpha. \quad r - \tfrac{1}{2}p\alpha = \beta. \quad s - \tfrac{1}{2}p\beta = \gamma.$$
$$\gamma - \tfrac{1}{4}\alpha\alpha = \zeta. \quad t - \tfrac{1}{2}\alpha\beta = \eta. \quad v - \tfrac{1}{4}\beta\beta = \theta.$$
$$\zeta\theta - \tfrac{1}{4}\eta\eta = \lambda.$$

Then for n take of the Terms 2ζ, η, 2θ, fome common Integer Divifor, that is not a Square, and that likewife is not divifible by a fquare Number, and which alfo divided by the Number 4, fhall leave Unity; provided any one of the Terms p, r, t be odd. For k take fome Integer Divifor of the Quantity $\frac{\lambda}{2nn}$ if p be even, or the half of an odd Divifor if p be odd, or 0 if λ be 0. For Q take the Quantity $\frac{1}{2}\alpha + \frac{1}{2}nkk$. For l fome Divifor of the Quantity $\frac{Qr - QQp - t}{n}$ if Q be an Integer; or the half of an odd Divifor if Q be a Fraction that has for its Denominator the Number 2; or 0, if that Dividual $\frac{Qr - QQp - t}{n}$ be nothing. And for R the Quantity $\frac{1}{2}r - \frac{1}{2}Qp + nkl$. Then try if $RR - v$ can be divided by n, and the Root of the Quotient extracted; and befides, if that Root be equal as well to the Quantity

Q R

$\frac{QR - \frac{1}{2}t}{nl}$ as to the Quantity $\frac{QQ + pR - nll - s}{2nk}$. If all these happen, call that Root m; and in room of the Equation proposed, write this, $x^3 + \frac{1}{2}pxx + Qx + r = \pm \sqrt{n} \times \overline{kxx + lx + m}$. For this Equation, by squaring its Parts, and taking from both Sides the Terms on the Right-Hand, will produce the Equation proposed. Now if all these Things do not happen in the Case proposed, the Reduction will be impossible, provided it appears beforehand that the Equation cannot be reduced by a rational Divisor.

For Example, let there be proposed the Equation $x^6 - 2ax^5 + 2bbx^4 + 2abbx^3 \begin{matrix} -2aabb \\ +2a^3b \\ -4ab^3 \end{matrix} xx \begin{matrix} +3aab^4 \\ -a^4bb \end{matrix} = 0$, and by writing $-2a$, $+2bb$, $+2abb$, $-2aabb + 2a^3b - 4ab^3$, 0, and $3aab^4 - a^4bb$ for p, q, r, s, t, and v respectively, there will come out $2bb - aa = \alpha$. $4abb - a^3 = \beta$. $2a^3b + 2aabb - 4ab^3 - a^4 = \gamma$. $-b^4 + 2a^3b + 3aabb - 4ab^3 - \frac{5}{4}a^4 = \zeta$. $-\frac{1}{2}a^5 - 3a^3bb - 4ab^4 = \eta$, and $-aab^4 + a^4bb - \frac{1}{4}a^6 = \theta$. And the common Divisor of the Terms 2ζ, η, and 2θ, is $aa - 2bb$, or $2bb - aa$, according as aa or $2bb$ is the greater. But let aa be greater than $2bb$, and $aa - 2bb$ will be n. For n must always be Affirmative. Moreover, $\frac{\zeta}{n}$ is $-\frac{5}{4}aa + 2ab + \frac{1}{2}bb$, $\frac{\eta}{n}$ is $-\frac{1}{2}a^3 + 2abb$ and $\frac{\theta}{n}$ is $-\frac{1}{4}a^4 + \frac{1}{2}aabb$, and consequently $\frac{\zeta}{2n} \times \frac{\theta}{n} - \frac{\eta\eta}{8nn}$ or $\frac{\lambda}{2nn}$, is $\frac{1}{8}a^6 - \frac{1}{4}a^5b - \frac{1}{8}a^4bb + \frac{1}{2}a^3b^3 - \frac{1}{8}aab^4$, the Divisors whereof are 1, a, aa; but because $\sqrt{n} \times k$ cannot be of more than one Dimension, and the $\sqrt{n}$ is of one, therefore k will be of none; and consequently can only be a Number. Wherefore, rejecting a and aa, there remains only 1 for k. Besides, $\frac{1}{2}\alpha + \frac{1}{2}nkk$ gives 0 for Q, and $\frac{Qr - QQp - t}{n}$ is also nothing; and consequently l, which ought to be its Divisor, will be nothing. Lastly, $\frac{1}{2}r - \frac{1}{2}pQ + nkl$ gives abb for R. And RR $- v$ is $-2aab^4 + a^4bb$, which may be divided by n or $aa - 2bb$, and the Root of the Quotient

$aabb$

$aabb$ be extracted, and that Root taken Negatively, *viz.* $-ab$, is not unequal to the indefinite Quantity $\frac{QR-\frac{1}{2}pt}{nl}$, or $\frac{0}{0}$, but equal to the definite Quantity $\frac{QQ+pR-nll-s}{2nk}$. Wherefore that Root $-ab$ will be m, and in the room of the Equation proposed, there may be writ $x^3-\frac{1}{2}pxx+Qx+R=\sqrt{n}\times\overline{kxx+lx+m}$, that is, $x^3-axx+abb=\sqrt{aa-2bb}\times\overline{xx-ab}$. The Truth of which Conclusion you may prove by squaring the Parts of the Equation found, and taking away the Terms on the Right Hand from both Sides. For from that Operation will be produced the Equation $x^6-2ax^5+2bbx^4+2abbx^3-2aabbxx+2a^3bxx-4ab^3xx+3aab^4-a^4bb=0$, which was proposed to be reduced.

If the Equation is of eight *Dimensions, let it be* $x^8+px^7+qx^6+rx^5+sx^4+tx^3+vxx+wx+z=0$, and make $q-\frac{1}{4}pp=\alpha$. $r-\frac{1}{2}p\alpha=\beta$. $s-\frac{1}{2}p\beta-\frac{1}{4}\alpha\alpha=\gamma$. $t-\frac{1}{2}p\gamma-\frac{1}{2}\alpha\beta=\delta$. $v-\frac{1}{2}\alpha\gamma-\frac{1}{4}\beta\beta=\varepsilon$. $w-\frac{1}{2}\beta\gamma=\zeta$, and $z-\frac{1}{4}\gamma\gamma=\eta$. And seek of the Terms 2δ, 2ε, 2ζ, 8η, a common Divisor that shall be an Integer, and neither a Square Number, nor divisible by a Square Number, and which also divided by 4 shall leave Unity, provided any of the alternate Terms, p, r, t, w be odd. If there be no such common Divisor, it is certain, that the Equation cannot be reduced by the Extraction of a Quadratick Surd Root; and if it cannot be so reduced, there will scarce be found a common Divisor of all those four Quantities. The Operation therefore hitherto is a Sort of an Examination, whether the Equation be reducible or not; and consequently, since that Sort of Reductions are seldom possible, it will most commonly end the Work.

And, by a like Reason, if the Equation be of ten, twelve, or more Dimensions, the Impossibility of its Reduction may be known.

As if it be $x^{10}+px^9+qx^8+rx^7+sx^6+tx^5+vx^4+ax^3+bx^2+cx+d=0$, you must make $q-\frac{1}{4}pp=\alpha$, $r-\frac{1}{2}p\alpha=\beta$, $s-\frac{1}{2}p\beta-\frac{1}{4}\alpha\alpha=\gamma$, $t-\frac{1}{2}p\gamma-\frac{1}{2}\alpha\beta=\delta$, $v-\frac{1}{2}p\delta-\frac{1}{2}\alpha\gamma-\frac{1}{4}\beta\beta=\varepsilon$, $a-\frac{1}{2}\alpha\delta-\frac{1}{2}\beta\gamma=\zeta$, $b-\frac{1}{2}\beta\delta-\frac{1}{4}\gamma\gamma=\eta$, $c-\frac{1}{2}\gamma\delta=\theta$, $d-\frac{1}{4}\delta\delta=\kappa$, and seek such a common Divisor to the five Terms,

Terms, 2ϵ, 2ζ, 8η, 4θ, 8κ, as is an Integer, and not a Square, and which shall also leave 1 when divided by 4, if any one of the Terms p, r, t, a, c be odd.

So if there be an Equation of *twelve* Dimensions, as $x^{12} + px^{11} + qx^{10} + rx^{9} + sx^{8} + tx^{7} + vx^{6} + ax^{5} + bx^{4} + cx^{3} + dx^{2} + ex + f = 0$, make $q - \frac{1}{4}pp = \alpha$, $r - \frac{1}{2}p\alpha = \beta$, $s - \frac{1}{2}p\beta - \frac{1}{4}\alpha\alpha = \gamma$, $t - \frac{1}{2}p\gamma - \frac{1}{2}\alpha\beta = \delta$, $v - \frac{1}{2}p\delta - \frac{1}{2}\alpha\gamma - \frac{1}{4}\beta\beta = \epsilon$, $a - \frac{1}{2}p\epsilon - \frac{1}{2}\alpha\delta - \frac{1}{2}\beta\gamma = \zeta$, $b - \frac{1}{2}\alpha\epsilon - \frac{1}{2}\beta\delta - \frac{1}{4}\gamma\gamma = \eta$, $c - \frac{1}{2}\beta\epsilon - \frac{1}{2}\gamma\delta = \theta$, $d - \frac{1}{2}\gamma\epsilon - \frac{1}{4}\delta\delta = \kappa$, $e - \frac{1}{2}\delta\epsilon = \lambda$, $f - \frac{1}{4}\epsilon\epsilon = \mu$, and you must seek a common Integer Divisor of the six Terms 2ζ, 8η, 4θ, 8κ, 4λ, 8μ, that is not a Square, but being divided by 4 shall leave Unity, provided any one of the Terms p, r, t, a, c, e be odd.

And thus you may go on *ad infinitum*, and the proposed Equation when it has no common Divisor, will be alway irreduceable by the Extraction of the surd quadratick Root. But if at any Time such a Divisor n being found, there are Hopes of a future Reduction, it may be tried by following the Steps of the Operation we shewed in an Equation of *eight* Dimensions.

Seek a Square Number, to which after it is multiplied by n, the last Term z of the Equation being added under its proper Sign, shall make a Square Number. But that may be expeditiously performed if you add to z, when n is an even Number, or to $4z$ when it is odd, these Quantities successively n, $3n$, $5n$, $7n$, $9n$, $11n$, and so on till the Sum becomes equal to some Number in the Table of Square Numbers, which I suppose to be ready at Hand. And if no such Square Number occurs before the Square Root of that Sum, augmented by the Square Root of the Excess of that Sum above the last Term of the Equation, is four times greater than the greatest of the Terms of the proposed Equation p, q, r, s, t, v, &c. there will be no Occasion to try any farther. For then the Equation cannot be reduced. But if such a Square Number does accordingly occur, let its Root be S, if n is even, or 2 S if n be odd; and call $\sqrt{\frac{SS - z}{n}} = h$.

But s and h ought to be Integers if n is even, but if n is odd, they may be Fractions that have 2 for their Denominator. And if one is a Fraction, the other ought to be so too. Which also is to be observed of the Numbers R and m, Q and l, p and k hereafter to be found. And all the Numbers

 S and

S and h, that can be found within the prescribed Limit, must be collected in a Catalogue.

Afterwards, for k all the Numbers are to be successively tried, which do not make $nk + \frac{1}{2}p$ four times greater than the greatest Term of the Equation, and you must in all Cases put $\frac{nkk + \alpha}{2} = Q$. Then you are to try successively for l all the Numbers that do not make $nl + Q$ four times greater than the greatest Term of the Equation, and in every Trial put $\frac{-npkk + 2\beta}{4} + nkl = R$. Lastly, for m you must try successively all the Numbers which do not make $nm + R$ four times greater than the greatest of the Terms of the Equation, and you must see whether in any Case if you make $s - QQ - pR + nll = 2H$, and $H + nkm = S$, let S be some of the Numbers which were before brought into the Catalogue for S; and besides, if the other Number answering to that S, which being set down for h in the same Catalogue, will be equal to these three, $\frac{2RS - w}{2nm}$, $\frac{2QS + RR - v - nmm}{2nl}$, and $\frac{pS + 2QR - t - 2nlm}{2nk}$. If all these Things shall happen in any Case, instead of the Equation proposed, you must write this $x^4 + \frac{1}{2}px^3 + Qxx + Rx + S = \sqrt{n} \times \overline{kx^3 + lxx + mx + h}$.

For Example, Let there be proposed the Equation $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10xx - 10x - 5 = 0$, and you will have $q - \frac{1}{4}pp = -1 - 4 = -5 = \alpha$. $r - \frac{1}{2}p\alpha = -10 + 10 = 0 = \beta$. $s - \frac{1}{2}p\beta - \frac{1}{4}\alpha\alpha = 5 - \frac{25}{4} = -\frac{5}{4} = \gamma$. $t - \frac{1}{2}p\gamma - \frac{1}{2}\alpha\beta = -5 + \frac{5}{2} = -\frac{5}{2} = \delta$. $v - \frac{1}{2}\alpha\gamma - \frac{1}{4}\beta\beta = -10 - \frac{25}{8} = -\frac{105}{8}$. $w - \frac{1}{4}\beta\gamma = -10 = \zeta$. $z - \frac{1}{4}\gamma\gamma = -5 - \frac{25}{64} = -\frac{345}{64} = \eta$. Therefore 2δ, 2ϵ, 2ζ, 8η respectively are -5, $-\frac{105}{4}$, -20, and $-\frac{345}{8}$, and their common Divisor 5, which divided by 4, leaves 1, as it ought, because the Term s is odd. Since therefore the common Divisor n, or 5, is found, which gives hope to a future Reduction, and because it is odd, to $4z$, or -20, I successively add n, $3n$, $5n$, $7n$, $9n$, &c. or 5, 15, 25, 35, 45, &c. and there arises -15, 0, 25, 60, 105, 160, 225, 300, 385, 480, 585, 700, 825, 960, 1105, 1260, 1425, 1600. Of which only 0, 25, 225, and 1600 are Squares. Where-

Wherefore the Halves of these Roots 0, $\frac{5}{2}$, $\frac{15}{2}$, 20, are to be collected in a Table for the Values of S, and the Values of $\sqrt{\frac{SS - z}{n}}$, that is, 1, $\frac{3}{2}$, $\frac{7}{2}$, 9, respectively for h. But because $S + nh$, if 20 be taken for S and 9 for h, becomes 65, a Number greater than four times the greatest Term of the Equation; therefore I reject 20 and 9, and write only the rest in the Table as follows:

h	$1 \cdot \frac{3}{2} \cdot \frac{7}{2} \cdot$
S	$0 \cdot \frac{5}{2} \cdot \frac{15}{2} \cdot$

Then I try for k all the Numbers which do not make $\frac{1}{2}p + nk$, or $2 + 5k$, greater than 40, (four times the greatest Term of the Equation) that is, the Numbers -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, putting $\frac{nkk + \alpha}{2}$, or $\frac{5kk - 5}{2}$, that is, the Numbers $\frac{315}{2}$, 120, $\frac{175}{2}$, 60, $\frac{75}{2}$, 20, $\frac{15}{2}$, 0, $-\frac{5}{2}$, 0, $\frac{15}{2}$, 20, $\frac{75}{2}$, 60, $\frac{175}{2}$, 120, respectively for Q. But even since $Q + nl$, and much more Q, ought not to be greater than 40, I perceive I am to reject $\frac{315}{2}$, 120, $\frac{175}{2}$, and 60, and their Correspondents -8, -7, -6, -5, 5, 6, 7, and consequently that only -4, -3, -2, -1, 0, 1, 2, 3, 4, must respectively be tried for k, and $\frac{75}{2}$, 20, $\frac{15}{2}$, 0, $-\frac{5}{2}$, 0, $\frac{15}{2}$, 20, $\frac{75}{2}$, respectively for Q. Let us therefore try -1 for k, and 0 for Q, and in this Case for l there will be successively to be tried all the Numbers which do not make $Q + nl$ greater than 40, that is, all the Numbers between 10 and -10; and for R you are respectively to try the Numbers $\frac{2\beta - npkk}{4} + nkl$, or $-5 - 5l$, that is, -55, -50, -45, -40, -35, -30, -25, -20, -15, -10, -5, 0, 5, 10, 15, 20, 25, 35, 40, 45, the three former of which and the last, because they are greater than 40, may be neglected. Let us try therefore -2 for l, and 5 for R, and in this Case for m there will be besides to be tried all the Numbers which do not make $R + nm$, or $5 + 5m$, greater than 40, that is, all the Numbers between 7 and -9, and see whether or not by putting $s - QQ - pR + nll$, that is $5 - 20 + 20$ or $5 = 2H$, it may be $H + nkm$ or $\frac{5}{2} - 5m = S$; that is, if any of these

these Numbers $\frac{-65}{2}$, $\frac{-55}{2}$, $\frac{-45}{2}$, $\frac{-35}{2}$, $\frac{-25}{2}$, $\frac{-15}{2}$, $\frac{-5}{2}$, $\frac{5}{2}$, $\frac{15}{2}$, $\frac{25}{2}$, $\frac{35}{2}$, $\frac{45}{2}$, $\frac{55}{2}$, $\frac{65}{2}$, $\frac{75}{2}$, $\frac{85}{2}$, is equal to any of the Numbers o, $\pm\frac{5}{2}$, $\pm\frac{15}{2}$, which were first brought into the Catalogue for S. And we meet with four of these $-\frac{15}{2}$, $-\frac{5}{2}$, $\frac{5}{2}$, $\frac{15}{2}$, to which answer $\pm\frac{7}{2}$, $\pm\frac{3}{2}$, $\pm\frac{3}{2}$, $\pm\frac{7}{2}$, written for h in the same Table, as also 2, 1, 0, -1 substituted for m. But let us try $-\frac{5}{2}$ for S, 1 for m, and $\pm\frac{3}{2}$ for h, and you will have $\frac{2RS-w}{2nm} = \frac{-25+10}{10} = -\frac{3}{2}$, and $\frac{2QS+RR-v-nmm}{2nl} = \frac{25+10-5}{-20} = -\frac{3}{2}$, and $\frac{pS+2QR-t-2nlm}{2nk} = \frac{-10+5+20}{-10} = -\frac{3}{2}$. Wherefore, since there comes out in all Cases $-\frac{3}{2}$, or h, I conclude all the Numbers to be rightly found, and consequently that in room of the Equation proposed, you must write $x^4 + \frac{1}{2}px^3 + Qxx + Rx + S = \sqrt{n} \times \overline{kx^3 + lxx + mx + h}$, that is, $x^4 + 2x^3 + 5x - 2\frac{1}{2} = \sqrt{5} \times \overline{-x^3 - 2xx + x - 1\frac{1}{2}}$. For by squaring the Parts of this, there will be produced that Equation of eight Dimensions, which was at first proposed.

But if by trying all the Cases of the Numbers, all the aforesaid Values of h had not in any Case consented, it would be an Argument that the Equation could not be reduced by the Extraction of the Surd Quadratick Root.

But something might be here remarked for the abbreviating of the Work, which however I pass over for the sake of Brevity, seeing the Use of so great Reductions is very little, and I was willing to shew rather the Possibility of the Thing, than a Practice that was commodious. These therefore are the Reductions of Equations by the Extraction of the *Surd Quadratick Root.*

I might now joyn the Reductions of Equations by the *Extraction of the Surd Cubick Root*, but these, as being seldom of Use, for Brevity I pass by.

Yet there are some Reductions of *Cubick Equations* commonly known, which, if I should wholly pass over, the Reader might perhaps think us deficient. Let there be proposed the

the Cubick Equation $x^3 * + qx + r = 0$, the second Term whereof is wanting. For that every Cubick Equation may be reduced to this Form, is evident from what we have said above. Let x be supposed $= a + b$. Then will be $a^3 + 3aab + 3abb + b^3$ (that is x^3) $+ qx + r = 0$. Let $3aab + 3abb$ (that is, $3abx$) $+ qx$ be $= 0$, and then will $a^3 + b^3 + r$ be $= 0$. By the former Equation b is $= -\frac{q}{3a}$, and cubically $b^3 = -\frac{q^3}{27a^3}$. Therefore, by the latter, $a^3 - \frac{q^3}{27a^3} + r$ is $= 0$, or $a^6 + ra^3 = \frac{q^3}{27}$, and by the Extraction of the adfected Quadratick Root, $a^3 = -\frac{1}{2}r \pm \sqrt{\frac{1}{4}rr + \frac{q^3}{27}}$. Extract the Cubick Root and you will have a. And above, you had $-\frac{q}{3a} = b$, and $a + b = x$. Therefore $a - \frac{q}{3a}$ is the Root of the Equation proposed.

For Example, let there be proposed the Equation $y^3 - 6yy + 6y + 12 = 0$. To take away the second Term of this Equation, make $x + 2 = y$, and there will arise $x^3 * - 6x + 8 = 0$. Where q is $= -6$, $r = 8$, $\frac{1}{4}rr = 16$, $\frac{q^3}{27} = -8$, $a^3 = -4 \pm \sqrt{8}$, $a - \frac{q}{3a} = x$, and $x + 2 = y$, that is, $2 + \sqrt[3]{-4 \pm \sqrt{8}} + \frac{2}{\sqrt[3]{-4 \pm \sqrt{8}}} = y$.

And after this Way the Roots of all Cubical Equations may be extracted wherein q is Affirmative; or also wherein q is Negative, and $\frac{q^3}{27}$ not greater than $\frac{1}{4}rr$, that is, *when two of the Roots of the Equation are impossible.* But where q is Negative, and $\frac{q^3}{27}$ at the same time greater than $\frac{1}{4}rr$, $\sqrt{\frac{1}{4}rr - \frac{q^3}{27}}$ becomes an impossible Quantity; and so so the Root of the Equation x or y will, in this Case, be impossible, *viz.* in this Case *there are three possible Roots*, which all of them are alike with respect to the Terms of the Equations q and r, and are indifferently denoted by the Letters x and y, and consequently all of them may be extracted by the same Method, and expressed the same Way as any one is extracted or expressed; *but it is impossible to express all three by the Law aforesaid.* The Quantity $a - \frac{q}{3a}$, whereby x is denoted, cannot be manifold, and for that Reason the Supposition that x, in this Case wherein it is threefold, may be equal to the Binomial $a - \frac{q}{3a}$, or $a + b$, the Cubes of whose Terms $a^3 + b^3$ may together be $= r$, and the triple Rectangle $3ab$ be $= q$, is plainly impossible; and it is no Wonder that from an impossible Hypothesis, an impossible Conclusion should follow.

There is, moreover, another Way of expressing these Roots, *viz.* from $a^3 + b^3 + r$, that is, from nothing take $a^3 + r$, or $\frac{1}{2}r \pm \sqrt{\frac{1}{4}rr + \frac{q^3}{27}}$, and there will remain $b^3 = -\frac{1}{2}r \mp \sqrt{\frac{1}{4}rr + \frac{q^3}{27}}$. Therefore a is $=$

$\sqrt[3]{-\frac{1}{2}r + \sqrt{\frac{1}{4}rr + \frac{q^3}{27}}}$, and $b =$

$\sqrt[3]{-\frac{1}{2}r - \sqrt{\frac{1}{4}rr + \frac{q^3}{27}}}$; or $a =$

$\sqrt[3]{-\frac{1}{2}r - \sqrt{\frac{1}{4}rr + \frac{q^3}{27}}}$, and $b =$

$\sqrt[3]{-\frac{1}{2}r + \sqrt{\frac{1}{4}rr + \frac{q^3}{27}}}$, and consequently the Sum of these $\sqrt[3]{-\frac{1}{2}r + \sqrt{\frac{1}{4}rr + \frac{q^3}{27}}} +$

$\sqrt[3]{-\frac{1}{2}r - \sqrt{\frac{1}{4}rr + \frac{q^3}{27}}}$ will be $= x$.

Moreover, the Roots of Biquadratick Equations may be extracted and expressed by means of Cubick ones.

But first you must take away the second Term of the Equation. Let the Equation that then results be $x^4+qxx+rx+s=0$. Suppose this to be generated by the Multiplication of these two $xx+ex+f=0$, and $xx-ex+g=0$, that is, to be the same with this $x^4 * \begin{matrix}+f\\+g\\-ee\end{matrix} xx \begin{matrix}+eg\\-ef\end{matrix} x + fg = 0$, and comparing the Terms you will have $f+g-ee=q$, $eg-ef=r$, and $fg=s$. Wherefore $q+ee=f+g$, $\frac{r}{e}=g-f$, $\frac{q+ee+\frac{r}{e}}{2}=g$, $\frac{q+ee-\frac{r}{e}}{2}=f$, $\frac{qq+2eeq+e^4-\frac{rr}{ee}}{4}(=fg)=s$, and by Reduction $e^6+2qe^4 \begin{matrix}+qq\\-4s\end{matrix} ee-rr=0$. For ee write y, and you will have $y^3+2qyy \begin{matrix}+qq\\-4s\end{matrix} y-rr=0$, a Cubick Equation, whose second Term may be taken away, and then the Root extracted either by the precedent Rule or otherwise. Then that Root being had, you must go back again, by putting $\sqrt{y}=e$, $\frac{q+ee-\frac{r}{e}}{2}=f$, $\frac{q+ee+\frac{r}{e}}{2}=g$, and the two Equations $xx+ex+f=0$, and $xx-ex+g=0$, their Roots being extracted, will give the four Roots of the Biquadratick Equation $x^4+qxx+rx+s=0$, *viz.* $x=-\frac{1}{2}e\pm\sqrt{\frac{1}{4}ee-f}$, and $x=\frac{1}{2}e\pm\sqrt{\frac{1}{4}ee-g}$. Where note, that if the four Roots of the Biquadratick Equation are possible, the three Roots of the Cubick Equation $y^3+2qyy \begin{matrix}+qq\\-4s\end{matrix} y-rr=0$ will be possible also, and consequently cannot be extracted by the precedent Rule. *And thus, if the affected Roots of an Equation of five or more Dimensions are converted into Roots that are not affected, the middle Terms of the Equation being some way or other taken away, that Ex-*

pression

pression of the Roots will be always impossible, where more than one Root in an Equation of odd Dimensions are possible, or more than two in an Equation of even Dimensions, which cannot be reduced by the Extraction of the Surd Quadratick Root, by the Method laid down above.

Monsieur *Des Cartes* taught how to reduce a Biquadratick Equation by the Rules last delivered. *E. g.* Let there be proposed the Equation reduced above, $x^4 - x^3 - 5xx + 12x - 6 = 0$. Take away the second Term, by writing $v + \frac{1}{4}$ for x, and there will arise $v^4 - \frac{43}{8}vv + \frac{75}{8}v - \frac{851}{256} = 0$. To take away the Fractions, write $\frac{1}{4}z$ for v, and there will arise $z^4 - 86zz + 600z - 851 = 0$. Here is $-86 = q$, $600 = r$, and $-851 = s$, and consequently $y^3 + 2qyy \begin{matrix} +qq \\ -4s \end{matrix} y - rr = 0$, substituting what is equivalent, will become $y^3 - 172yy + 10800y - 360000 = 0$. Where trying all the Divisors of the last Term 1, −1, 2, −2, 3, −3, 4, −4, 5, −5, and so onwards to 100, you will find at length $y = 100$. Which yet may be found far more expeditiously by our Method above delivered. Then having got y, its Root 10 will be e, and

$$\frac{q + ee - \frac{r}{e}}{2}, \text{ that is, } \frac{-86 + 100 - 60}{2} \text{ or } -23 \text{ will be } f,$$

and $\frac{q + ee + \frac{r}{e}}{2}$ or 37 will be g, and consequently the Equations $xx + ex + f = 0$, and $xx - ex + g = 0$, writing z for x, and substituting equivalent Quantities, will become $zz + 10z - 23 = 0$, and $zz - 10z + 37 = 0$. Restore v in the room of $\frac{1}{4}z$, and there will arise $vv + 2\frac{1}{2}v - \frac{23}{16} = 0$, and $vv - 2\frac{1}{2}v + \frac{37}{16} = 0$. Restore, moreover, $x - \frac{1}{4}$ for v, and there will come out $xx + 2x - 2 = 0$, and $xx - 3x + 3 = 0$, two Equations, the four Roots whereof $x = -1 \pm \sqrt{3}$, and $x = 1\frac{1}{2} \pm \sqrt{-\frac{3}{4}}$, are the same with the four Roots of the Biquadratick Equation proposed at the Beginning, $x^4 - x^3 - 5xx + 12x - 6 = 0$. But these might have been more easily found by the Method of finding Divisors, explained before.

A P-

APPENDIX

THE LINEAR CONSTRUCTION OF *EQUATIONS.*

HITHERTO I have shewn the *Properties, Trasmutations, Limits,* and *Reductions* of all Sorts of Equations. I have not always joyned the Demonstrations, because they seemed too easy to need it, and sometimes cannot be laid down without too much Tediousness. It remains now only to shew, how, after Equations are reduced to their most commodious Form, their Roots may be extracted in Numbers. And here the chief *Difficulty* lies in obtaining the two or three first Figures; which may be most commodiously done by either the Geometrical or Mechanical Construction of an Equation. Wherefore I shall subjoin some of these Constructions.

The Antients, as we learn from *Pappus*, at first in vain endeavoured at the Trisection of an Angle, and the finding out of two mean Proportionals by a right Line and a Circle. Afterwards they began to consider several other Lines, as the Conchoid, the Cissoid, and the Conick Sections, and by some of these to solve those Problems. At length, having more

more throughly examined the Matter, and the Conick Sections being received into Geometry, they distinguished Problems into three Kinds; *viz.* Into *Plane ones*, which deriving their Original from Lines on a Plane, may be solved by a right Line and a Circle, into *Solid ones*, which were solved by Lines deriving their Original from the Consideration of a Solid, that is, of a Cone: And *Linear ones*, to the Solution of which were required Lines more compounded. And according to this Distinction, we are not to solve solid Problems by other Lines than the Conick Sections; especially if no other Lines but right ones, a Circle, and the Conick Sections, must be received into Geometry. But the Moderns advancing yet much farther, have received into Geometry all Lines that can be expressed by Equations, and have distinguished, according to the Dimensions of the Equations, those Lines into Kinds; and have made it a Law, that you are not to construct a Problem by a Line of a superior Kind, that may be constructed by one of an inferior one. In the Contemplation of Lines, and finding out their Properties, I approve of their Distinction of them into Kinds, according to the Dimensions of the Equations by which they are defined. But it is not the Equation, but the Description that makes the Curve to be a Geometrical one. The Circle is a Geometrical Line, not because it may be expressed by an Equation, but because its Description is a Postulate. It is not the Simplicity of the Equation, but the Easiness of the Description, which is to determine the Choice of our Lines for the Construction of Problems. For the Equation that expresses a Parabola, is more simple than That that expresses a Circle, and yet the Circle, by reason of its more simple Construction, is admitted before it. The Circle and the Conick Sections, if you regard the Dimension of the Equations, are of the same Order, and yet the Circle is not numbered with them in the Construction of Problems, but by reason of its simple Description, is depressed to a lower Order, *viz.* that of a right Line; so that it is not improper to construct that by a Circle that may be constructed by a right Line. But it is a Fault to construct that by the Conick Sections which may be constructed by a Circle. Either therefore you must fix the Law to be observed in a Circle from the Dimensions of Equations, and so take away as vitious the Distinction between Plane and Solid Problems; or else you must grant, That that Law is not so strictly to be observed in Lines of superior Kinds, but that some, by reason of their more simple Description, may be preferred

to

to others of the ſame Order, and may be numbered with Lines of inferior Orders in the Conſtruction of Problems. In *Conſtructions* that are equally Geometrical, the moſt ſimple are always to be preferred. This Law is beyond all Exception. But Algebraick Expreſſions add nothing to the Simplicity of the Conſtruction. The bare Deſcriptions of the Lines only are here to be conſidered. Theſe alone were conſidered by thoſe Geometricians who joyned a Circle with a right Line. And as theſe are eaſy or hard, the Conſtruction becomes eaſy or hard. And therefore it is foreign to the Nature of the Thing, from any thing elſe to eſtabliſh Laws about Conſtructions. Either therefore let us, with the Antients, exclude all Lines beſides a right Line, the Circle, and perhaps the Conick Sections, out of Geometry, or admit all, according to the Simplicity of the Deſcription. If the Trochoid were admitted into Geometry, we might, by its Means, divide an Angle in any given Ratio. Would you therefore blame thoſe who ſhould make uſe of this Line to divide an Angle in the Ratio of one Number to another, and contend that this Line was not defined by an Equation, but that you muſt make uſe of ſuch Lines as are defined by Equations? If therefore, when an Angle was to be divided, for Inſtance, into 10001 Parts, we ſhould be obliged to bring a Curve defined by an Equation of above an hundred Dimenſions to do the Buſineſs; which no Mortal could deſcribe, much leſs underſtand: and ſhould prefer this to the Trochoid, which is a Line well known, and deſcribed eaſily by the Motion of a Wheel or a Circle, who would not ſee the Abſurdity? Either therefore the Trochoid is not to be admitted at all into Geometry, or elſe, in the Conſtruction of Problems, it is to be preferred to all Lines of a more difficult Deſcription. And there is the ſame Reaſon for other Curves. For which Reaſon we approve of the Triſections of an Angle by a Conchoid, which *Archimedes* in his Lemma's, and *Pappus* in his Collections, have preferred to the Inventions of all others in this Caſe; becauſe we ought either to exclude all Lines, beſides the Circle and right Line, out of Geometry, or admit them according to the Simplicity of their Deſcriptions, in which Caſe the Conchoid yields to none, except the Circle. Equations are Expreſſions of Arithmetical Computation, and properly have no Place in Geometry, except as far as Quantities truly Geometrical (that is, Lines, Surfaces, Solids, and Proportions) may be ſaid to be ſome equal to others. Multiplications, Diviſions, and ſuch

such sort of Computations, are newly received into Geometry, and that unwarily, and contrary to the first Design of this Science. For whosoever considers the Construction of Problems by a right Line and a Circle, found out by the first Geometricians, will easily perceive that Geometry was invented that we might expeditiously avoid, by drawing Lines, the Tediousness of Computation. Therefore these two Sciences ought not to be confounded. The Antients did so industriously distinguish them from one another, that they never introduced Arithmetical Terms into Geometry. And the Moderns, by confounding both, have lost the Simplicity in which all the Elegancy of Geometry consists. Wherefore that is *Arithmetically* more simple which is determined by the more simple Equations, but that is *Geometrically* more simple which is determined by the more simple drawing of Lines; and in Geometry, that ought to be reckoned best which is Geometrically most simple. Wherefore, I ought not to be blamed, if, with that Prince of Mathematicians, *Archimedes*, and other Antients, I make use of the Conchoid for the Construction of solid Problems. But if any one thinks otherwise, let him know, that I am here sollicitous not for a Geometrical Construction, but any one whatever, by which I may the nearest Way find the Roots of the Equations in Numbers. For the sake whereof I here premise this Lemmatical Problem.

To place the right Line BC *of a given Length, so between two other given Lines* AB, AC, *that being produced, it shall pass through the given Point* P. [See *Figure* 90.]

If the Line BC turn about the Pole P, and at the same time moves on its End C upon the right Line AC, its other End B shall describe the Conchoid of the Antients. Let this cut the Line AB in the Point B. Join PB, and its Part BC will be the right Line which was to be drawn. And, by the same Law, the Line BC may be drawn, where, instead of AC, some Curve Line is made use of.

If any do not like this Construction by a Conchoid, another, done by a Conick Section, may be substituted in its room. From the Point P to the right Line AD, AE, draw PD, PE, making the Parallelogram EADP, and from the Points C and D to the right Lines AB let fall the Perpendiculars CF, DG, as also from the Point E to the right Line

Line AC, produced towards A, let fall the Perpendicular EH, and making $AD = a$, $PD = b$, $BC = c$, $AG = d$, $AB = x$, and $AC = y$, you will have $AD : AG :: AC : AF$, and consequently $AF = \frac{dy}{a}$. Moreover, you will have $AB : AC :: PD : CD$, or $x : y :: b : a - y$. Therefore $by = ax - yx$, which is an Equation expressive of an Hyperbola. And again, by the 13th of the 2d *Elem.* BCq will be $= ACq + ABq - 2FAB$, that is, $cc = yy + xx - \frac{2dxy}{a}$. Both Sides of the former Equation being multiplied by $\frac{2d}{a}$, take them from both Sides of this, and there will remain $cc - \frac{2bdy}{a} = yy + xx - 2dx$, an Equation expressing a Circle, where x and y are at right Angles. Wherefore, if you make these two Lines an Hyperbola and a Circle, by the Help of these Equations, by their Intersection you will have x and y, or AB and AC, which determine the Position of the right Line BC. But those right Lines will be compounded after this Way. [*See* Figure 91.]

Draw any two right Lines, KL equal to AD, and KM equal to PD, containing the right Angle MKL. Compleat the Parallelogram KLMN, and with the Asymptotes LN, MN, describe through the Point K the Hyperbola IKX.

On KM produced towards K, take KP equal to AG, and KQ equal to BC. And on KL produced towards K, take KR equal to AH, and RS equal to RQ. Compleat the Parallelogram PKRT, and from the Center T, at the Interval TS, describe a Circle. Let that cut the Hyperbola in the Point X. Let fall to KP the Perpendicular XY, and XY will be equal to AC, and KY equal to AB. Which two Lines, AC and AB, or one of them, with the Point P, determine the Position sought of the right Line BC. To demonstrate which Construction, and its Cases, according to the different Cases of the Problem, I shall not here insist.

I say, by this Construction, if you think fit, you may solve the Problem. But this Solution is too compounded to serve for any particular Uses. It is a bare Speculation, and

and Geometrical Speculations have just as much Elegancy as Simplicity, and deserve just so much Praise as they can promise Use. For which Reason, I prefer a Construction by the Conchoid, as much the simpler, and not less Geometrical; and which is of especial Use in the Resolution of Equations as by us proposed. Premising therefore the preceding Lemma, we Geometrically construct Cubick and Biquadratick Problems [*as which may be reduced to Cubick ones*] as follows. [*See* Figures 92 and 93.]

Let there be proposed the Cubick Equation $x^3 * + qx + r = 0$, *whose second Term is wanting*, but the third is *denoted under its Sign* $+q$, *and the fourth by* $+r$.

Draw any right Line, K A, which call n. On K A, produced on both Sides, take K B $= \frac{q}{n}$ to the same Side as K A, if it be $+q$, otherwise to the contrary Part. Bisect B A in C, and on K, as a Center with the Radius K C, describe the Circle C X, and in it accommodate the right Line C X equal to $\frac{r}{nn}$, producing it each Way. Join A X, which produce also both Ways. Lastly, between these Lines C X and A X inscribe E Y of the same Length as C A, and which being produced, may pass through the Point K; then shall X Y be the Root of the Equation. [*See* Figure 94.] And of these Roots, those will be Affirmative which fall from X towards C, and those Negative which fall on the contrary Side, if it be $+r$, but contrarily if it be $-r$.

Demonstration.

To demonstrate which, I premise these Lemma's.

LEMMA I.

Y X *is to* A K *as* C X *to* K E. For draw K F parallel to C X; then because of the similar Triangles A C X, A K F, and E Y X, E K F, it will be A C to A K as C X to K F, and Y X to Y E or A C as K F to K E, and therefore by perturbated Equality Y X to A K as C X to K E. Q. E. D.

LEMMA II.

Y X *is to* A K *as* C Y *to* A K + K E. For by Composition of Proportion Y X is to A K as Y X + C X (*i. e.* C Y) to A K + K E. Q. E. D.

LEMMA

LEMMA III.

KE — BK *is to* YX *as* YX *to* AK.

For (by 12. *Elem.* 2.) YK*q* — CK*q* is = CY*q* — CY × CX = CY × YX. That is, if the Theorem be resolved into Proportionals, CY to YK — CK as YK + CK to YX. But YK — CK is = YK — YE + CA — CK = KE — BK. And YK + CK = YK — YE + CA + CK = KE + AK. Wherefore CY is to KE — BK as KE + AK to YX. But by *Lemma* 2. it was CY to KE + AK as YX to AK. Wherefore by Equality it is YX to KE — BK as AK to YX. Or KE — BK to YX as YX to AK. Q. E. D.

These Things being premised, the Theorem will be thus demonstrated.

In the *first Lemma* it was YX to AK as CX to KE, or KE × YX = AK × CX. In the *third Lemma* it was proved, that KE — BK was to YX as YX to AK. Wherefore, if the Terms of the first Ratio be multiplied by YX, it will be KE × YX — BK × YX to XY*q* as YX : AK that is, AK × CX — BK × YX to YX*q* as YX to AK, and by multiplying the Extremes and Means into themselves, it will be AK*q* × XC — AK × BK × YX = YX *cube.* Lastly, for YX, AK, BK, and CX, re-substituting x, n, $\frac{q}{n}$, and $\frac{r}{nn}$, this Equation will arise, *viz.* $r - qx = x^3$. Q. E. D. I need not stay to shew you the Variations of the Signs, for they will be determined according to the different Cases of the Problem.

Let now an Equation be proposed wanting the third Term, as $x^3 + pxx + r = 0$; and in order to construct it, n being assumed, take in any right Line two Lengths KA $= \frac{r}{nn}$, and KB $= p$, and let them be taken the same Way if r and p have like Signs; but otherwise, take them towards contrary Sides. Bisect BA in C, and on K, as a Center, with the Radius KC, describe a Circle, into which accommodate CX $= n$, producing it both Ways. Join also AX, and produce it both Ways. Lastly, between these Lines CX and AX inscribe EY = CA, so that if produced it may pass through the Point K, and KE will be the Root of the Equation. And the Roots will be Affirmative, when the Point

Point Y falls on that Side of X which lies towards C; and Negative, when it falls on the contrary Side of X, provided it be $+r$; but if it be $-r$, it will be the Reverse of this.

To demonstrate this Proposition, look back to the Figures and Lemma's of the former; and then you will find it thus.

By *Lemma* 1. it was YX to AK as CX to KE, or $YX \times KE = AK \times CX$, and by *Lemma* 3, $KE - KB$ to YX as YX to AK, or, (taking KB towards contrary Parts) $KE + KB$ to YX as YX to AK, and therefore $KE + KB$ multiplied by KE will be to $YX \times KE$ (or $AK \times CX$) as YX to AK, or as CX to KE. Wherefore multiplying the Extreams and Means into themselves, KE *cube* $+ KB \times KEq$ is $= AK \times CXq$; and then for KE, KB, AK, and CK, restoring their Values assigned above, $x^3 + pxx = r$.

Let now an Equation having three Dimensions, and wanting no Term, be proposed in this Form, $x^3 + pxx + qx + r = 0$, *some of whose Roots shall be Affirmative, and some Negative.* [See *Figure* 95.]

And first suppose q *Negative*, then in any right Line, as KB, let two Lengths be taken, as $KA = \frac{r}{q}$, and $KB = p$, and take them the same Way, if p and $\frac{r}{q}$ have contrary Signs; but if their Signs are alike, then take the Lengths contrary Ways from the Point K. Bisect AB in C, and there erect the Perpendicular CX equal to the Square Root of the Term q; then between the Lines AX and CX, produced infinitely both Ways, inscribe the right Line $EY = AC$, so that being produced, it may pass through K; so shall KE be the Root of the Equation, which will be Affirmative when the Point X falls between A and E; but Negative when the Point E falls on that Side of the Point X which is towards A.

But if Q had been Affirmative, then in the Line KB you must have taken those two Lengths thus, *viz.* $KA = \sqrt{\frac{-r}{p}}$, and $KB = \frac{q}{KA}$, and the same Way from K, if $\sqrt{\frac{-r}{p}}$ and $\frac{q}{KA}$ have different Signs; but contrary Ways, if the Signs are of the same Nature. BA also must be be bisected in C; and there the Perpendicular CX erected equal to the Term p; and between the Lines AX and CX, infinitely drawn out both Ways, the right Line EY must also be inscribed equal to AC, and made to pass through the Point K, as before; then will XY be the Root of the Equation; Negative when the Point X should fall between A and E, and affirmative when the Point Y falls on the Side of the Point X towards C.

The Demonstration of the first Case.

By the first Lemma, KE was to CX as AK to YX, and (by Composition) so KE + AK, *i. e.* KY + KC is to CX + YX, *i. e.* CY. But in the right-angled Triangle KCY, YCq = YKq — KCq = $\overline{KY+KC}\times\overline{KY-KC}$; and by resolving the equal Terms into Proportionals, KY + KC is to CY as CY is to KY — KC; or KE + AK is to CY as CY is to EK — KB. Wherefore since KE was to XC in this Proportion, by Duplication KEq will be to CXq as KE + AK to KE — KB, and by multiplying the Extreams and Means by themselves, KE *cube* — KB × KEq is = CXq × KE × CXq × AK. And by restoring the former Values $x^3 - pxx = qx + r$.

The Demonstration of the second Case.

By the first Lemma, KE is to CX as AK is to YX, then by multiplying the Extreams and Means by themselves, KE × YX is = CX × AK. Therefore in the preceding Case, put KE × YX for CX × AK, and it will be KE *cube* — KB × KEq = CXq × KE + CX × KE × YX; and by dividing all by KE, there will be KEq — KB × KE = CXq + CX × YX; then multiplying all by AK, and you will have AK × KEq — AK × KB × KE = AK × CXq + AK × CX × YX. And again, put KE × YX instead of its equal CX × AK, then AK × KEq — AK × KB × KE = EK × CX × YX + KE × YXq; whence all being divided by KE there will arise AK × KE — AK × KB = YX × CX + YXq; and when all are multiplied by YX there will be AK × KE × YX — AK × KB × YX = YXq × CX + YX *cube*. And instead of KE × YX in the first Term put CX × AK, and then CX × AKq — AK × BK × YX = CX

$CX \times YXq + YX$ *cube*, or, which is the same Thing, YX *cube* $+ CX \times YXq + AK \times KB \times YX - CX \times AKq = 0$. And by substituting for YX, CX, AK and KB, their Values $x, p, \sqrt{\frac{-r}{p}}, q\sqrt{\frac{p}{-r}}$, there will come out, $x^3 + pxx + qx + r = 0$, the Equation to be constructed.

These Equations are also solved, by drawing a right Line from a given Point, in such a Manner that the Part of it, which is intercepted between another right Line and a Circle, both given in Position, may be of a given Length. [See Figure 96.]

For, let there be proposed a Cubick Equation $x^3 * + qx + r = 0$, *whose second Term is wanting.*

Draw the right Line KA at Pleasure, which call n. In KA produced both Ways, take $KB = \frac{q}{n}$ on the same Side of the Point K as the Point A is if q be Negative, if not, on the contrary. Bisect BA in C, and from the Center A, with the Distance AC, describe a Circle CX. To this inscribe the right Line $CX = \frac{r}{nn}$, and through the Points K, C, and X describe the Circle KCXG. Join AX, and produce it till it again cuts the Circle KCXG last described in the Point G. Lastly, between this Circle KCXG, and the right Line KC produced both Ways, inscribe the right Line EY = AC, so that EY produced pass through the Point G. And EG will be one of the Roots of the Equation. But those Roots are Affirmative which fall in the greater Segment of the Circle KGC, and Negative which fall in the lesser KFC, if r is Negative, and the contrary will be when r is Affirmative.

In order to demonstrate this Construction, let us premise the following *Lemmas.*

LEMMA I.

All Things being supposed as in the Construction, CE *is to* KA *as* CE + CX *is to* AY, *and as* CX *to* KA.

For the right Line KG being drawn, AC is to AK as CX is to KG, because the Triangles ACX and AKG are Similar. The Triangles YEC, YKG are also Similar; for the Angle at Y is common to both Triangles, and the Angles

G and

G and C are in the same Segment EGCK of the Circle KGC, and therefore equal. Whence CE will be to EY as KG to KY, that is, CE to AC as KG to KY, because EY and AC were supposed equal. And by comparing this with the Proportionality above, it will follow by perturbated Equality that CE is to KA as CX to KY, and alternately CE is to CX as KA to KY. Whence, by Composition, CE + CX will be to CX as KA + KY to KY, that is, AY to KY, and alternately CE + CX is to AY as CX is to KY, that is, as CE to KA. Q. E. D.

LEMMA II.

Let fall the Perpendicular CH *upon the right Line* GY, *and the Rectangle* 2 HEY *will be equal to the Rectangle* CE × CX.

For the Perpendicular GL being let fall upon the Line AY, the Triangles KGL, ECH have right Angles at L and H, and the Angles at K and E are in the same Segment CKEG of the Circle CGK, and are therefore equal; consequently the Triangles are Similar. And therefore KG is to KL as EC to EH. Moreover, AM being let fall from the Point A perpendicular to the Line KG, because AK is equal to AG, KG will be bisected in M; and the Triangles KAM and KGL are Similar, because the Angle at K is common, and the Angles at M and L are right ones; and therefore AK is to KM as KG is to KL. But as AK is to KM so is 2 AK to 2 KM, or KG; (and because the Triangles AKG and ACX are Similar) so is 2 AC to CX; also (because AC = EY) so is 2 EY to CX. Therefore 2 EY is to CX as KG to KL. But KG was to KL as EC to EH, therefore 2 EY is to CX as EC to EH, and so the Rectangle 2 HEY (by multiplying the Extreams and Means by themselves) is equal to EC × CX. Q. E. D.

Here we took the Lines AK and AG to be equal. For the Rectangles CAK and XAG are equal (by *Cor.* to 36 *Prop.* of the 3d Book of *Euc.*) and therefore as CA is to XA so is AG to AK. But XA and CA are equal by Hypothesis; therefore AG = AK.

LEMMA III.

All Things being as above, the three Lines BY, CE, KA *are continual Proportionals.*

For

For (by *Prop.* 12. *Book* 2. *Elem.*) C Y q is $=$ E Y q $+$ C E q $+$ 2 E Y $\times$ E H. And by taking E Y q from both Sides, C Y q $-$ E Y q is $=$ C E q $+$ 2 E Y $\times$ E H. But 2 E Y $\times$ E H is $=$ C E $\times$ C X (by *Lem.* 2.) and by adding G E q to both Sides, C E q $+$ 2 E Y $\times$ E H becomes $=$ C E q $+$ C E $\times$ C X. Therefore C Y q $-$ E Y q is $=$ C E q $+$ C E $\times$ C X, that is, $\overline{CY + EY} \times \overline{CY - EY} = CEq + CE \times CX$. And by resolving the equal Rectangles into proportional Sides, it will be as C E $+$ C X is to C Y $+$ E Y, so is C Y $-$ E Y to C E. But the three Lines E Y, C A, C B, are equal, and thence C Y $+$ E Y $=$ C Y $+$ C A $=$ A Y, and C Y $-$ E Y $=$ C Y $-$ C B $=$ B Y. Write A Y for C Y $+$ E Y, and B Y for C Y $-$ E Y, and it will be as C E $+$ C X is to Y A so is B Y to C E. But (*by Lem.* 1.) C E is to K A as C E $+$ C X is to A Y, therefore C E is to K A as B Y is to C E, that is, the three Lines B Y, C E, and K A are continual Proportionals. Q. E. D.

Now, by the Help of these three Lemmas, we may *demonstrate* the Construction of the preceding Problem, thus:

By Lem. 1. C E is to K A as C X is to K Y, so K A $\times$ C X is $=$ C E $\times$ K Y, and by dividing both Sides by C E, $\frac{KA \times CX}{CE}$ becomes $=$ K Y. To these equal Sides add B K, and B K $+ \frac{KA \times CX}{CE}$ will be $=$ B Y. Whence (by *Lem.* 3.) B K $+ \frac{KA \times CX}{CE}$ is to C E as C E is to K A, and thence, by multiplying the Extreams and Means by themselves, C E q is $= BK \times KA + \frac{KAq \times CX}{CE}$, and both Sides being multiplied by C E, C E *cub.* becomes $=$ K B $\times$ K A $\times$ C E $+$ K A q $\times$ C X. C E was called x, the Root of the Equation, K A was $= n$, K B $= \frac{q}{n}$, and C X $= \frac{r}{nn}$. These being substituted instead of C E, K A, K B and C X, there will arise $x^3 = qx + r$, or $x^3 - qx - r = 0$, the Equation to be constructed; when q and r are Negatives, K A and K B having been taken on the same Side of the Point

Point K, and the Affirmative Root being in the greater Segment C G K. This is *one Case* of the Construction to be demonstrated. Draw K B on the contrary Side, that is, let its Sign be changed, or the Sign of $\frac{q}{n}$, or, which is the same Thing, the Sign of the Term q, and there will be had the Construction of the Equation $x^3 + qx - r = 0$. *Which is the other Case.* In these Cases C X, and the Affirmative Root C E, fall towards the same Parts of the Line A K. Let C X and the Negative Root fall towards the same Parts when the Sign of C X, or of $\frac{r}{nn}$, or (which is the same Thing) the Sign of r is changed; and this will be the *third Case* $x^3 + qx + r = 0$, where all the Roots are Negative. And again, when the Sign of K B, or of $\frac{q}{n}$, or only of q, is changed, it will be the *fourth Case* $x^3 - qx + r = 0$. The Constructions of all these Cases may be run through, and particularly demonstrated after the same Manner as the first was. We having demonstrated one Case, thought it sufficient to touch slightly the rest. These are demonstrated with the same Words, by changing only the Situation of the Lines.

Now let the Cubick Equation $x^3 + pxx * + r = 0$, *whose third Term is wanting, be to be constructed.*

In the same Figure n being taken of any Length, take in any infinite right Line A Y, $KA = \frac{r}{nn}$ and $KB = p$ and take them on the same Side of the Point K, if the Signs of the Terms p and r are the same, otherwise on contrary Sides. Bisect B A in C, and from the Center K with the Distance K C describe the Circle C X G. And to it inscribe the right Line C X equal to n the assumed Length. Join A X and produce it to G, so that A G may be equal to A K, and through the Points K, C, X, G describe a Circle. And, lastly, between this Circle and the right Line K C, produced both Ways, inscribe the right Line $EY = AC$, so that being produced it may pass through the Point G; then the right Line K Y being drawn, will be one of the Roots of the Equation. And those Roots are Affirmative which fall on that Side of the Point K on which the Point A is on, if r is

Affir-

Affirmative; but if r is Negative, then the Affirmative Roots fall on the contrary Side. And if the Affirmative Roots fall on one Side, the Negative fall on the other.

This Construction is demonstrated by the Help of the three last *Lemmas* after this Manner:

By the third Lemma, B Y, C E, K A are continual Proportionals; and by *Lemma* 1. as C E is to K A so is C X to K Y. Therefore B Y is to C E as C X to K Y. B Y is = K Y — K B. Therefore K Y — K B is to C E as C X is to K Y. But as K Y — K B is to C E so is $\overline{KY - KB} \times$ K Y to C E × K Y, by *Prop.* 1. *Book* 6. *Euc.* and because of the Proportionals C E to K A as C X to K Y it is C E × K Y = K A × C X. Therefore $\overline{KY - KB} \times KY$ is to K A × CX (as K Y — K B to C E, that is) as C X to K Y. And by multiplying the Extreams and Means by themselves $\overline{KY - KB} \times KYq$ becomes = K A × C X q; that is, K Y *cub.* — K B × K Y q = K A × C X q. But in the Construction K Y was x the Root of the Equation, K B was put = p, K A = $\frac{r}{nn}$, and C X = n. Write therefore x, p, $\frac{r}{nn}$, and n for K Y, K B, K A, and C X respectively, and $x^3 - pxx$ will become = r, or $x^3 - pxx - r = 0$.

This Construction may be resolved into these four Cases of Equations, $x^3 - pxx - r = 0$, $x^3 - pxx + r = 0$, $x^3 + pxx - r = 0$, and $x^3 + pxx + r = 0$. The first Case I have already demonstrated; the rest are demonstrated with the same Words, only changing the Situation of the Lines. To wit, as in taking K A and K B on the same Side of the Point K, and the Affirmative Root K Y on the contrary Side, has already produced K Y *cub.* — K B × K Y q = K A × C X q, and thence $x^3 - pxx - r = 0$; so by taking K B on the other Side the Point K there will be produced, by the like Reasoning, K Y *cub.* + K B × K Y q = K A × C X q, and thence $x^3 + pxx - r = 0$. And in these two Cases, if the Situation of the Affirmative Root K Y be changed, by taking it on the other Side of the Point K, by a like Series of Argumentation you will fall upon the other two Cases, K Y *cub.* + K B × K Y q = — K A × C X q, CXq, or $x^3+pxx+r=0$, and KY *cub.* — KB × KYq = — KA × CXq, or $x^3-pxx+r=0$. Which were all the Cases to be demonstrated.

Now let this Cubick Equation $x^3+pxx+qx+r=0$ *be proposed, wanting no Term (unless perhaps the third).* Which is constructed after this Manner: [*See Figures* 97 and 98.]

Take the length n at Pleasure. Draw any right Line GC $=\frac{n}{2}$, and at the Point G erect a Perpendicular GD $=\sqrt{\frac{r}{p}}$, and if the Terms p and r have contrary Signs, from the Center C, with the Interval CD describe a Circle PBE. If they have the same Signs from the Center D, with the Space GC, describe an occult Circle, cutting the right Line GA in H; then from the Center C, with the Distance GH, describe the Circle PBE. Then make GA $=-\frac{q}{n}-\frac{r}{np}$ on the same Side the Point G that C is on, provided the Quantity $-\frac{q}{n}-\frac{r}{np}$ (the Signs of the Terms p, q, r in the Equation to be constructed being well observed) should come out Affirmative; otherwise, draw GA on the other Side of the Point G, and at the Point A erect the Perpendicular AY, between which and the Circle PBE already described, inscribe the right Line EY equal to the Term p, so that being produced, it may pass through the Point G; which being done, the Line EG will be one of the Roots of the Equation to be constructed. Those Roots are Affirmative when the Point E falls between the Points G and Y, and Negative, when the Point E falls without, if p is Affirmative; and the contrary, if Negative.

In order to demonstrate this Construction, let us premise the following *Lemmas.*

LEMMA I.

Let EF *be let fall perpendicular to* AG, *and the right Line* EC *be drawn;* EGq + GCq is = ECq + 2CGF.

For

For (by *Prop.* 12. *Book* 2. *Elem.*) EGq is $= ECq + GCq + 2\,GCF$. Let GCq be added on both Sides, and $EGq + GCq$ will become $= ECq + 2\,GCq + 2\,GCF$. But $2\,GCq + 2\,GCF$ is $= 2\,GC \times \overline{GC + CF} = 2\,CGF$. Therefore $EGq + GCq = ECq + 2\,CGF$. Q. E. D.

Lemma II.

In the first Case of the Construction, where the Circle PBE *passes through the Point* D, $EGq - GDq$ *is* $= 2\,CGF$.

For by the *first Lemma* $EGq + GCq$ is $= ECq + 2\,CGF$, and by taking CGq from both Sides, EGq is $= ECq - GCq + 2\,CGF$. But $ECq - GCq$ is $= CDq - GCq = GDq$. Therefore $EGq = GDq + 2\,CGF$, and by taking GDq from both Sides, $EGq - GDq$ is $= 2\,CGF$. Q. E. D.

Lemma III.

In the second Case of the Construction, where the Circle PBE *does not pass through the Point* D, $EGq + GDq$ is $= 2\,CGF$.

For in the *first Lemma*, $EGq + GCq$ was $= ECq + 2\,CGF$. Take ECq from both Sides, and it becomes $EGq + GCq - ECq = 2\,CGF$. But $GC = DH$, and $EC = CP = GH$. Therefore $GCq - ECq = DHq - GHq = GDq$, and so $EGq + GDq = 2\,CGF$. Q. E. D.

Lemma IV.

$GY \times 2\,CGF$ *is* $= 2\,CG \times AGE$.

For by reason of the similar Triangles GEF and GYA, as GF is to GE so is AG to GY, that is, (by *Prop.* 1. *Book* 6. *Elem.*) as $2\,CG \times AG$ is to $2\,CG \times GY$. Let the Extreams and Means be multiplied by themselves, and $2\,CG \times GY \times GF$ becomes $= 2\,CG \times AG \times GE$. Q. E. D.

Now, by the Help of these Lemmas, the Construction of the Problem may be thus demonstrated.

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In the *first* Case, $EGq - GDq$ is $= 2\,CGF$ (by Lemma 2.) and by multiplying all by GY, $EGq \times GY - GDq \times GY$ becomes $= 2\,CGF \times GY =$ (by Lemma 4.) $2\,CG \times AGE$. Instead of GY write $EG + EY$, and $EG\,cub. + EY \times EGq - GDq \times EG - GDq \times EY$ becomes $= 2\,CGA \times EG$, or $EG\,cub. + EY \times \overline{EGq - GDq - 2\,CGA} \times EG - GDq \times EY = 0$.

In the *second* Case, $EGq + GDq$ is $= 2\,CGF$ (by Lemma 3.) and by multiplying all by GY, $EGq \times GY + GDq \times GY$ becomes $= 2\,CGF \times GY = 2\,CG \times AGE$, by Lemma 4. Instead of GY write $EG + EY$, and $EG\,cub. + EY \times EGq + GDq \times EG + GDq \times EY$ will become $= 2\,CGA \times EG$, or $EG\,cub. + EY \times \overline{EGq + GDq - 2\,CGA} \times EG + GDq \times EY = 0$.

But the Root of the Equation EG was called x, $GD = \sqrt{\frac{r}{p}}$, $EY = p$, $2\,CG = n$, and $GA = -\frac{q}{n} - \frac{r}{np}$, that is, in the first Case, where the *Signs* of the Terms p and r are different; but in the second Case, where the Sign of one of the two, p or r, is changed, there is $-\frac{q}{n} + \frac{r}{np}$ $= GA$. Let therefore EG be put $= x$, $GD = \sqrt{\frac{r}{p}}$, $EY = p$, $2\,CG = n$, and $GA = -\frac{q}{n} \mp \frac{r}{np}$, and in the *first* Case it will be $x^3 + px^2 + \overline{q + \frac{r}{p} - \frac{r}{p}} \times x - r = 0$; that is, $x^3 + px^2 + qx - r = 0$; but in the *second* Case, $x^3 + pxx + \overline{q + \frac{r}{p} - \frac{r}{p}} \times x + r = 0$, that is, $x^3 + px^2 + qx + r = 0$. Therefore in both Cases EG is the true Value of the Root x. Q.E.D.

But either Case may be distinguished into its several Particulars; as the former into these, $x^3 + px^2 + qx - r = 0$, $x^3 + px^2 - qx - r = 0$, $x^3 - px^2 + qx + r = 0$, $x^3 - px^2 - qx + r = 0$, $x^3 + px^2 - r = 0$, and $x^3 - px^2 + r = 0$; the latter into these, $x^3 + px^2 + qx + r = 0$, $x^3 + px^2 - qx + r = 0$, $x^3 - px^2 + qx - r = 0$, $x^3 - px^2 - qx - r = 0$, $x^3 + px^2 + r = 0$, and $x^3 -$

$x^3 - px^2 - r = 0$. The Demonstration of all which Cases may be carried on in the same Words with the two already demonstrated, by only changing the Situation of the Lines.

These are the chief Constructions of Problems, by inscribing a right Line given in Length so between a Circle and a right Line given in Position, that the inscribed right Line produced may pass through a given Point. And such a right Line may be inscribed by describing the *Conchoid* of the Antients, of which let that Point, through which the right Line given ought to pass, be the Pole, the other right Line given in Position be the Ruler or Asymptote, and the Interval be the Length of the inscribed Line. For this Conchoid will cut the Circle in the Point E, through which the right Line to be inscribed must be drawn. But it will be sufficient in Practice to draw the right Line between a Circle and a right Line given in Position by any Mechanick Method.

But in these Constructions observe, that the Quantity n is undetermined and left to be taken at Pleasure, that the Construction may be more conveniently fitted to particular Problems. We shall give Examples of this in finding two mean Proportionals, and in trisecting an Angle.

Let x and y be two mean Proportionals to be found between a and b. Because a, x, y, b are continual Proportionals, a^2 will be to x^2 as x to b, therefore $x^3 = aab$, or $x^3 - aab = 0$. Here the Terms p and q of the Equation are wanting, and $-aab$ is in the room of the Term r; therefore in the first Form of the Constructions, where the right Line EY tending to the given Point K, is drawn between other two right Lines EX and YC, given in Position, and the right Line CX supposed $= \frac{r}{nn}$ that is $= \frac{-aab}{nn}$, let n be taken equal to a, and then CX will become $= -b$. From whence the following Construction comes out. [*See* Figure 99.]

I draw any Line, KA $= a$, and bisect it in C, and from the Center K, with the Distance KC, describe the Circle CX, to which I inscribe the right Line CX $= b$, and between AX and CX, infinitely produced, I so inscribe EY $=$ CA, that EY being produced, may pass through the Point K. So KA, XY, KE, CX will be continual Proportionals, that is, XY and KE two mean Proportionals between a and b. This Construction is known. [*See* Figure 100.]

But

But in the other Form of the Constructions, where the right Line E Y converging to the given Point G is inscribed between the Circle G E C X and the right Line A K, and C X is $= \frac{r}{nn}$, that is, (in this Problem) $= \frac{-aab}{nn}$, I put, as before, $n = a$, and then C X will be $= b$, and the rest are done as follows. [*See* Figure 101.]

I draw any right Line K A $= a$, and bisect it in C, and from the Center A, with the Distance A K, I describe the Circle K G, to which I inscribe the right Line K G $= 2b$, constituting the *Isosceles* Triangle A K G. Then, through the Points C, K, G I describe the Circle, between the Circumference of which and the right Line A K produced, I inscribe the right Line E Y $=$ C K tending to the Point G. Which being done, A K, E C, K Y, $\frac{1}{2}$ K G are continual Proportionals, that is, E C and K Y are two mean Proportionals between the given Quantities a and b.

Let there be an Angle to be divided into three equal Parts; [See *Figure* 102.] *and let that Angle be* A C B, *and the Parts thereof to be found be* A C D, E C D, *and* E C B.

From the Center C, with the Distance C A, let the Circle A D E B be described, cutting the right Lines C A, C D, C E, C B in A, D, E, B. Let A D, D E, E B be joined, and A B cutting the right Lines C D, C E at F and H, and let D G, meeting A B in G, be drawn parallel to C E. Because the Triangles C A D, A D F, and D F G are Similar, C A, A D, D F, and F G are continual Proportionals. Therefore if A C be $= a$, and A D $= x$, D F will be equal to $\frac{xx}{a}$, and F G $= \frac{x^3}{aa}$. But A B is $=$ B H $+$ H G $+$ F A $-$ G F $= 3$ A D $-$ G F $= 3x - \frac{x^3}{aa}$. Let A B be $= b$, then b becomes $= 3x - \frac{x^3}{aa}$, or $x^3 - 3aax + aab = 0$. Here p, the second Term of the Equation, is wanting, and instead of q and r we have $-3aa$ and aab. Therefore in the first Form of the Constructions, where p was $= 0$, K A $= n$, K B $= \frac{q}{n}$, and C X $= \frac{r}{nn}$, that is, in this Pro-

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blem, $KB = -\frac{3aa}{n}$, and $CX = \frac{aab}{nn}$, that these Quantities may come out as simple as possible, I put $n = a$, and so KB becomes $= -3a$, and $CX = b$. Whence this *Construction* of the Problem comes out.

Draw any Line, $KA = a$, and on the contrary Side make $KB = 3a$. [*See* Figure 103.] Bisect BA in C, and from the Center K, with the Distance KC, describe a Circle, to which inscribe the right Line $CX = b$, and the right Line AX being drawn between that infinitely produced and the right Line CX, inscribe the right Line $EY = AC$, and so that it being produced, will pass through the Point K. So XY will be $= x$. But (see the last Figure) because the Circle ADEB = Circle CXA, and the Subtense AB = Subtense CX, and the Parts of the Subtenses BH and XY are equal; the Angles ACB, and CKX will be equal, as also the Angles BCH, XKY; and so the Angle XKY will be one third Part of the Angle CKX. Therefore the third Part XKY of any given Angle CKX is found by inscribing the right Line $EY = AC$ the Diameter of the Circle, between the Chords CX and AX infinitely produced, and converging towards K the Center of the Circle.

Hence, if from K, the Center of the Circle, you let fall the Perpendicular KH upon the Chord CX, the Angle HKY will be one third Part of the Angle HKX; so that if any Angle HKX were given, the third Part thereof HKY may be found by letting fall from any Point X of any Side KX, the Line HX perpendicular to the other Side HK, and by drawing XE parallel to HK, and by inscribing the right Line $YE = 2XK$ between XH and XE, so that it being produced may pass through the Point K. *Or thus.* [See *Figure* 104.]

Let any Angle AXK be given. To one of its Sides AX raise a Perpendicular XH, and from any Point K of the other Side XK let there be drawn the Line KE, the Part of which EY (lying between the Side AX produced, and the Perpendicular XH) is double the Side XK, and the Angle KEA will be one third of the given Angle AXK. Again, the Perpendicular EZ being raised, and KF being drawn, whose Part ZF, between EF and EZ, let be double to KE, and the Angle KFA will be one third of the Angle KEA; and so you may go on by a continual Tri-

section

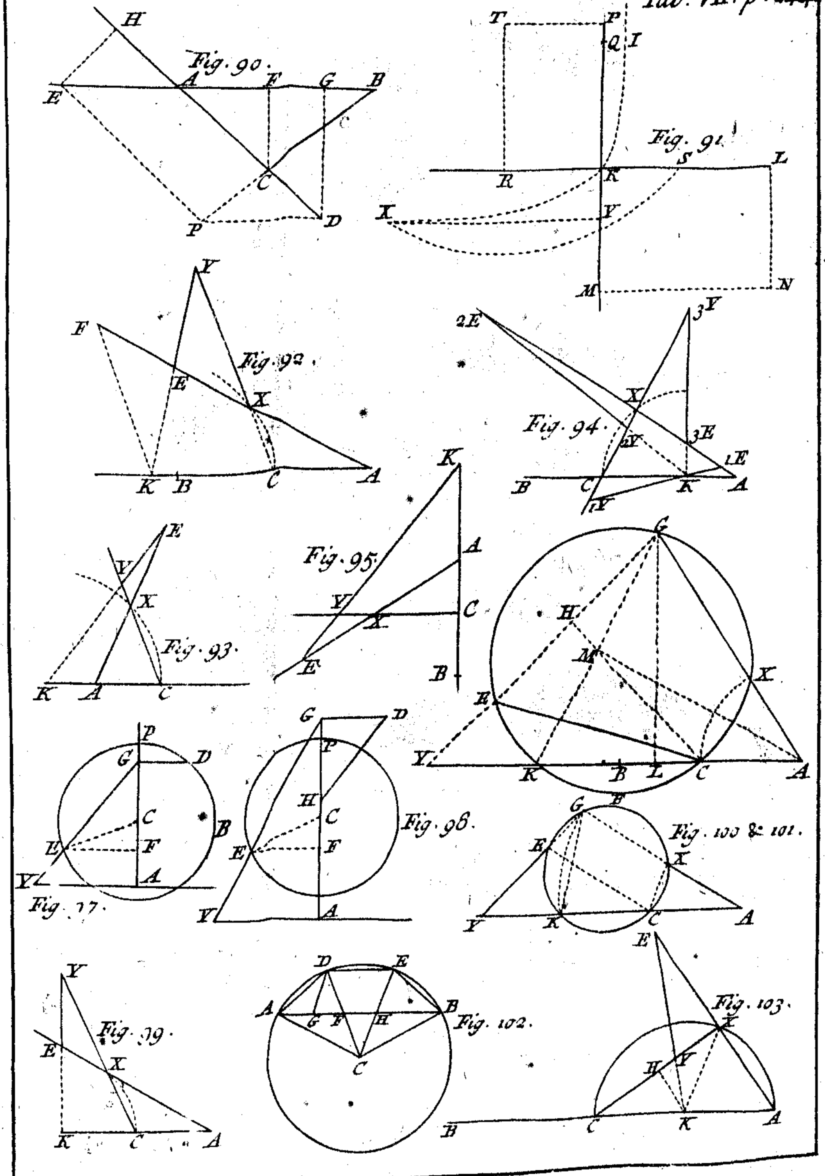
Fig. 90.
Fig. 91.
Fig. 92.
Fig. 94.
Fig. 93.
Fig. 95.
Fig. 97.
Fig. 98.
Fig. 100 & 101.
Fig. 99.
Fig. 102.
Fig. 103.

section of an Angle *ad infinitum*. This Method is in the 32d *Prop.* of the 4th Book of *Pappus*.

But if you would trisect an Angle by the other Form of Constructions, where the right Line is to be inscribed between another right Line and a Circle: Here also will K B be $= \frac{q}{n}$, and C X $= \frac{r}{nn}$, that is, in the Problem we are now about, K B $= \frac{-3aa}{n}$, and C X $= \frac{aab}{nn}$; and so by putting $n = a$, K B will be $= -3a$, and C X $= b$. Whence this *Construction* comes out.

From any Point K let there be drawn two right Lines towards the same Way, K A $= a$, and K B $= 3a$. [*See* Figure 105.] Bisect A B in C, and from the Center A with the Distance A C describe a Circle. To which inscribe the right Line C X $= b$. Join A X, and produce it till it cuts the Circle again in G. Then between this Circle and the right Line K C, infinitely produced, inscribe the Line E Y = A C, and passing through the Point G; and the right Line E C being drawn, will be equal to x the Quantity sought, by which the third Part of the given Angle will be subtended.

This Construction arises from the Form above; which, however, comes out better thus: Because the Circles A D E B and K X G are equal, and also the Subtenses C X and A B, the Angles C A X, or K A G, and A C B are equal, therefore C E is the Subtense of one third Part of the Angle K A G. Whence in any given Angle K A G, that its third Part C A E may be found, inscribe the right Line E Y equal to the Semi-Diameter A G of the Circle K C G, between the Circle and the Side K A, of the Angle, infinitely produced, and tending to the Point G. Thus *Archimedes*, in *Lemma* 8. taught to trisect an Angle. The same Constructions may be more easily explained than I have done here; but in these I would shew how, from the general Constructions of Problems I have already explained, we may derive the most simple Constructions of particular Problems.

Besides the Constructions here set down, we might add many more. [*See* Figure 106.] *As if there were two mean Proportionals to be found between a and b.* Draw any right Line A K $= b$, and perpendicular to it A B $= a$. Bisect A K in I, and in A K put A H equal to the Subtense B I; and also in the Line A B produced, A C = Subtense B H. Then

Then in the Line A K on the other Side of the Point A take A D of any Length and D E equal to it, and from the Centers D and E, with the Distances D B and E C, describe two Circles, B F and C G, and between them draw the right Line F G equal to the right Line A I, and converging at the Point A, and A F will be the first of the two mean Proportionals that were to be found.

The Antients taught how to find two mean Proportionals by the *Cissoid*; but no Body that I know of hath given a good manual Description of this Curve. [*See* Figure 107.] Let A G be the Diameter, and F the Center of a Circle to which the *Cissoid* belongs. At the Point F let the Perpendicular F D be erected, and produced *in infinitum*. And let F G be produced to P, that F P may be equal to the Diameter of the Circle. Let the rectangular Ruler P E D be moved, so that the Leg E P may always pass through the Point P, and the other Leg E D must be equal to the Diameter A G, or F P, with its End D, always moving in the Line F D; and the middle Point C of this Leg will describe the *Cissoid* G C K which was desired, as has been already shewn. Wherefore, if between any two Quantities, a and b, there be two mean Proportionals to be found: Take A M $= a$, raise the Perpendicular M N $= b$. Join A N; and move the Rule P E D, as was just now shewn, until its Point C fall upon the right Line A N. Then let fall C B perpendicular to A P, take t to B H, and v to B G, as M N is to B C, and because A B, B H, B G, B C are continual Proportionals, a, t, v, b will also be continual Proportionals.

By the Application of such a Ruler other solid Problems may be constructed.

Let there be proposed the Cubick Equation $x^3 \pm pxx + qx - r = 0$; where let q be always Affirmative, r Negative, and p of any Sign. Make A G $= \frac{r}{q}$, and bisect it in F, and take F R and G L $= \frac{p}{2}$, and that towards A if it be $+p$, if not towards P. Moreover, erect the Perpendicular F D, and in it take F Q $= \sqrt{q}$; to this erect also the Perpendicular Q C. And in the Leg E D of the Ruler, take E D and E C respectively equal to A G and A R, and let the Leg of the Ruler be applied to the Scheme, so that the Point D may touch the right Line F D, and the Point C the right

right Line QC, then if the Parallelogram BQ be compleated, LB will be the sought Root x of the Equation.

Thus far, I think, I have expounded the Construction of solid Problems by Operations whose manual Practice is most simple and expeditious. So the Antients, after they had obtained a Method of solving these Problems by a Composition of solid Places, thinking the Constructions by the Conick Sections useless, by reason of the Difficulty of describing them, sought easier Constructions by the Conchoid, Cissoid, the Extension of Threads, and by any Mechanick Application of Figures, preferring useful Things though Mechanical, to useless Speculations in Geometry, as we learn from *Pappus*. So the great *Archimedes* himself neglected the Trisection of an Angle by the Conick Sections, which had been handled by other Geometricians before him, and taught how to trisect an Angle in his Lemmas after the Method we have already explained. If the Antients had rather construct Problems by Figures not received in Geometry in that Time, how much more ought these Figures now to be preferred which are received by many into Geometry as well as the Conick Sections.

However, I do not agree to this new Sort of Geometricians, who receive all Figures into Geometry. Their Rule of admitting all Lines to the Construction of Problems in that Order in which the Equations, whereby the Lines are defined, ascend to the Number of Dimensions, is arbitrary and has no Foundation in Geometry. Nay, it is false; for according to this Rule, the Circle should be joined with the Conick Sections, but all Geometers join it with the right Line; and this being an inconstant Rule, takes away the Foundation of admitting into Geometry all Analytick Lines in a certain Order. In my Judgment, no Lines ought to be admitted into plain Geometry besides the right Line and the Circle, unless some Distinction of Lines might be first invented, by which a circular Line might be joined with a right Line, and separated from all the rest. But truly plain Geometry is not then to be augmented by the Number of Lines. For all Figures are plain that are admitted into plain Geometry, that is, those which the Geometers postulate to be described *in plano*. And every plain Problem is that which may be constructed by plain Figures. So therefore admitting the Conick Sections and other Figures more compounded into plain Geometry, all the solid and more than solid Problems that can be constructed by these

Figures

Figures will become plane. But all plane Problems are of the ſame Order. A right Line is Analytically more ſimple than a Circle; nevertheleſs, Problems which are conſtructed by right Lines alone, and thoſe that are conſtructed by Circles, are of the ſame Order. Theſe Things being poſtulated, a Circle is reduced to the ſame Order with a right Line. And much more the Ellipſe, which differs much leſs from a Circle than a Circle from a right Line, by poſtulating in like manner the Deſcription thereof *in plano*, will be reduced to the ſame Order with the Circle. If any, in conſidering the Ellipſe, ſhould fall upon ſome ſolid Problem, and ſhould conſtruct it by the Help of the ſame Ellipſe, and a Circle: This would be counted a plane Problem, becauſe the Ellipſe was ſuppoſed to be deſcribed *in plano*, and all the Conſtruction beſides will be ſolved by the Deſcription of the Circle only. Wherefore, for the ſame Reaſon, every plane Problem whatever may be conſtructed by a given Ellipſe. For Example, [*See* Figure 108.] If the Center O of the given Ellipſe A D F G be required, I would draw the two Parallels A B, C D meeting the Ellipſe in A, B, C, D; and alſo two other Parallels E F, G H meeting the Ellipſe in E, F, G, H, and I would biſect them in I, K, L, M, and produce I K, L M, till they meet in O. This is a real Conſtruction of a plane Problem by an Ellipſe. It imports nothing that an Ellipſe is Analytically defined by an Equation of two Dimenſions. Nor that it be generated Geometrically by the Section of a ſolid Figure. The Hypotheſis, only conſidering it as already deſcribed *in plano*, reduces all ſolid Problems conſtructed by it to the Order of plane ones, and concludes, that all plane ones may be rightly conſtructed by it. And this is the State of a *Poſtulate*. Whatever may be ſuppoſed done, it is permitted to aſſume it, as already done and given. Therefore let this be a Poſtulate to deſcribe an Ellipſe *in plano*, and then all thoſe Problems that can be conſtructed by an Ellipſe, may be reduced to the Order of plane ones, and all plane Problems may be conſtructed by the Ellipſe.

It is neceſſary therefore that either plane and ſolid Problems be confounded among one another, or that all Lines be flung out of plane Geometry, beſides the right Line and the Circle, unleſs it happens that ſometime ſome other is given in the State of conſtructing ſome Problem. But certainly none will

will permit the Orders of Problems to be confused. Therefore the Conick Sections and all other Figures must be cast out of plane Geometry, except the right Line and the Circle, and those which happen to be given in the State of the Problems. Therefore all these Descriptions of the Conicks *in plano*, which the Moderns are so fond of, are foreign to Geometry. Nevertheless, the Conick Sections ought not to be flung out of Geometry. They indeed are not described Geometrically *in plano*, but are generated in the plane Superficies of a Geometrical Solid. A Cone is constituted geometrically, and cut by a Geometrical Plane. Such a Segment of a Cone is a Geometrical Figure, and has the same Place in solid *Geometry*, as the Segment of a Circle has in Plane, and for this Reason its Base, which they call a Conick Section, is a Geometrical Figure. Therefore a Conick Section hath a Place in Geometry so far as it is the Superficies of a Geometrical Solid; but is Geometrical for no other Reason than that it is generated by the Section of a Solid, and therefore was not in former Times admitted but only into solid Geometry. But such a Generation of the Conick Sections is difficult, and generally useless in Practice, to which Geometry ought to be most serviceable. Therefore the Antients betook themselves to various Mechanical Descriptions of Figures *in plano*. And we, after their Example, have framed the preceding Constructions. Let these Constructions be Mechanical; and so the Constructions by Conick Sections described *in plano* (as is wont now to be done) are Mechanical. Let the Constructions by Conick Sections given be Geometrical; and so the Constructions by any other given Figures are Geometrical, and of the same Order with the Constructions of plane Problems. There is no Reason that the Conick Sections should be preferred in Geometry before any other Figures, unless so far as they are derived from the Section of a Cone; they being altogether unserviceable in Practice in the Solution of Problems. But least I should wholly neglect Constructions by the Conick Sections, it will be proper to say something concerning them, in which also we will consider some commodious manual Description.

The Ellipse is the most simple of the Conick Sections, most known, and nearest of Kin to a Circle, and easiest described by the Hand *in plano*. Many prefer the Parabola before it, for the Simplicity of the Equation by which it is expressed. But by this Reason the Parabola ought to be preferred before the Circle it ſelf, which it never is. Therefore the reaſoning from the Simplicity of the Equation will not hold. The modern Geometers are too fond of the Speculation of Equations. The Simplicity of theſe is of an Analytick Conſideration. We treat of Compoſition, and Laws are not given to Compoſition from Analyſis. Analyſis does lead to Compoſition: But it is not true Compoſition before its freed from Analyſis. If there be never ſo little Analyſis in Compoſition, that Compoſition is not yet real. Compoſition in it ſelf is perfect, and far from a Mixture of Analytick Speculations. The Simplicity of Figures depend upon the Simplicity of their Geneſis and Ideas, and it is not an Equation but a Deſcription (either Geometrical or Mechanical) by which a Figure is generated and rendered more eaſy to the Conception. Therefore we give the Ellipſe the firſt Place, and ſhall now ſhew how to conſtruct Equations by it.

Let there be any Cubick Equation propoſed, $x^3 = px^2 + qx + r$, *where* p, q, *and* r *ſignify given Co-efficients of the Terms of the Equation, with their Signs* $+$ *and* $-$, *and either of the Terms* p *and* q, *or both of them, may be wanting.* For ſo we ſhall exhibit the Conſtructions of all Cubick Equations in one Operation, which follows:

From the Point B in any given right Line, take any two right Lines, BC and BE, on the ſame Side the Point B, and alſo BD, ſo that it may be a mean Proportional between them. [*See* Figure 109.] And call BC, n, in the ſame right Line alſo take $BA = \frac{q}{n}$, and that towards the Point C, if $-q$, if not, the contrary Way. At the Point A erect a Perpendicular AI, and in it take $AF = p$, $FG = AF$, $FI = \frac{r}{nn}$, and FH to FI as BC is to BE. But FH and FI are to be taken on the ſame Side of the Point F towards G, if the Terms p and r have the ſame Signs; and if they have not the ſame Signs, towards the Point A. Let the Parallelograms IACK and HAEL be compleated, and from the Center K, with the Diſtance KG, let a Circle be deſcribed. Then in the Line HL let there be taken HR on either Side the Point H, which let be to HL as BD to BE; let GR be drawn, cutting EL in S, and let the Line GRS be moved with its Point R falling on the Line

Line HL, and the Point S upon the Line EL, until its third Point G in describing the Ellipse, meet the Circle, as is to be seen in the Position of $\gamma\rho\sigma$. For half the Perpendicular γX let fall from γ the Point of meeting to AE will be the Root of the Equation. But G or γ the End of the Rule GRS, or $\gamma\rho\sigma$, can meet the Circle in as many Points as there are possible Roots. And those Roots are Affirmative which fall towards the same Parts of the Line EA, as the Line FI drawn from the Point F does, and those are Negative which fall towards the contrary Parts of the Line AE if r is Affirmative; and contrarily if r is Negative.

But this Construction is *demonstrated* by the Help of the following *Lemmas.*

LEMMA I.

All being supposed as in the Construction, $2CAX - AXq$ is $= \gamma Xq - 2AI \times \gamma X + 2AG \times FI$.

For from the Nature of the Circle, $K\gamma q - CXq$ is $= \overline{\gamma X - AI}|^2$. But $K\gamma q$ is $= GIq + ACq$, and $CXq = \overline{AX - AC}|^2$, that is, $= AXq - 2CAX + ACq$, and so their Difference $GIq + 2CAX - AXq$ is $= \overline{\gamma X - AI}|^2 = \gamma Xq - 2AI \times \gamma X + AIq$. Subtract GIq from both, and there will remain $2CAX - AXq = \gamma Xq - 2AI \times \gamma X + AIq - GIq$. But (by *Prop.* 4. *Book* 2. *Elem.*) AIq is $= AGq + 2AGI + GIq$, and so $AIq - GIq$ is $= AGq + 2AGI$, that is, $= 2AG \times \frac{1}{2}AG + GI$, or $= 2AG \times FI$, and thence $2CAX - AXq$ is $= \gamma Xq - 2AI \times \gamma X + 2AG \times FI$. Q. E. D.

LEMMA II.

All Things being constructed as above $2EAX - AXq$ *is* $= \frac{FI}{FH} X\gamma q - \frac{2FI}{FH} AH \times X\gamma + 2AG \times FI$.

For it is known, that the Point γ, by the Motion of the Ruler $\gamma\rho\sigma$ assigned above, describes an Ellipse, the Center whereof is L, and the two Axis's coincide with the two right Lines LE and LH, of which that which is in LE is $= 2\gamma\rho$, or $= 2GR$, and the other which is in LH is $= 2\gamma\sigma$, or $= 2GS$. And the Ratio of these to one another is the same as that of the Line HR to the Line HL, or of the Line BD to the Line BE. Whence the *Latus Transversum* is to the principal *Latus Rectum*, as BE is to BC, or as FI is to FH. Wherefore since γ T is ordinately applied to HL, it will be from the Nature of the Ellipse $GSq - LTq = \frac{FI}{FH} T\gamma$ squared. But LT is $=$ $AE - AX$, and $T\gamma = X\gamma - AH$. Let the Squares of which be put instead of LTq and $T\gamma q$, and then $GSq - AEq + 2EAX - AXq$ will become $= \frac{FI}{FH} \times X\gamma q - 2AH \times X\gamma + AHq$. But $GSq - AEq$ is $= \overline{GH + LS}|^2$, because GS is the Hypothenuse of a Rectangled Triangle, the Sides whereof are equal to AE and $GH + LS$. And (by reason of the similar Triangles RGH and RSL) LS is to GH as LR is to HR, and by Composition $GH + LS$ is to GH as HL is to HR, and by squaring the Proportions $\overline{CH + LS}|^2$ is to GHq as HLq is to HRq, that is, (by Construction) as BEq is to BDq, that is, as BE is to BC, or as FI is to FH, and so $\overline{GH + LS}|^2$ is $= \frac{FI}{FH} GHq$. Therefore $GSq - AEq$ is $= \frac{FI}{FH} GHq$, and so $\frac{FI}{FH} GHq + 2EAX - AXq = \frac{FI}{FH} \times X\gamma q - 2AH \times X\gamma + AHq$. Subtract $\frac{FI}{FH} GHq$ from both Sides, and there will remain $2EAX - AXq = \frac{FI}{FH} \times X\gamma q - 2AH \times X\gamma + AHq - GHq$. But AH is $= AG + GH$, and so $AHq = AGq + 2AGH + GHq$, and by subtracting GHq from both, there will remain $AHq - GHq = AGq + 2AGH$, that is, $= 2AG \times \overline{\tfrac{1}{2}AG + GH} = 2AG \times FH$, and therefore $2EAX - AXq$ is $= \frac{FI}{FH} \times X\gamma q -$

2 AH

$2\,AH \times X\gamma + 2\,AG \times FH$, that is, $= \frac{FI}{FH}\,X\gamma q - \frac{2\,FI}{FH}\,AH \times X\gamma + 2\,AG \times FI$. Q. E. D.

LEMMA III.

All Things standing as before, AX will be to $X\gamma -$ AG as $X\gamma$ is to 2 BC.

For if from the Equals in the *second Lemma* there be subtracted the Equals in the *first Lemma*, there will remain $2\,CE \times AX = \frac{HI}{FH}\,X\gamma q - \frac{2\,FI}{FH}\,AH \times X\gamma + 2\,AI \times X\gamma$. Let both Sides be multiplied by FH, and $2\,FH \times CE \times AX$ will become $= HI \times X\gamma q - 2\,FI \times AH \times X\gamma + 2\,AI \times FH \times X\gamma$. But AI is $= HI + AH$, and so $2\,FI \times AH - 2\,FH \times AI = 2\,FI \times AH - 2\,FHA - 2\,FHI$. But $2\,FI \times HA - 2\,FHA = 2\,AHI$, and $2\,AHI - 2\,FHI = 2\,HI \times AF$. Therefore $2\,FI \times AH - 2\,FH \times AI = 2\,HI \times AF$, and so $2\,FH \times CE \times AX = HI \times X\gamma q - 2\,HI \times AF \times X\gamma$. And thence as HI is to FH, so is $2\,CE \times AX$ to $X\gamma q - 2\,AF \times X\gamma$. But by Construction HI is to FH as CE is to BC, and consequently as $2\,CE \times AX$ is to $2\,BC \times AX$, and therefore $2\,BC \times AX$ will be $= X\gamma q - 2\,AF \times X\gamma$, (by *Prop.* 9. *Book* 5. *Elem.*) But because the Rectangles are equal, the Sides are proportional, AX to $X\gamma - 2\,AF$, (that is, $X\gamma -$ AG) as $X\gamma$ is to 2 BC. Q. E. D.

LEMMA IV.

The same Things being still supposed, 2 FI *is to* $AX - 2\,AB$ *as* $X\gamma$ *is to* 2 BC.

For if from the Equals in the *third Lemma*, to wit, $2\,BC \times AX = X\gamma q - 2\,AF \times X\gamma$, the Equals in the *first Lemma* be subtracted, there will remain $-2\,AB \times AX + AXq = 2\,FI \times X\gamma - 2\,AG \times FI$, that is, $AX \times \overline{AX - 2\,AB} = 2\,FI \times \overline{X\gamma - AG}$. But because the Rectangles are Equal, the Sides are Proportional, 2 FI is to $AX - 2\,AB$ as AX is to $X\gamma -$ AG, that is, (by the *third Lemma*) as $X\gamma$ is to 2 BC. Q. E. D.

At length, by the Help of these Lemmas, the Construction of the Problem is thus demonstrated.

By the *fourth Lemma*, $X\gamma$ is to $2BC$ as $2FI$ is to $AX - 2AB$, that is, (by *Prop.* 1. *Book* 6. *Elem.*) as $2BC \times 2FI$ is to $2BC \times \overline{AX - 2AB}$, or to $2BC \times AX - 2BC \times 2AB$. But by the *third Lemma*, AX is to $X\gamma - 2AF$ as $X\gamma$ is to $2BC$, or $2BC \times AX = X\gamma q - 2AF \times X\gamma$, and consequently $X\gamma$ is to $2BC$ as $2BC \times 2FI$ is to $X\gamma q - 2AF \times X\gamma - 2BC \times 2AB$. And by multiplying the Means and Extreams into themselves, $X\gamma$ *cub.* $- 2AF \times X\gamma q - 4BC \times AB \times X\gamma = 8BCq \times FI$. And by adding $2AF \times X\gamma q + 4BC \times AB \times X\gamma$ to both Sides $X\gamma$ *cub.* is $= 2AF \times X\gamma q + 4BC \times AB \times X\gamma + 8BCq \times FI$. But $\frac{1}{2}X\gamma$ in the Construction to be demonstrated was equal to the Root of the Equation $= x$, and $AF = p$, $BC = n$, $AB = \frac{q}{n}$, and $FI = \frac{r}{nn}$, and therefore $BC \times AB = q$. And $BCq \times FI = r$. Which being substituted, will make $x^3 = px^2 + qx + r$. Q. E. D.

Corol. Hence if AF and AB be supposed equal to nothing, by the third and fourth *Lemma*, $2FI$ will be to AX as AX is to $X\gamma$, and $X\gamma$ to $2BC$. From whence arises the Invention of two mean Proportionals between any two given Quantities, FI and BC.

Scholium, Hitherto I have only expounded the Construction of a Cubick Equation by the Ellipse; but the Rule is of a more universal Nature, extending it self indifferently to all the Conick Sections. For, if instead of the Ellipse you would use the Hyperbola, take the Lines BC and BE on the contrary Side of the Point B, then let the Points A, F, G, I, H, K, L, and R be determined as before, except only that FH ought to be taken on the Side of F not towards I, and that HR ought to be taken in the Line AI not in HL, on each Side the Point H, and instead of the right Line GRS, two other right Lines are to be drawn from the Point L to the two Points R and R for Asymptotes to the Hyperbola. With these Asymptotes LR, LR describe an Hyperbola through the Point G, and a Circle from the Center K with the Distance GK: And the halves of the Perpendiculars let fall from their Intersections to the right Line AE will be the Roots of the Equation proposed. All which, the Signs + and − being rightly changed, are demonstrated as above.

But

But if you would uſe the Parabola, the Point E will be removed to an infinite Diſtance, and ſo not to be taken any where, and the Point H will coincide with the Point F, and the Parabola will be to be deſcribed about the Axis H L with the principal *Latus Rectum* B C through the Points G and A, the Vertex being placed on the ſame Side of the Point F, on which the Point B is in reſpect of the Point C.

Thus the Conſtructions by the Parabola, if you regard Analytick Simplicity, are the moſt ſimple of all. Thoſe by the Hyperbola next, and thoſe which are ſolved by the Ellipſe, have the third Place. But if in deſcribing of Figures, the Simplicity of the manual Operation be reſpected, the Order muſt be changed.

But it is to be obſerved in theſe Conſtructions, that by the Proportion of the principal *Latus Rectum* to the *Latus Tranſverſum*, the Species of the Ellipſe and Hyperbola may be determined, and that Proportion is the ſame as that of the Lines B C and B E, and therefore may be aſſumed: But there is but one Species of the Parabola, which is obtained by putting B E infinitely long. So therefore we may conſtruct any Cubick Equation by a Conick Section of any given Species. To change Figures given in Specie into Figures given in Magnitude, is done by encreaſing or diminiſhing in a given Ratio, all the Lines by which the Figures were given in Specie, and ſo we may conſtruct all Cubick Equations by any given Conick Section whatever. Which is more fully explained thus.

Let there be propoſed any Cubick Equation $x^3 = p\,x\,x \;.\; q\,x \;.\; r$, *to conſtruct it by the Help of any given Conick Section.* [See *Figures* 110 and 111.]

From any Point B in any infinite right Line B C E, take any two Lengths B C, and B E towards the ſame Way, if the Conick Section is an Ellipſe, but towards contrary Ways if it be an Hyperbola. But let B C be to B E as the principal *Latus Rectum* of the given Section, is to the *Latus Tranſverſum*, and call B C, n, take B A $= \frac{q}{n}$, and that towards C, if q be Negative, and contrarily if Affirmative. At the Point A erect a Perpendicular A I, and in it take A F $= p$, and F G $=$ A F; and F I $= \frac{r}{nn}$. But let F I be taken towards G if the Terms p and r have the ſame Signs, if not, towards A. Then make as F H is to F I ſo is B C to B E, and

and take this F H from the Point F towards I, if the Section is an Ellipse, but towards the contrary Way if it is an Hyperbola. But let the Parallelograms I A C K and H A E L be compleated, and all these Lines already described transferred to the given Conick Section; or, which is the same Thing, let the Curve be described about them, so that its Axis or principal transverse Diameter might agree with the right Line L H, and the Center with the Point L. These Things being done, let the Line K L be drawn as also G L cutting the Conick Section in g. In L K take L k, which let be to L K as L g to L G, and from the Center k, with the Distance k g, describe a Circle. From the Points where it cuts the given Curve, let fall Perpendiculars to the Line L H, whereof let T γ be one. Lastly, towards γ take T Y, which let be to T γ as L G to L g, and this T Y produced will cut A B in X, and $\frac{1}{2}$ X Y will be one of the Roots of the Equation. But those Roots are Affirmative which lie towards such Parts of A B as F I lies from F, and those are Negative which lie on the contrary Side, if r is $+$, and the contrary if r is $-$

After this Manner are Cubick Equations constructed by given Ellipses and Hyperbolas: But if a Parabola should be given, the Line B C is to be taken equal to the *Latus Rectum* it self. Then the Points A, F, G, I, and K, being found as above, a Circle must be described from the Center K with the Distance K G, and the Parabola must be so applied to the Scheme already described, (or the Scheme to the Parabola) that it may pass through the Points A and G, and its Axis through the Point F parallel to A C, the Vertex falling on the same Side of the Point F as the Point B falls of the Point C; these being done, if Perpendiculars were let fall from the Points where the Parabola intersects the Circle to the Line B C, their Halves will be equal to the Roots of the Equation to be constructed.

And take Notice, that where the second Term of the Equation is wanting, and so the *Latus Rectum* of the Parabola is the Number 2, the Construction comes out the same as that which *Des Cartes* produced in his Geometry, with this Difference only, that these Lines are the double of them.

This is a general Rule of Constructions. But where particular Problems are proposed, we ought to consult the most simple Forms of Constructions. For the Quantity n remains free, by the taking of which the Equation may, for the most part,

part, be rendered more ſimple. One Example of which I will give.

Let there be given an Ellipſe, and let there be two mean Proportionals to be found between the given Lines a and b. Let the firſt of them be x, and $a \,.\, x \,.\, \frac{xx}{a} \,.\, b$ will be continual Proportionals, and ſo $ab = \frac{x^3}{a}$, or $x^3 = aab$, is the Equation which you muſt conſtruct. Here the Terms p and q are wanting, and the Term $r = aab$, and therefore B A and A F are $= 0$, and F I is $= \frac{aab}{nn}$. That the laſt Term may be more ſimple, let n be aſſumed $= a$, and let F I be $= b$. And then the Conſtruction will be thus:

From any Point A in any infinite right Line A E [*See Figure* 112.] take A C $= a$, and on the ſame Side of the Point A take A C to A E as the principal *Latus Rectum* of the Ellipſe is to the *Latus Tranſverſum*. Then in the Perpendicular A I take A I $= b$, and A H to A I as A C to A E. Let the Parallelograms I A C K, H A E L be compleated. Join L A and L K. Upon this Scheme lay the given Ellipſe, and it will cut the right Line A L in the Point g. Make L k to L K as L g to L A. From the *Center* k, with the Diſtance $k\,g$, deſcribe a *Circle cutting* the Ellipſe in γ. Upon A E let fall the Perpendicular γ X, cutting H L in T, and let that be produced to Y, that T Y may be to T γ as L A to L g. And ſo $\frac{1}{2}$ X Y will be equal to x the firſt of the two mean Proportionals. Q. E. I.

A New, Exact, and Easy Method, of finding the Roots of any Equations Generally, and that without any previous Reduction. *By* Edm. Halley, *Savilian Professor of Geometry.* [*Published in the* Philosophical Transactions, Numb. 210. A. D. 1694.]

THE principal Use of the Analytick Art, is to bring Mathematical Problems to Equations, and to exhibit those Equations in the most simple Terms that can be. But this Art would justly seem in some Degree defective, and not sufficiently Analytical, if there were not some Methods, by the Help of which, the Roots (be they Lines or Numbers) might be gotten from the Equations that are found, and so the Problems in that respect be solved. The Antients scarce knew any Thing in these Matters beyond Quadratick Equations. And what they writ of the Geometrick Construction of solid Problems, by the Help of the Parabola, Cissoid, or any other Curve, were only particular Things designed for some particular Cases. But as to Numerical Extraction, there is every where a profound Silence; so that whatever we perform now in this Kind, is entirely owing to the Inventions of the Moderns.

And first of all, that great Discoverer and Restorer of the Modern Algebra, *Francis Vieta*, about 100 Years since, shewed a general Method for extracting the Roots of any Equation, which he published under the Title of, *A Numerical Resolution of Powers*, &c. *Harriot*, *Oughtred*, and others, as well of our own Country, as Foreigners, ought to acknowledge whatsoever they have written upon this Subject, as taken from *Vieta*. But what the Sagacity of Mr. *Newton*'s Genius has performed in this Business we may rather conjecture (than be fully assured of) from that short Specimen given by Dr. *Wallis* in the 94th Chapter of his *Algebra*. And we must be forced to expect it, till his great Modesty shall yield to the Intreaties of his Friends, and suffer those curious Discoveries to see the Light.

Not

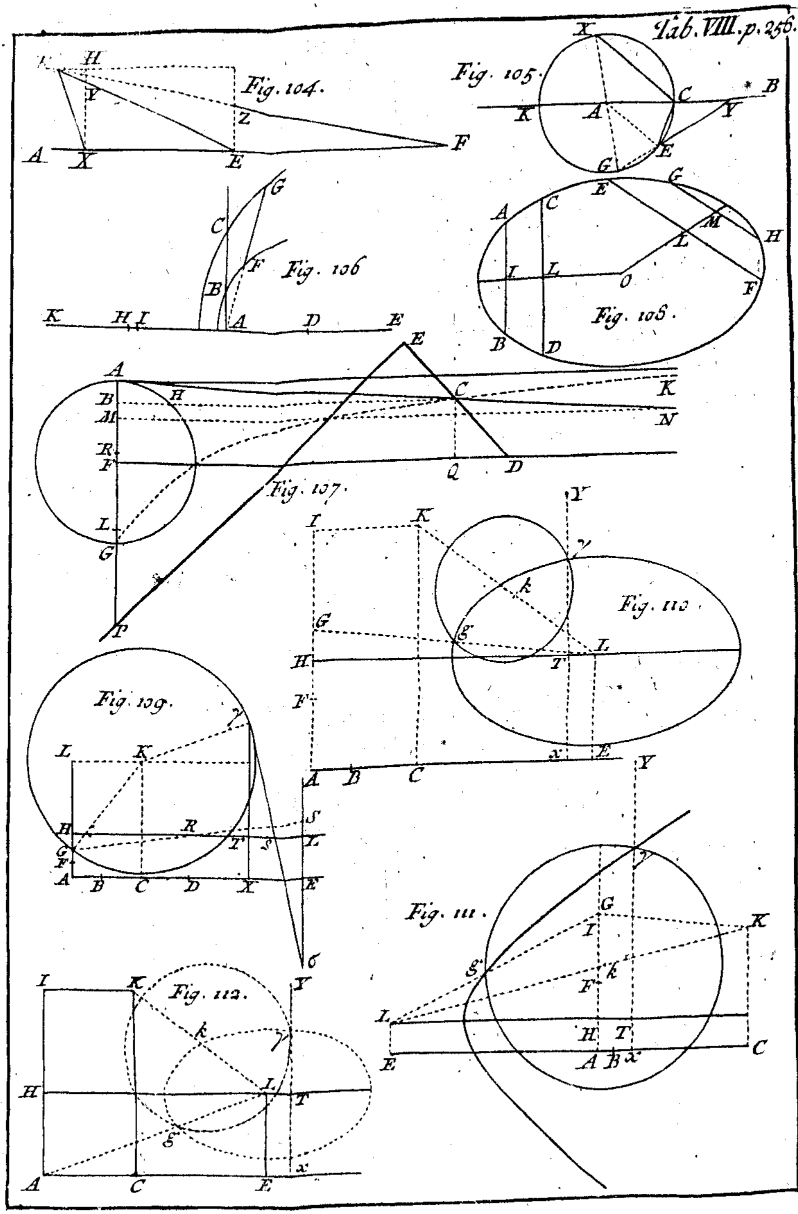
Fig. 104.
Fig. 105.
Fig. 106
Fig. 107.
Fig. 108.
Fig. 109.
Fig. 110.
Fig. 111.
Fig. 112.

Not long since, (*viz. A. D.* 1690.) that excellent Person, Mr. *Joseph Ralphson*, F. R. S. published his *Universal Analysis of Equations*, and illustrated his Method by Plenty of Examples; by all which he has given Indications of a Mathematical Genius, from which the greatest Things may be expected.

By his Example, M. *de Lagney*, an ingenious Professor of Mathematicks at *Paris*, was encouraged to attempt the same Argument; but he being almost altogether taken up in extracting the Roots of pure Powers (especially the Cubick) adds but little about affected Equations, and that pretty much perplexed too, and not sufficiently demonstrated: Yet he gives two very compendious Rules for the Approximation of a Cubical Root; one a Rational, and the other an Irrational one. *Ex. gr.* That the Side of the Cube $aaa + b$ is between $a + \frac{ab}{3aaa+b}$, and $\sqrt{\frac{1}{4}aa + \frac{b}{3a}} + \frac{1}{2}a$. And the Root of the 5th Power, $a^5 + b$, he makes $= \frac{1}{2}a + \sqrt{\sqrt{\frac{1}{4}a^4 + \frac{b}{5a}} - \frac{1}{4}aa}$ (where note, that it is $\frac{1}{4}aa$, not $\frac{1}{2}aa$, as it is erroneously printed in the French Book.) These Rules were communicated to me by a Friend, I having not seen the Book; but having by Trial found the Goodness of them, and admiring the Compendium, I was willing to find out the Demonstration. Which having done, I presently found that the same Method might be accommodated to the Resolution of all Sorts of Equations. And I was the rather inclined to improve these Rules, because I saw that the whole Thing might be explained in a *Synopsis*; and that by this means, at every repeated Step of the Calculus, the Figures already found in the Root, would be at least trebled, which all other Ways are encreased but in an equal Number with the given ones. Now, the forementioned Rules are easily demonstrated from the Genesis of the Cube, and the 5th Power. For, supposing the Side of any Cube $= a + e$, the Cube arising from thence is $aaa + 3aae + 3aee + eee$. And consequently, if we suppose aaa the next less Cube, to any given Non-Cubick Number, then eee will be less than Unity, and the Remainder b, will $=$ the other Members of the Cube, $3aae + 3aee + eee$. Whence rejecting eee upon the Account of its Smallness, we have $b = 3aae + 3aee$. And since aae is much greater than aee, the Quantity

$\frac{b}{3aa}$ will not much exceed e; so that putting $e = \frac{b}{3aa}$ then the Quantity $\frac{b}{3aa + 3ae}$ (to which e is nearly equal) will be found $= \frac{b}{3aa + \frac{3ab}{3aa}}$, or $\frac{b}{3aa + \frac{b}{a}}$, that is, $\frac{ab}{3aa + b} = e$. And so the Side of the Cube $aaa + s$ will be $a + \frac{ab}{3aa + b}$, which is the *Rational Formula* of M. *de Lagney*. But now, if aaa were the next greater Cubick Number to that given, the Side of the Cube $aaa - b$, will, after the same Manner, be found to be $a - \frac{ab}{3aa - b}$. And this easy and expeditious Approximation to the Cubick Root, is only (a very small Matter) erroneous in point of *Defect*, the Quantity e, the Remainder of the Root thus found, coming something less than really it is.

As for the *Irrational Formula*, it is derived from the same Principle, *viz.* $b = 3aae + 3aee$, or $\frac{b}{3a} = ae + ee$, and so $\sqrt{\frac{1}{4}aa + \frac{b}{3a}} = \frac{1}{2}a + e$, and $\sqrt{\frac{1}{4}aa + \frac{b}{3a}} + \frac{1}{2}a = a + e$, the Root sought. Also the Side of the Cube $aaa - b$, after the same Manner, will be found to be $\frac{1}{2}a + \sqrt{\frac{3}{4}aa - \frac{b}{3a}}$. And this *Formula* comes something nearer to the Scope, being erroneous in point of *Excess*, as the other was in *Defect*, and is more accommodated to the Ends of Practice, since the Restitution of the Calculus is nothing else but the continual Addition or Subtraction of the Quantity $\frac{aee}{3a}$, according as the Quantity e can be known. So that we should rather write $\sqrt{\frac{1}{4}a + \frac{b - eee}{3a}} + \frac{1}{2}a$, in the for-

mer

mer Case, and in the latter, $\frac{1}{2}a + \sqrt{\frac{1}{4}aa + \frac{eee - b}{3a}}$.
But by either of the two *Formulas* the Figures already known in the Root to be extracted are at least tripled; which I conclude will be very grateful to all the Students in Arithmetick, and I congratulate the Inventor upon the Account of his *Discovery*.

But that the Use of these Rules may be the better perceived, I think it proper to subjoyn an Example or two. Let it be proposed to find the Side of the double Cube, or $aaa + b = 2$. Here $a = 1$, and $\frac{b}{3a} = \frac{1}{3}$ and so $\frac{1}{2} + \sqrt{\frac{7}{12}}$, or 1,26, be found to be the true Side nearly. Now, the Cube of 1,26, is 2,000376, and so $0,63 + \sqrt{,3969 - \frac{,000376}{3,78}}$: or $0,63 + \sqrt{,3968005291005291} = 1,259921049895 -$; which in 13 Figures gives the Side of the double Cube with very little Trouble, *viz.* by one only Division, and the Extraction of the Square Root; when as by the common Way of working, how much Pains it would have cost, the Skilful very well know. This Calculus a Man may continue as far as he pleases, by encreasing the Square by the Addition of the Quantity $\frac{eee}{3a}$; which Correction, in this Case, will give but the Encrease of Unity in the 14th Figure of the Root.

Example II. Let it be proposed to find the Sides of a Cube equal to that English Measure commonly called a Gallon which contains 231 solid Ounces. The next less Cube is 216, whose Side $6 = a$, and the Remainder $15 = b$; and so for the first Approximation, we have $3 + \sqrt{9 + \frac{5}{6}} =$ the Root. And since $\sqrt{9,8333\ldots}$ is 3,1358...., it is plain, that $6,358 = a + e$. Now, let $6,1358 = a$; and we shall then have for its Cube 231,000853894712, and according to the Rule, $3,0679 + \sqrt{9,41201041 - \frac{,000853894712}{18,4070}}$ is most accurately equal to the Side of the given Cube, which, within the Space of an Hour, I determined by Calculation to be 6,135792439661958971, which is exact in the 18th Figure, defective

defective in the 19th. And this *Formula* is deservedly preferable to the *Rationale*, upon the Account of the great Divisor, which is not to be managed without a great deal of Labour; whereas the Extraction of the Square Root proceeds much more easily, as manifold Experience has taught me.

But the Rule for the Root of a pure Surfolid, or the 5th Power, is of something a higher Enquiry, and does much more perfectly yet do the Business: for it does at least Quintuple the given Figures of the Root, neither is the Calculus very large or operose. Though the Author no where shews his Method of Invention, or any Demonstration, although it seems to be very much wanting; especially since all Things are not right in the printed Book, which may easily deceive the unskilful. Now the 5th Power of the Side $a+e$ is composed of these Members, $a^5+5a^4e+10a^3e^2+10a^2e^3+5ae^4+e^5=a^5+b$; from whence $b=5a^4e+10a^3e^2+10a^2e^3+5ae^4$, rejecting e^5 because of its smallness. Whence $\frac{b}{5a}=a^3e+2a^2e^2+2ae^3+e^4$, and adding on both Sides $\frac{1}{4}a^4$, we shall have $\sqrt{\frac{1}{4}a^4+\frac{b}{5a}}=\sqrt{\frac{1}{4}a^4+a^3e+2a^2e^2+2ae^3+e^4}=\frac{1}{2}aa+ae+ee$. Then subtracting $\frac{1}{4}aa$ from both Sides, $\frac{1}{2}a+e$ will $=\sqrt{\sqrt{\frac{1}{4}a^4+\frac{b}{5a}}-\frac{1}{4}aa}$; to which, if $\frac{1}{2}a$ be added, then will $a+e=\frac{1}{2}a+\sqrt{\sqrt{\frac{1}{4}a^4+\frac{b}{5a}}-\frac{1}{4}aa}=$ the Root of the Power a^5+b. But if it had a^5-b (the Quantity a being too great) the Rule would have been thus, $\frac{1}{2}a+\sqrt{\sqrt{\frac{1}{4}a^4-\frac{b}{5a}}-\frac{1}{4}aa}$. And this Rule approaches wonderfully, so that there is hardly any need of Restitution.

But while I considered these Things with my self, I light upon a general Method for the *Formulas* of all Powers whatsoever, and (which being handsome and concise enough) I thought I would not conceal from the Publick.

These

These Formulas, (as well the *Rational* as the *Irrational* ones) are thus.

$$\sqrt{aa+b}=\sqrt{aa+b}, \text{ or } a+\frac{ab}{2aa+\frac{1}{2}b}.$$

$$\sqrt[3]{a^3+b}=\tfrac{1}{2}a+\sqrt{\tfrac{1}{4}aa+\frac{b}{3a}}, \text{ or } a+\frac{ab}{3aaa+b}.$$

$$\sqrt[4]{a^4+b}=\tfrac{2}{3}a+\sqrt{\tfrac{1}{9}aa+\frac{b}{6aa}}, \text{ or } a+\frac{ab}{4a^4+\frac{3}{2}b}.$$

$$\sqrt[5]{a^5+b}=\tfrac{3}{4}a+\sqrt{\tfrac{1}{16}aa+\frac{b}{10a^3}}, \text{ or } a+\frac{ab}{5a^5+2b}.$$

$$\sqrt[6]{a^6+b}=\tfrac{4}{5}a+\sqrt{\tfrac{1}{25}aa+\frac{b}{15a^4}}, \text{ or } a+\frac{ab}{6a^6+\frac{5}{2}b}.$$

$$\sqrt[7]{a^7+b}=\tfrac{5}{6}a+\sqrt{\tfrac{1}{36}aa+\frac{b}{21a^5}}, \text{ or } a+\frac{ab}{7a^7+3b}.$$

And so also of the other higher Powers. But if a were assumed bigger than the Root sought, (which is done with some Advantage, as often as the Power to be resolved is much nearer, the Power of the *next greater* whole Number, than of the *next less*) in this Case, *Mutatis Mutandis*, we shall have the same Expressions of the Roots, *viz.*

$$\sqrt{aa-b}=\sqrt{aa-b}, \text{ or } a-\frac{ab}{2aa-\frac{1}{2}b}.$$

$$\sqrt[3]{a^3-b}=\tfrac{1}{2}a+\sqrt{\tfrac{1}{4}aa-\frac{b}{3a}}, \text{ or } a-\frac{ab}{3a^3-b}.$$

$$\sqrt[4]{a^4-b}=\tfrac{2}{3}a+\sqrt{\tfrac{1}{9}aa-\frac{b}{6aa}}, \text{ or } a-\frac{ab}{4a^4-\frac{3}{2}b}.$$

$$\sqrt[5]{a^5-b}=\tfrac{3}{4}a+\sqrt{\tfrac{1}{16}aa-\frac{b}{10a^3}}, \text{ or } a-\frac{ab}{5a^5-2b}.$$

$$\sqrt[6]{a^6-b}=\tfrac{4}{5}a+\sqrt{\tfrac{1}{25}aa-\frac{b}{15a^4}}, \text{ or } a-\frac{ab}{6a^6-\frac{5}{2}b}.$$

$$\sqrt[7]{a^7-b}=\tfrac{5}{6}a+\sqrt{\tfrac{1}{36}aa-\frac{b}{21a^5}}, \text{ or } a-\frac{ab}{7a^7-3b}.$$

And

And within these two Terms the true Root is ever found, being something nearer to the *Irrational* than the *Rational* Expression. But the Quantity e found by the *Irrational Formula*, is always too great, as the Quotient resulting from the *Rational Formula*, is always too little. And consequently, if we have $+b$, the *Irrational Formula* gives the Root something greater than it should be, and the *Rational* something less. But contrariwise if it be $-b$.

And thus much may suffice to be said concerning the Extraction of the Roots of pure Powers; which notwithstanding, for common Uses, may be had much more easily by the Help of the Logarithms. But when a Root is to be determined very accurately, and the Logarithmick Tables will not reach so far, then we must necessarily have Recourse to these, or such like Methods. Farther, the Invention and Contemplation of these *Formulas*, leading me to a certain universal Rule of adfected Equations, (which I hope will be of Use to all the Students in *Algebra* and *Geometry*) I was willing here to give some Account of this Discovery, which I will do with all the Perspicuity I can. I had given at N°. 188. of the *Transactions*, a very easy and general Construction of all adfected Equations, not exceeding the Biquadratick Power; from which Time I had a very great Desire of doing the same in Numbers. But quickly after, Mr. *Ralphson* seemed in great Measure to have satisfied this Desire, until Mr. *Lagney*, by what he had performed in his Book, intimated, that the Thing might be done more compendiously yet. Now, my Method is thus:

Let z, the Root of any Equation, be imagined to be composed of the Parts $a+$, or $-e$, of which, let a be assumed as near z as is possible; which is notwithstanding not *necessary*, but only *commodious*. Then from the Quantity $a+e$, or $a-e$, let there be formed all the Powers of z, found in the Equation, and the Numerical Co-efficients be respectively affixed to them: Then let the Power to be *resolved* be subtracted from the Sum of the given Parts (in the first Column where e is not found) which they call the *Homogeneum Comparationis*, and let the Difference be $\pm b$. In the next Place, take the Sum of all the Co-efficients of e in the second Column, to which put $=s$. Lastly, in the third Column let there be put down the Sum of all the Co-efficients of ee, which Sum call t. Then will the Root z stand thus in the *Rational Formula*, viz. $z=a+\frac{sb}{ss\pm tb}$; and thus

thus in the *Irrational Formula*, viz. $z = a \mp \frac{\frac{1}{2}s \pm \sqrt{\frac{1}{4}ss \mp bt}}{t}$; which perhaps it may be worth while to illuſtrate by ſome Examples. And inſtead of an *Inſtrument* let this *Table* ſerve, which ſhews the Geneſis of the ſeveral Powers of $a \pm e$, and if need be, may eaſily be continued farther; which, for its Uſe, I may rightly call a *General Analytical Speculum.* The forementioned Powers ariſing from a continual Multiplication by $a + e$ ($= z$) come out thus with their adjoined Coefficients.

Tabula Poteſtatum.

		s	t	u	w	x	y	
$lz^7 =$	la^7	$+ 7la^6e$	$+ 21la^5ee$	$+ 35la^4e^3$	$+ 35la^3e^4$	$+ 21la^2e^5$	$+ 7lae^6$	$+ le^7$
$kz^6 =$	ka^6	$+ 6ka^5e$	$+ 15ka^4ee$	$+ 20ka^3e^3$	$+ 15ka^2e^4$	$+ 6kae^5$	$+ ke^6$	
$bz^5 =$	ba^5	$+ 5ba^4e$	$+ 10ba^3ee$	$+ 10ba^2e^3$	$+ 5bae^4$	$+ be^5$		
$gz^4 =$	ga^4	$+ 4ga^3e$	$+ 6ga^2ee$	$+ 4gae^3$	$+ ge^4$			
$fz^3 =$	fa^3	$+ 3fa^2e$	$+ 3faee$	$+ fe^3$				
$dz^2 =$	da^2	$+ 2dae$	$+ dee$					
$cz =$	ca	$+ ce$						

But now, if it be $a - e = z$, the Table is composed of the same Members, only the odd Powers of e, as e, e^3, e^5, e^7 are Negative, and the even Powers, as e^2, e^4, e^6, Affirmative. Also, let the Sum of the Co-efficients of the Side e, be $= s$; the Sum of the Co-efficients of the Square $ee = t$, the Sum of the Co-efficient of $e^3 = u$, of $e^4 = w$, of $e^5 = x$, of $e^6 = y$, &c. But now, since e is supposed only a small Part of the Root that is to be enquired, all the Powers of e will be much less than the correspondent Powers of a, and so far the first Hypothesis; all the superior ones may be rejected; and forming a new Equation, by substituting $a + e = z$, we shall have (as was said) $\pm b = \pm se \pm tee$. The following Examples will make this more clear.

Example I. Let the Equation $z^4 - 3z^2 + 75z = 10000$ be proposed. For the first Hypothesis, let $a = 10$, and so we have this Equation;

$$\begin{array}{lllll}
z^4 = + & a^4 & 4a^3e + 6a^2ee & 4ae^3e + e^4 \\
-dz^2 = -da^2 & & dae - dee & \\
+cz = +ca & & ce & \\
= + 10000 & & 4000e + 600ee & 40e^3 + e^4 \\
- & 300 & 60e - 3ee & \\
+ & 750 & 75e & \\
- & 10000 & & \\
\hline
+ & 450 & -4015e + 597ee - 40e^3 + e^4 = 0 & \\
 & & \quad s \qquad\quad t \qquad\quad u &
\end{array}$$

The Signs + and —, with respect to the Quantities e and e^3, are left as doubtful, until it be known whether e be Negative or Affirmative; which Thing creates some Difficulty, since that in Equations that have several Roots, the *Homogenea Comparationis* (as they term them) are oftentimes encreased by the minute Quantity a, and on the contrary, *that* being encreased, *they* are diminished. But the Sign of e is determined from the Sign of the Quantity b. For taking away the *Resolvend* from the *Homogeneal* formed of a; the Sign of se (and consequently of the prevailing Parts in the Composition of it) will always be contrary to the Sign of the Difference b. Whence it will be plain, whether it must be $+ e$, or $- e$; and consequently, whether a be taken greater or less than the *true Root*. Now the Quantity e is $= \frac{\frac{1}{2}s - \sqrt{\frac{1}{4}ss - bt}}{t}$, when b and t have the same Sign, but

when

When the Signs are different, e is $= \frac{\sqrt{\frac{1}{4}ss + bt} - \frac{1}{2}s}{t}$. But after it is found that it will be $-e$, let the Powers e, e^3, e^5, &c. in the Affirmative Members of the Equation be made Negative, and in the Negative be made Affirmative; that is, let them be written with the contrary Sign. On the other hand, if it be $+e$ let those forementioned Powers be made Affirmative in the Affirmative, and Negative in the Negative Members of the Equation.

Now we have in this Example of ours, 10450 instead of the Resolved 10000, or $b = +450$, whence it is plain, that a is taken greater than the Truth, and consequently, that it is $-e$. *Hence the Equation comes to be*, $10450 - 4015e + 597ee - 4e^3 + e^4 = 10000$. That is, $450 - 4015e + 597ee = 0$; and so $450 = 4015e - 597ee$, or $b = se - tee$, whose Root $e = \frac{\frac{1}{2}s - \sqrt{\frac{1}{4}ss - bt}}{t}$, or $\frac{s}{2t} - \sqrt{\frac{ss}{4tt} - \frac{b}{t}}$; that is, *in the* present Case,

$e = \frac{2007\frac{1}{2} - \sqrt{3761406\frac{1}{4}}}{597}$, from whence we have the Root sought, 9,886, which is near the Truth. But then substituting this for a second *Supposition*, there comes $a + e = z$, most accurately, 9,8862603936495 .... scarce exceeding the Truth by 2 in the last Figure, *viz.* when $\frac{\sqrt{\frac{1}{4}ss + bt} - \frac{1}{2}s}{t} = e$. And this (if need be) may be yet much farther verified, by subtracting (if it be $+e$) the Quantity $\frac{\frac{1}{2}ue^3 + \frac{1}{2}e^4}{\sqrt{\frac{1}{4}ss + tb}}$, from the Root before found; or (if it be $-e$) by adding $\frac{\frac{1}{2}ue^3 - \frac{1}{2}e^4}{\sqrt{\frac{1}{4}ss - tb}}$ to that Root. Which Compendium is so much the more valuable, in that sometimes from the first Supposition alone, but always from the second, the Calculus may be continued (keeping the same Co-efficients) as far as one pleases. *It may be noted*, that the fore-mentioned Equation has also a Negative Root, *viz.* $z = 10,26$ .... which any one that has a Mind, may determine more accurately.

EXAMPLE II. Suppose $z^3 - 17z + 54z = 350$, and let $a = 10$. Then according to the Prescript of the Rule,

$$\begin{aligned} + z^3 &= a^3 + 3a^2e + 3aee + e^3 \\ - dz^2 &= da^2 - 2dae - de^2 \\ + cz &= ca + ce \end{aligned}$$

$$\begin{array}{llll} & & b & s & t \\ \text{That is,} & + 1000 & + 300e & + 30e^2 & + e^3 \\ & - 1700 & - 340e & - 17e^2 & \\ & + 540 & + 54e & & \\ & - 350 & & & \\ \hline \text{Or,} & - 510 & + 14e & + 13ee & + e^3 = 0 \end{array}$$

Now, since we have -510, it is plain, that a is assumed less than the Truth, and consequently that e is Affirmative. And from (the Equation) $510 = 14e + 13e^2$, comes $e = \frac{\sqrt{bt + \frac{1}{4}ss} - \frac{1}{2}s}{t} = \frac{\sqrt{6679} - 7}{13}$. Whence $z = 15{,}7\ldots$, which is too much, because of a taken wide. Therefore, Secondly, let $a = 15$, and by the like Way of Reasoning we shall find $e = \frac{\frac{1}{2}s - \sqrt{\frac{1}{4}ss - tb}}{t} = \frac{109\frac{1}{2} - \sqrt{11710\frac{1}{4}}}{28}$, and consequently, $z = 14{,}954068$. If the Operation were to be repeated the third Time, the Root will be found conformable to the Truth as far as the 25th Figure; but he that is contented with fewer, by writing $tb \pm te^3$ instead of tb, or subtracting or adding $\frac{\frac{1}{2}e^3}{\sqrt{\frac{1}{4}ss + tb}}$ to the Root before found, will presently obtain his End. Note, the Equation proposed is not explicable by any other Root, because the *Resolvend* 350 is greater than the Cube of $\frac{17}{3}$, or $\frac{d}{3}$.

EXAMPLE III. Let us take the Equation $z^4 - 80z^3 + 1998z^2 - 14937z + 5000 = 0$, which Dr. *Wallis* uses *Chap.* 62. of his *Algebra*, in the Resolution of a very difficult Arithmetical Problem, where, by *Vieta's* Method, he has obtained the Root most accurately; and Mr. *Ralphson* brings it also as an Example of his Method, *Page* 25, 26. Now this Equation is of the Form which may have several Affirmative Roots, and (which increases the Difficulty) the *Coefficients* are very great in respect of the *Resolvend* given.

But

But that it may be the easier managed, let it be divided, and according to the known Rules of *Pointing*, let $-z^4 + 8z^3 - 20z^2 + 15z = 0,5$ (where the Quantity z is $\frac{1}{10}$ of z in the Equation proposed) and for the first Supposition, let $a = 1$. Then $+2 - 5e - 2e^2 + 4e^3 - e^4 - 0,5 = 0$; that is, $1\frac{1}{2} = 5e + 2ee$; hence $e = \frac{\sqrt{\frac{1}{4}ss + bt} - \frac{1}{2}s}{t}$ is $= \frac{\sqrt{37} - 5}{4}$, and so $z = 1,27$; whence it is manifest, that 12,7 is near the true Root of the Equation proposed. Now, Secondly, let us suppose $z = 12,7$, and then according to the Directions of the Table of Powers, there arises

b	s	t	u	
$-26014,4641$	$-8193,532e$	$-967,74e^2$	$-50,8e^3$	$-e^4$
$+163870,640$	$+38709,60\ e$	$+3048\ e^2$	$+80\ e^3$	
$-322257,42$	$-50749,2\ e$	$-1998\ e^2$		
$+189699,9$	$+14937,\ e$			
$-5000.$				
$+298,6559$	$-5296,132e$	$+82,26e^2$	$+29,2e^3$	$-e^4 = 0.$

And so $-298,6559 = -5296,132e + 82,26ee$, whose Root e (according to the Rule) $= \frac{\frac{1}{2}s - \sqrt{\frac{1}{4}ss - bt}}{t}$, comes to $\frac{2648,066 - \sqrt{6987686,106022}}{82,26} = ,05644080331 \ldots = e$ less than the Truth. But that it may be corrected, it is to be considered, that $\frac{\frac{1}{2}ue^3 - \frac{1}{2}e^4}{\sqrt{\frac{1}{4}ss - bt}}$, or $\frac{,0026201 \ldots\ldots}{2643,423 \ldots\ldots}$ is $,00000099$, and consequently e corrected, is $= 0564470448$. And if you desire yet more Figures of the Root, from the e corrected, let there be made $tue^3 - te^4 = 0,43105602423 \ldots$, and $\frac{\frac{1}{2}s - \sqrt{\frac{1}{4}ss - bt - tue^3 + te^4}}{t}$, or which is all one,

$$\frac{2648,066 - \sqrt{6987685,67496597577 \ldots}}{82,26} =$$

$,05644179448074402 = e$; whence $a + e = z$ the Root is most accurately $12,75644179448074402 \ldots$, as Dr. *Wallis* found in the forementioned Place; where it may be observed, that

that the Repetition of the *Calculus* does ever triple the true Figures in the assumed a, which the first Correction, or $\frac{\frac{1}{2}ue^3-\frac{1}{2}e^4}{\sqrt{\frac{1}{4}ss-bt}}$ does quintuple; which is also commodiously done by the *Logarithms*. But the other Correction after the first, does also double the Number of Figures, so that it renders the *assumed* altogether Seven-fold; yet the first Correction is abundantly sufficient for Arithmetical Uses, for the most Part.

But as to what is said concerning the Number of Places rightly taken in the Root, I would have understood so, that when a is but $\frac{1}{10}$ Part distant from the true Root, then the first Figure is rightly assumed; if it be within $\frac{1}{100}$ Part, then the two first Figures are rightly assumed; if within $\frac{1}{1000}$, and then the three first are so; which consequently, managed according to our Rule, do presently become nine Figures.

It remains now that I add something concerning our *Rational Formula*, viz. $e=\frac{sb}{ss+tb}$, which seems expeditious enough, and is not much inferior to the former, since it will triple the given Number of Places. Now, having formed an Equation from $a \pm e = z$, as before, it will presently appear, whether a be taken greater or lesser than the Truth; since se ought always to have a Sign contrary to the Sign of the Difference of the *Resolvend*, and its *Homogeneal* produced from a. Then supposing $+b+se+a-tee=0$, the Divisor is $ss-tb$, as often as t and b have the same Signs; but it is $ss+bt$, when they have different ones. But it seems most commodious for Practice, to write the *Theorem* thus, $e=\frac{b}{s}+\frac{tb}{s}$, since this Way the Thing is done by one Multiplication and two Divisions, which otherwise would require three Multiplications, and one Division.

Let us take now one Example of this Method, from the Root (of the forementioned Equation) 12,7 ...., where

$$\underset{+b}{298{,}6559} - \underset{-s}{5296{,}132\,e} + \underset{+t}{82{,}26\,ee} + \underset{+u}{29{,}2\,e^3} - e^4 = 0,$$

and so $\frac{b}{s}-\frac{tb}{s}=e$; that is, (let it be as s to t, so b to

$\frac{\text{[illegible]}}{s} = 5296,132$) 298,6559 into 82,26 (4,63875 . . . wherefore the Divisor is $s - \frac{tb}{s} = 5291,49325$ . . . . .) 298,6559 0,056441 . . . . . $= e$, that is, to five true Figures, added to the Root that was taken. But this *Formula* cannot be corrected, as the foregoing *Irrational* one was; and so if more Figures of the Root are desired, it is the best to make a new Supposition, and repeat the *Calculus* again: And then a new Quotient, tripling the known Figures of the Root, will abundantly satisfy even the most scrupulous.

F I N I S.

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