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LATITUDE DEVELOPMENTS CONNECTED WITH GEODESY AND CARTOGRAPHY

WITH TABLES

INCLUDING A TABLE FOR

LAMBERT EQUAL-AREA MERIDIONAL PROJECTION

BY

OSCAR S. ADAMS
Geodetic Computer

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FOREWORD.

There are five different kinds of latitude that come under consideration in the application of mathematical analysis to questions of geodesy and cartography. It is the aim of this publication to express the difference between the geodetic or astronomic latitude and each of the various four other kinds of latitude in a series of the sines of the multiple arcs. This difference in each case is obtained in an expression in the sines of the multiple arcs of the geodetic or astronomic latitude and also in a series of the sines of the multiple arcs of the other latitude in question.

The analysis connected with the development of both the isometric or conformal latitude [a] and of the authalic or equal-area latitude[a] is given in some degree of detail, since it is a good example of the application of mathematical analysis to such questions.

The series are derived in their general form in the first instance in which no geodetic constant appears except the eccentricity. At the end of the text in this publication the numerical values of the various coefficients are given computed for the Clarke spheroid of 1866. This is the spheroid that is used for all geodetic purposes in North America. Finally, tables are given of the results of the computations for this spheroid calculated for every half degree of latitude. These results and tables will be useful in connection with all geodetic and cartographic questions in which it is desired to take into consideration the spheroidal shape of the earth. It is believed that no previous table has been computed for the Clarke spheroid of 1866, at least none for half degrees of latitude. It is thought that the idea of the authalic latitude is new in the science of cartography. It has been applied in the computation of the elements of an Albers' equal-area projection for the United States, and it has been found materially to simplify the calculations to be performed.

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[a] For the full definition of these terms see pp. 8 and 10.

It is thought that in a later publication on equivalent or equal-area projections this latitude may be applied in the theory of the various types of projection belonging to this class.

In addition to the latitude tables there are given tables for transformation from latitude and longitude to arc distance and azimuth from a point on the equator. After these is given a table of the radial distance for a Lambert azimuthal equal-area projection upon a meridional plane, and finally a table of the coordinates for such a projection.

It is hoped that the analysis employed in the derivation of the formulas may be of interest to those who have to deal with the applications of mathematical theory to such problems as arise in practice. A few examples of such applications are of more value than any amount of the theory without the practical working out of the results in specific cases. "Learn to do by doing" is a safe maxim at all times. Finally, the numerical form of the results should appeal to those who wish to use these latitudes in questions of geodesy or cartography. The numerical forms of the expansions are given both in numbers and in logarithms. If a multiplying machine is available the coefficients expressed in numbers are more useful, but in case such a machine is not at hand it is necessary to resort to logarithms. With a five-place table of sines the results ought to be good to tenths of a second, and a six-place table should give results good to hundredths of a second. The results of computation given in the tables were derived from an eight-place table of natural sines.

LATITUDE DEVELOPMENTS CONNECTED WITH GEODESY AND CARTOGRAPHY, WITH TABLES, INCLUDING A TABLE FOR LAMBERT EQUAL-AREA MERIDIONAL PROJECTION.

By Oscar S. Adams,
Geodetic Computer, U. S. Coast and Geodetic Survey.

DERIVATION OF DEFINITIONS.

In considering subjects connected with geodesy and cartography there are five different kinds of latitude that are found to be of interest and of use in practical applications. We shall now proceed to apply analysis in the derivation of the definitions of these latitudes.

If the meridian ellipse is defined by equations in the parametric form

$$x = a \cos \theta$$
$$y = b \sin \theta,$$

then θ is called the parametric latitude, a is the semimajor axis, and b is the semiminor axis of the meridian ellipse.

The geodetic or astronomic latitude is the angle which the normal at a given point of the ellipse makes with the axis of x. This latitude, denoted by φ, will then be defined analytically by the expression

$$\tan \varphi = -\frac{dx}{dy},$$

since it is perpendicular to the tangent at the point x, y. But from the parametric equations we get

$$-\frac{dx}{dy} = \frac{a \sin \theta}{b \cos \theta} = \frac{a}{b} \tan \theta.$$

Hence

$$\tan \varphi = \frac{a}{b} \tan \theta,$$

or

$$\tan \theta = \frac{b}{a} \tan \varphi.$$

The eccentricity, ϵ, of the ellipse is defined by the equation

$$\epsilon^2 = \frac{a^2 - b^2}{a^2}.$$

From this expression we get

$$\frac{b}{a} = (1 - \epsilon^2)^{1/2};$$

therefore

$$\tan \theta = (1 - \epsilon^2)^{1/2} \tan \varphi.$$

The geocentric latitude is the angle formed with the axis of x by the radius vector from the center of the ellipse to the point x, y. Denoting this latitude by ψ, we define it by the form

$$\begin{aligned} \tan \psi &= \frac{y}{x} \\ &= \frac{b}{a} \tan \theta \\ &= (1 - \epsilon^2)^{1/2} \tan \theta; \end{aligned}$$

or, in terms of φ, by substituting the value of $\tan \theta$ in terms of φ, this becomes

$$\tan \psi = (1 - \epsilon^2) \tan \varphi.$$

In the theory of the conformal representation of the spheroid, the function, x, that forms with the longitude, λ, a set of isometric coordinates is defined by the integral

$$x = -\int_{p}^{\pi/2} \frac{(1 - \epsilon^2)\, dp}{(1 - \epsilon^2 \cos^2 p) \sin p},$$

in which p is the geodetic colatitude.[a]

$$x = -\int_{p}^{\pi/2} \frac{\cos \frac{p}{2}}{\sin \frac{p}{2}} \frac{dp}{2} + \int_{p}^{\pi/2} \frac{-\sin \frac{p}{2}}{\cos \frac{p}{2}} \frac{dp}{2}$$

$$-\frac{\epsilon}{2}\int_{p}^{\pi/2} \frac{-\epsilon \sin p\, dp}{1 + \epsilon \cos p} + \frac{\epsilon}{2}\int_{p}^{\pi/2} \frac{\epsilon \sin p\, dp}{1 - \epsilon \cos p},$$

[a] See "General Theory of the Lambert Conformal Conic Projection," Special Publication No. 53, United States Coast and Geodetic Survey.

or by integration

$$x = \log_e \sin \frac{p}{2} - \log_e \cos \frac{p}{2} + \frac{\epsilon}{2} \log_e (1 + \epsilon \cos p)$$
$$- \frac{\epsilon}{2} \log_e (1 - \epsilon \cos p)$$
$$= \log_e \tan \frac{p}{2} + \frac{\epsilon}{2} \log_e \left(\frac{1 + \epsilon \cos p}{1 - \epsilon \cos p} \right).$$

On passing to exponentials this becomes

$$e^x = \tan \frac{p}{2} \cdot \left(\frac{1 + \epsilon \cos p}{1 - \epsilon \cos p} \right)^{\epsilon/2}.$$

For the sphere, putting $\epsilon = 0$, the corresponding coordinate, y, is defined by the integral

$$y = -\int_z^{\pi/2} \frac{dz}{\sin z},$$

in which z is the colatitude on the sphere.

$$y = -\int_z^{\frac{\pi}{2}} \frac{\cos \frac{z}{2}}{\sin \frac{z}{2}} \frac{dz}{2} + \int_z^{\frac{\pi}{2}} \frac{-\sin \frac{z}{2}}{\cos \frac{z}{2}} \frac{dz}{2}$$

or by integration

$$y = \log_e \tan \frac{z}{2}$$

or

$$e^y = \tan \frac{z}{2}.$$

If we now define z by the equation

$$\tan \frac{z}{2} = \tan \frac{p}{2} \cdot \left(\frac{1 + \epsilon \cos p}{1 - \epsilon \cos p} \right)^{\epsilon/2},$$

we shall have determined a conformal representation of the spheroid upon a sphere. If we denote the isometric latitude by χ, and in place of z substitute its value $\frac{\pi}{2} - \chi$ and for p its value $\frac{\pi}{2} - \varphi$, we get the definition of the isometric latitude in the form

$$\tan\left(\frac{\pi}{4} + \frac{\chi}{2}\right) = \tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right) \cdot \left(\frac{1 - \epsilon \sin \varphi}{1 + \epsilon \sin \varphi} \right)^{\epsilon/2}.$$

This latitude χ has been called the isometric latitude because it is determined by means of the isometric coordinate.

Besides these latitudes there is another latitude that arises when the spheroid is projected upon a sphere of equivalent surface in such a way that the representation is equivalent or equal area in every part. The element of area upon the spheroid is given by the equation

$$dS=\frac{a^2(1-\epsilon^2)\cos\varphi}{(1-\epsilon^2\sin^2\varphi)^2}\,d\varphi\,d\lambda.$$

The area of a section of a lune of width $d\lambda$ from the Equator to the parallel of latitude ϕ is given by the following integral multiplied by $d\lambda$

$$S=\int_0^\varphi\frac{a^2(1-\epsilon^2)\cos\varphi}{(1-\epsilon^2\sin^2\varphi)^2}\,d\varphi.$$

But

$$a^2(1-\epsilon^2)=b^2;$$

hence

$$S=b^2\int_0^\varphi\frac{\cos\varphi\,d\varphi}{(1-\epsilon^2\sin^2\varphi)^2}=\frac{b^2}{4}\int_0^\varphi\frac{\cos\varphi\,d\varphi}{(1-\epsilon\sin\varphi)^2}$$

$$+\frac{b^2}{4}\int_0^\varphi\frac{\cos\varphi\,d\varphi}{(1+\epsilon\sin\varphi)^2}+\frac{b^2}{4}\int_0^\varphi\frac{\cos\varphi\,d\varphi}{1+\epsilon\sin\varphi}-\frac{b^2}{4}\int_0^\varphi\frac{-\cos\varphi\,d\varphi}{1-\epsilon\sin\varphi}.$$

By integration this becomes

$$S=\frac{b^2}{4\epsilon}\left[\frac{1}{1-\epsilon\sin\varphi}-\frac{1}{1+\epsilon\sin\varphi}+\log_e(1+\epsilon\sin\varphi)\right.$$

$$\left.-\log_e(1-\epsilon\sin\varphi)\right],$$

or

$$S=b^2\left[\frac{\sin\varphi}{2(1-\epsilon^2\sin^2\varphi)}+\frac{1}{4\epsilon}\log_e\left(\frac{1+\epsilon\sin\varphi}{1-\epsilon\sin\varphi}\right)\right].$$

If c is the radius of the sphere with area equivalent to that of the spheroid, the area of a section of a lune of width $d\lambda$ is equal to the value of the following integral multiplied by $d\lambda$:

$$S'=c^2\int_0^\beta\cos\beta\,d\beta,$$

in which β is the latitude on the sphere.

By integration this gives

$$S' = c^2 \sin \beta.$$

If the $d\lambda$ is taken the same in this case as in that of the spheroid, S' will equal S if β is defined by the equation

$$c^2 \sin \beta = b^2 \left[\frac{\sin \varphi}{2(1-\epsilon^2 \sin^2 \varphi)} + \frac{1}{4\epsilon} \log_e \left(\frac{1+\epsilon \sin \varphi}{1-\epsilon \sin \varphi} \right) \right].$$

If the right-hand member is developed in a series, it will be equal to the sum of the series of its two parts.

But we have

$$\frac{\sin \varphi}{2(1-\epsilon^2 \sin^2 \varphi)} = \frac{1}{2} \sin \varphi + \frac{1}{2} \epsilon^2 \sin^3 \varphi + \frac{1}{2} \epsilon^4 \sin^5 \varphi + \frac{1}{2} \epsilon^6 \sin^7 \varphi + \cdots$$

and

$$\frac{1}{4\epsilon} [\log_e(1+\epsilon \sin \varphi) - \log_e(1-\epsilon \sin \varphi)]$$

$$= \frac{1}{2} \sin \varphi + \frac{1}{6} \epsilon^2 \sin^3 \varphi + \frac{1}{10} \epsilon^4 \sin^5 \varphi + \frac{1}{14} \epsilon^6 \sin^7 \varphi + \cdots$$

Therefore

$$c^2 \sin \beta = b^2 \sin \varphi \left(1 + \frac{2\epsilon^2}{3} \sin^2 \varphi + \frac{3\epsilon^4}{5} \sin^4 \varphi + \frac{4\epsilon^6}{7} \sin^6 \varphi + \cdots \right)$$

We can determine the ratio of b^2 to c^2 by setting both β and φ equal to $\frac{\pi}{2}$, thus introducing the condition that the latitudes shall be equal for this value. This gives us the value

$$\frac{b^2}{c^2} = \frac{1}{1 + \frac{2\epsilon^2}{3} + \frac{3\epsilon^4}{5} + \frac{4\epsilon^6}{7} + \cdots}.$$

We have, then, as the definition of β the equation

$$\sin \beta = \sin \varphi \left(\frac{1 + \frac{2\epsilon^2}{3} \sin^2 \varphi + \frac{3\epsilon^4}{5} \sin^4 \varphi + \frac{4\epsilon^6}{7} \sin^6 \varphi + \cdots}{1 + \frac{2\epsilon^2}{3} + \frac{3\epsilon^4}{5} + \frac{4\epsilon^6}{7} + \cdots} \right).$$

The latitude defined in this manner has been called the authalic latitude to conform to Tissot's term for equivalent or equal-area projections used in his "Mémoire sur la Représentations des Surfaces."

In all projects of equivalent or equal-area mapping, if this latitude is used, the spheroid can be considered as the sphere of equivalent area. That is, the spheroid is first projected equivalently upon a sphere of equal surface and then this sphere is mapped upon the plane. The principle is similar to that employed in conformal mapping when the isometric latitude is employed. This simplifies the computations for Albers' equal-area projection or for the Lambert equal-area projections of any type. In all questions of spheroidal areas bounded by meridians and parallels, the computations can be made with any desired degree of exactness upon the authalic sphere.

RECAPITULATION OF DEFINITIONS.

As we have just shown in considering subjects connected with geodesy and cartography, there are five different kinds of latitude that have to be dealt with. A list of the symbols and definitions is given as follows:

1. φ = geodetic or astronomic latitude.
2. ψ = geocentric latitude.
3. θ = reduced or parametric latitude.
4. χ = isometric latitude.
5. β = authalic latitude.

The last four are defined in terms of the first as follows:

$$\tan\psi = (1-\epsilon^2)\tan\varphi,$$

$$\tan\theta = (1-\epsilon^2)^{1/2}\tan\varphi,$$

$$\tan\left(\frac{\pi}{4}+\frac{\chi}{2}\right) = \tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)\cdot\left(\frac{1-\epsilon\sin\varphi}{1+\epsilon\sin\varphi}\right)^{\epsilon/2},$$

$$\sin\beta = \sin\phi\left(\frac{1+\frac{2\epsilon^2}{3}\sin^2\varphi+\frac{3\epsilon^4}{5}\sin^4\varphi+\frac{4\epsilon^6}{7}\sin^6\varphi+\cdots}{1+\frac{2\epsilon^2}{3}+\frac{3\epsilon^4}{5}+\frac{4\epsilon^6}{7}+\cdots}\right).$$

In the application of these latitudes it is generally desirable to have given the difference between φ and the various other latitudes. It is most convenient to have this difference expressed in terms of the sines of the multiple arcs. The problem is then to determine these series in the most satisfactory manner. In most cases this can be accomplished by the use of the principles of the functions of a complex variable. We will now proceed to apply this process wherever possible.

DEVELOPMENT OF $\varphi-\psi$ IN TERMS OF φ.

We have

$$\tan(\varphi-\psi)=\frac{\tan\varphi-\tan\psi}{1+\tan\varphi\tan\psi}$$

but

$$\tan\psi=(1-\epsilon^2)\tan\varphi$$

hence

$$\tan(\varphi-\psi)=\frac{\epsilon^2\tan\varphi}{1+(1-\epsilon^2)\tan^2\varphi}$$

$$=\frac{\epsilon^2\sin 2\varphi}{2-\epsilon^2+\epsilon^2\cos 2\varphi.}$$

Let

$$m=\frac{\epsilon^2}{2-\epsilon^2},$$

then

$$\varphi-\psi=\tan^{-1}\left(\frac{m\sin 2\varphi}{1+m\cos 2\varphi}\right).$$

Now,

$$\log_e(1+m\,e^{2i\varphi})=m\,e^{2i\varphi}-\frac{m^2}{2}e^{4i\varphi}+\frac{m^3}{3}e^{6i\varphi}-\ \cdot\ \cdot\ \cdot\ ;$$

but

$$\log_e(1+m\,e^{2i\varphi})=\frac{1}{2}\log_e(1+2\,m\cos 2\varphi+m^2)$$

$$+i\tan^{-1}\left(\frac{m\sin 2\varphi}{1+m\cos 2\varphi}\right),$$

and

$$m\,e^{2i\varphi}-\frac{m^2}{2}e^{4i\varphi}+\frac{m^3}{3}e^{6i\varphi}-\ \cdot\ \cdot\ \cdot\ \cdot=m\cos 2\varphi+i\,m\sin 2\varphi$$

$$-\frac{m^2}{2}\cos 4\varphi-i\frac{m^2}{2}\sin 4\varphi+\frac{m^3}{3}\cos 6\varphi+i\frac{m^3}{3}\sin 6\varphi-\cdot\cdot\cdot\cdot$$

Therefore, equating the imaginary parts, we obtain

$$\tan^{-1}\left(\frac{m\sin 2\varphi}{1+m\sin 2\varphi}\right)=m\sin 2\varphi-\frac{m^2}{2}\sin 4\varphi$$

$$+\frac{m^3}{3}\sin 6\varphi-\cdot\cdot\cdot\ ;$$

or, finally,

$$\varphi-\psi=m\sin 2\varphi-\frac{m^2}{2}\sin 4\varphi+\frac{m^3}{3}\sin 6\varphi-\ \cdot\ \cdot\ \cdot,$$

DEVELOPMENT OF $\varphi-\psi$ IN TERMS OF ψ.

If we wish this same quantity expressed in a series of the sines of the multiple arcs of ψ, we proceed as follows:

$$\tan\varphi=\frac{1}{1-\epsilon^2}\tan\psi,$$

$$\tan(\varphi-\psi)=\frac{\epsilon^2\tan\psi}{1-\epsilon^2+\tan^2\psi}$$

$$=\frac{\epsilon^2\sin\psi\cos\psi}{1-\epsilon^2\cos^2\psi}$$

$$=\frac{\epsilon^2\sin 2\psi}{2-\epsilon^2-\epsilon^2\cos 2\psi}$$

$$=\frac{m\sin 2\psi}{1-m\cos 2\psi}$$

Now

$$\log_e(1-m\,e^{-2i\psi})=-m\,e^{-2i\psi}-\frac{m^2}{2}e^{-4i\psi}-\frac{m^3}{3}e^{-6i\psi}-\ \cdot\ \cdot\ \cdot\ ;$$

but

$$\log_e(1-m\,e^{-2i\psi})=\frac{1}{2}\log_e(1-2m\cos 2\psi+m^2)$$
$$+i\tan^{-1}\left(\frac{m\sin 2\psi}{1-m\cos 2\psi}\right),$$

and

$$-m\,e^{-2i\psi}-\frac{m^2}{2}e^{-4i\psi}-\frac{m^3}{3}e^{-6i\psi}-\cdots=-m\cos 2\psi+i\,m\sin 2\psi$$
$$-\frac{m^2}{2}\cos 4\psi+i\frac{m^2}{2}\sin 4\psi-\frac{m^3}{3}\cos 6\psi+i\frac{m^3}{3}\sin 6\psi-\cdots.$$

By equating the imaginary parts, we obtain

$$\tan^{-1}\left(\frac{m\sin 2\psi}{1-m\cos 2\psi}\right)=m\sin 2\psi+\frac{m^2}{2}\sin 4\psi+\frac{m^3}{3}\sin 6\psi+\cdots,$$

or

$$\varphi-\psi=m\sin 2\psi+\frac{m^2}{2}\sin 4\psi+\frac{m^3}{3}\sin 6\psi+\ \cdot\ \cdot\ \cdot\ .$$

This result could have been obtained directly from the previous development by changing the sign of m and by interchanging φ and ψ.

DEVELOPMENT OF $\varphi-\theta$ IN TERMS OF φ.

By substituting the definition of θ in the formula for $\tan(\varphi-\theta)$ we get

$$\tan(\varphi-\theta)=\frac{[1-(1-\epsilon^2)^{1/2}]\tan\varphi}{1+(1-\epsilon^2)^{1/2}\tan^2\varphi}$$

$$=\frac{[1-(1-\epsilon^2)^{1/2}]\sin 2\varphi}{1+(1-\epsilon^2)^{1/2}+[1-(1-\epsilon^2)^{1/2}]\cos 2\varphi}.$$

Now let

$$n=\frac{1-(1-\epsilon^2)^{1/2}}{1+(1-\epsilon^2)^{1/2}}$$

then

$$\tan(\varphi-\theta)=\frac{n\sin 2\varphi}{1+n\cos 2\varphi}.$$

Since this expression is similar in form to that which gave $\tan(\varphi-\psi)$ in terms of φ, except that we have n in place of m, by a similar procedure we get

$$\varphi-\theta=n\sin 2\varphi-\frac{n^2}{2}\sin 4\varphi+\frac{n^3}{3}\sin 6\varphi-\cdots.$$

DEVELOPMENT OF $\varphi-\theta$ IN TERMS OF θ.

When $\tan(\varphi-\theta)$ is expressed in terms of θ, we get

$$\tan(\varphi-\theta)=\frac{n\sin 2\theta}{1-n\cos 2\theta}.$$

From the previous development we see that this gives $\varphi-\theta$ in terms of θ in the form

$$\varphi-\theta=n\sin 2\theta+\frac{n^2}{2}\sin 4\theta+\frac{n^3}{3}\sin 6\theta+\cdots.$$

DEVELOPMENT OF $\theta-\psi$ IN TERMS OF θ AND IN TERMS OF ψ.

From the original relations we get

$$\tan\psi=(1-\epsilon^2)^{1/2}\tan\theta.$$

This relation is the same as that which exists between θ and φ; so we get at once

$$\theta-\psi=n \sin 2\theta-\frac{n^2}{2} \sin 4\theta+\frac{n^3}{3} \sin 6\theta-\cdots\cdot$$

and

$$\theta-\psi=n \sin 2\psi+\frac{n^2}{2} \sin 4\psi+\frac{n^3}{3} \sin 6\psi+\cdots\cdot$$

DEVELOPMENT OF $\varphi-\chi$ IN TERMS OF φ—FIRST METHOD.

When we come to the isometric latitude, we meet difficulties of a different order. In the first place let p be the complement of φ and let z be the complement of χ. The definition then becomes

$$\tan \frac{z}{2}=\tan \frac{p}{2} \cdot \left(\frac{1+\epsilon \cos p}{1-\epsilon \cos p}\right)^{\epsilon/2}.$$

Let

$$e^v=\left(\frac{1+\epsilon \cos p}{1-\epsilon \cos p}\right)^{\epsilon/2};$$

then

$$\tan \frac{z}{2}=e^v \tan \frac{p}{2}$$

$$\tan \left(\frac{z-p}{2}\right)=\frac{(e^v-1) \tan \frac{p}{2}}{1+e^v \tan^2 \frac{p}{2}}$$

$$=\frac{(e^v-1) \sin p}{e^v+1-(e^v-1) \cos p}.$$

Let

$$q=\frac{e^v-1}{e^v+1};$$

then

$$\tan \left(\frac{z-p}{2}\right)=\frac{q \sin p}{1-q \cos p}.$$

By analysis similar to that used before, we get

$$\frac{z-p}{2}=q\sin p+\frac{q^2}{2}\sin 2p+\frac{q^3}{3}\sin 3p+\cdots;$$

but

$$v=\frac{\epsilon}{2}\log_e(1+\epsilon\cos p)-\frac{\epsilon}{2}\log_e(1-\epsilon\cos p)$$

$$=\epsilon^2\cos p+\frac{\epsilon^4}{3}\cos^3 p+\frac{\epsilon^6}{5}\cos^5 p+\frac{\epsilon^8}{7}\cos^7 p+\cdots\cdot$$

and

$$q=\frac{e^v-1}{e^v+1}=\tanh\frac{v}{2}=\frac{v}{2}-\frac{v^3}{24}+\frac{v^5}{240}-\frac{17v^7}{40320}+\cdots\cdot$$

Including terms in the eighth power of ϵ, we get

$$q=\frac{\epsilon^2}{2}\cos p+\left(\frac{\epsilon^4}{6}-\frac{\epsilon^6}{24}\right)\cos^3 p+\left(\frac{\epsilon^6}{10}-\frac{\epsilon^8}{24}\right)\cos^5 p+\frac{\epsilon^8}{14}\cos^7 p+\cdots$$

Hence

$$z-p=\left[\epsilon^2\cos p+\left(\frac{\epsilon^4}{3}-\frac{\epsilon^6}{12}\right)\cos^3 p+\left(\frac{\epsilon^6}{5}-\frac{\epsilon^8}{12}\right)\cos^5 p\right.$$

$$\left.+\frac{\epsilon^8}{7}\cos^7 p\right]\sin p+\left[\frac{\epsilon^4}{4}\cos^2 p+\left(\frac{\epsilon^6}{6}-\frac{\epsilon^8}{24}\right)\cos^4 p\right.$$

$$\left.+\frac{23\epsilon^8}{180}\cos^6 p\right]\sin 2p+\frac{1}{3}\left(\frac{\epsilon^6}{4}\cos^3 p+\frac{\epsilon^8}{4}\cos^5 p\right)\sin 3p$$

$$+\frac{\epsilon^8}{32}\cos^4 p\sin 4p+\cdots\cdots$$

When this expression is reduced to terms in the sines of multiple arcs (see reduction table on p. 88), we get

$$z-p=\left(\frac{\epsilon^2}{2}+\frac{5\epsilon^4}{24}+\frac{3\epsilon^6}{32}+\frac{281\epsilon^8}{5760}+\cdots\cdot\right)\sin 2p+\left(\frac{5\epsilon^4}{48}+\frac{7\epsilon^6}{80}\right.$$

$$\left.+\frac{697\epsilon^8}{11520}+\cdots\cdot\right)\sin 4p+\left(\frac{13\epsilon^6}{480}+\frac{461\epsilon^8}{13440}+\cdots\cdot\right)\sin 6p$$

$$+\left(\frac{1237\epsilon^8}{161280}+\cdots\cdot\right)\sin 8p+\cdots\cdot$$

or in terms of φ and χ to the desired approximation

$$\varphi-\chi=\left(\frac{\epsilon^2}{2}+\frac{5\epsilon^4}{24}+\frac{3\epsilon^6}{32}+\frac{281\epsilon^8}{5760}\right)\sin 2\varphi-\left(\frac{5\epsilon^4}{48}+\frac{7\epsilon^6}{80}+\frac{697\epsilon^8}{11520}\right)\sin 4\varphi$$

$$+\left(\frac{13\epsilon^6}{480}+\frac{461\epsilon^8}{13440}\right)\sin 6\varphi-\frac{1237\epsilon^8}{161280}\sin 8\varphi.$$

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DEVELOPMENT OF $\varphi-\chi$ IN TERMS OF φ—SECOND METHOD.

The isometric latitude can be developed in terms of φ by another method that is very simple in application. From the definition

$$\tan\frac{z}{2}=\tan\frac{p}{2}\cdot\left(\frac{1+\epsilon\cos p}{1-\epsilon\cos p}\right)^{\epsilon/2}$$

we obtain at once

$$\log_e\tan\frac{z}{2}=\log_e\tan\frac{p}{2}+\epsilon^2\cos p+\frac{\epsilon^4}{3}\cos^3 p+\frac{\epsilon^6}{5}\cos^5 p+\frac{\epsilon^8}{7}\cos^7 p+\cdots\cdot$$

By Taylor's theorem we have

$$f\left(\frac{p}{2}+h\right)=f\left(\frac{p}{2}\right)+\frac{h}{1!}\left[\frac{df}{dh}\right]+\frac{h^2}{2!}\left[\frac{d^2f}{dh^2}\right]+\frac{h^3}{3!}\left[\frac{d^3f}{dh^3}\right]+\frac{h^4}{4!}\left[\frac{d^4f}{dh^4}\right]+\cdots,$$

in which the brackets denote the values of the derivatives of $f\left(\frac{p}{2}+h\right)$ with respect to h for $h=0$.

Let

$$f\left(\frac{p}{2}+h\right)=\log_e\tan\left(\frac{p}{2}+h\right).$$

Then

$$f\left(\frac{p}{2}\right)=\log_e\tan\frac{p}{2}$$

$$\frac{df}{dh}=\frac{\sec^2\left(\frac{p}{2}+h\right)}{\tan\left(\frac{p}{2}+h\right)}=\frac{1}{\sin\left(\frac{p}{2}+h\right)\cos\left(\frac{p}{2}+h\right)}=\frac{2}{\sin(p+2h)}$$

$$\left[\frac{df}{dh}\right]=\frac{2}{\sin p}$$

$$\frac{d^2f}{dh^2}=-4\operatorname{cosec}(p+2h)\cot(p+2h)$$

$$\left[\frac{d^2f}{dh^2}\right]=-4\operatorname{cosec} p\cot p=-\frac{4\cos p}{\sin^2 p}$$

$$\frac{d^3f}{dh^3}=8\operatorname{cosec}(p+2h)\cot^2(p+2h)+8\operatorname{cosec}^3(p+2h)$$

$$\left[\frac{d^3f}{dh^3}\right]=8 \operatorname{cosec} p \cot^2 p+8 \operatorname{cosec}^3 p=\frac{8 \cos^2 p}{\sin^3 p}+\frac{8}{\sin^3 p}$$

$$\frac{d^4p}{dh^4}=-16 \operatorname{cosec} (p+2h) \cot^3 (p+2h)$$
$$-80 \operatorname{cosec}^3 (p+2h) \cot (p+2h)$$

$$\left[\frac{d^4f}{dh^4}\right]=-16 \operatorname{cosec} p \cot^3 p-80 \operatorname{cosec}^3 p \cot p$$
$$=-\frac{16 \cos^3 p}{\sin^4 p}-\frac{80 \cos p}{\sin^4 p}.$$

Substituting these values in Taylor's series, we get

$$\log_e \tan\left(\frac{p}{2}+h\right)=\log_e \tan\frac{p}{2}+\frac{2h}{\sin p}-\frac{2 \cos p}{\sin^2 p}h^2$$
$$+\left(\frac{4 \cos^2 p}{3 \sin^3 p}+\frac{4}{3 \sin^3 p}\right)h^3-\left(\frac{2 \cos^3 p}{3 \sin^4 p}+\frac{10 \cos p}{3 \sin^4 p}\right)h^4+ \cdots\cdots$$

Now let us assume the relation

$$\frac{z}{2}=\frac{p}{2}+a\epsilon^2+b\epsilon^4+c\epsilon^6+d\epsilon^8+ \cdots\cdot,$$

then

$$h=a\epsilon^2+b\epsilon^4+c\epsilon^6+d\epsilon^8+ \cdots\cdots.$$

Substituting this value of h and retaining all powers of ϵ up to and including the eighth, we obtain

$$\log_e \tan\left(\frac{p}{2}+h\right)=\log_e \tan\frac{z}{2}=\log_e \tan\frac{p}{2}+\frac{2}{\sin p}(a\epsilon^2$$
$$+b\epsilon^4+c\epsilon^6+d\epsilon^8)-\frac{2 \cos p}{\sin^2 p}(a\epsilon^2+b\epsilon^4+c\epsilon^6+d\epsilon^8)^2$$
$$+\left(\frac{4 \cos^2 p}{3 \sin^3 p}+\frac{4}{3 \sin^3 p}\right)(a\epsilon^2+b\epsilon^4+c\epsilon^6+d\epsilon^8)^3$$
$$-\left(\frac{2 \cos^3 p}{3 \sin^4 p}+\frac{10 \cos p}{3 \sin^4 p}\right)(a\epsilon^2+b\epsilon^4+c\epsilon^6+d\epsilon^8)^4+ \cdots\cdot,$$

or

$$\log_e \tan\frac{z}{2}=\log_e \tan\frac{p}{2}+\frac{2}{\sin p}(a\epsilon^2+b\epsilon^4+c\epsilon^6+d\epsilon^8)$$
$$-\frac{2 \cos p}{\sin^2 p}[a^2\epsilon^4+2ab\epsilon^6+(b^2+2ac)\epsilon^8]+\left(\frac{4 \cos^2 p}{3 \sin^3 p}\right.$$
$$\left.+\frac{4}{3 \sin^3 p}\right)(a^3\epsilon^6+3a^2b\epsilon^8)-\left(\frac{2 \cos^3 p}{3 \sin^4 p}\right.$$
$$\left.+\frac{10 \cos p}{3 \sin^4 p}\right)a^4\epsilon^8+ \cdots\cdot.$$

But in the original series for $\log_e \tan \frac{z}{2}$ we have

$$\log_e \tan \frac{z}{2} = \log_e \tan \frac{p}{2} + \epsilon^2 \cos p + \frac{\epsilon^4}{3} \cos^3 p + \frac{\epsilon^6}{5} \cos^5 p + \frac{\epsilon^8}{7} \cos^7 p + \cdots$$

These two series must be identically equal and so we can equate the coefficients of the same powers of ϵ. In this way we get

$$\frac{2a}{\sin p} = \cos p,$$

$$\frac{2b}{\sin p} - \frac{2a^2 \cos p}{\sin^2 p} = \frac{\cos^3 p}{3},$$

$$\frac{2c}{\sin p} - \frac{4ab \cos p}{\sin^2 p} + \frac{4a^3 \cos^2 p}{3 \sin^3 p} + \frac{4a^3}{3 \sin^3 p} = \frac{\cos^5 p}{5},$$

$$\frac{2d}{\sin p} - \frac{2(b^2 + 2ac) \cos p}{\sin^2 p} + \frac{4a^2 b \cos^2 p}{\sin^3 p} + \frac{4a^2 b}{\sin^3 p} - \frac{2a^4 \cos^3 p}{3 \sin^4 p}$$
$$- \frac{10a^4 \cos p}{3 \sin^4 p} = \frac{\cos^7 p}{7}.$$

From these equations, we can in succession determine the values of a, b, c, and d.

(For the reductions see the reduction table p. 88.)

$$a = \frac{1}{2} \sin p \cos p = \frac{1}{4} \sin 2p,$$

$$b = \frac{5}{12} \sin p \cos^3 p = \frac{5}{48} \sin 2p + \frac{5}{96} \sin 4p,$$

$$c = \frac{13}{30} \sin p \cos^5 p - \frac{1}{12} \sin p \cos^3 p = \frac{3}{64} \sin 2p$$
$$+ \frac{7}{160} \sin 4p + \frac{13}{960} \sin 6p,$$

$$d = \frac{1237}{2520} \sin p \cos^7 p - \frac{3}{16} \sin p \cos^5 p = \frac{281}{11520} \sin 2p$$
$$+ \frac{697}{23040} \sin 4p + \frac{461}{26880} \sin 6p + \frac{1237}{322560} \sin 8p.$$

Substituting these values of a, b, c, and d and rearrangeing, we get to the desired approximation

$$z = p + \left(\frac{\epsilon^2}{2} + \frac{5\epsilon^4}{24} + \frac{3\epsilon^6}{32} + \frac{281\epsilon^8}{5760}\right) \sin 2p + \left(\frac{5\epsilon^4}{48} + \frac{7\epsilon^6}{80} + \frac{697\epsilon^8}{11520}\right) \sin 4p + \left(\frac{13\epsilon^6}{480} + \frac{461\epsilon^8}{13440}\right) \sin 6p + \frac{1237\epsilon^8}{161280} \sin 8p.$$

Substitute

$$z = \frac{\pi}{2} - \chi \text{ and } p = \frac{\pi}{2} - \varphi,$$

and we get, as before, the approximation

$$\varphi - \chi = \left(\frac{\epsilon^2}{2} + \frac{5\epsilon^4}{24} + \frac{3\epsilon^6}{32} + \frac{281\epsilon^8}{5760}\right) \sin 2\varphi - \left(\frac{5\epsilon^4}{48} + \frac{7\epsilon^6}{80} + \frac{697\epsilon^8}{11520}\right) \sin 4\varphi + \left(\frac{13\epsilon^6}{480} + \frac{461\epsilon^8}{13440}\right) \sin 6\varphi - \frac{1237\epsilon^8}{161280} \sin 8\varphi.$$

DEVELOPMENT OF $\varphi - \chi$ IN TERMS OF φ—THIRD METHOD.

The difference between φ and χ can be developed directly by Maclaurin's theorem. Let us take the definition in the form

$$\log_e \tan \frac{z}{2} = \log_e \tan \frac{p}{2} + \epsilon^2 \cos p + \frac{\epsilon^4}{3} \cos^3 p + \frac{\epsilon^6}{5} \cos^5 p + \frac{\epsilon^8}{7} \cos^7 p + \cdots.$$

In this expression, setting $\epsilon^2 = h$, we get

$$\log_e \tan \frac{z}{2} = \log_e \tan \frac{p}{2} + h \cos p + \frac{h^2}{3} \cos^3 p + \frac{h^3}{5} \cos^5 p + \frac{h^4}{7} \cos^7 p + \cdots.$$

Differentiating this expression, considering z as a function of h or ϵ^2, we get in succession

$$\operatorname{cosec} z \frac{dz}{dh} = \cos p + \frac{2h}{3} \cos^3 p + \frac{3h^2}{5} \cos^5 p + \frac{4h^3}{7} \cos^7 p,$$

$$\operatorname{cosec} z \frac{d^2 z}{dh^2} - \operatorname{cosec} z \cot z \left(\frac{dz}{dh}\right)^2 = \frac{2}{3} \cos^3 p + \frac{6h}{5} \cos^5 p + \frac{12h^2}{7} \cos^7 p,$$

$$\operatorname{cosec} z \frac{d^3z}{dh^3} - 3 \operatorname{cosec} z \cot z \frac{dz}{dh}\frac{d^2z}{dh^2} + \operatorname{cosec} z \cot^2 z \left(\frac{dz}{dh}\right)^3 + \operatorname{cosec}^3 z \left(\frac{dz}{dh}\right)^3 = \frac{6}{5}\cos^5 p + \frac{24h}{7}\cos^7 p,$$

$$\operatorname{cosec} z \frac{d^4z}{dh^4} - 4 \operatorname{cosec} z \cot z \frac{dz}{dh}\frac{d^3z}{dh^3} - 3 \operatorname{cosec} z \cot z \left(\frac{d^2z}{dh^2}\right)^2 + 6 \operatorname{cosec} z \cot^2 z \left(\frac{dz}{dh}\right)^2 \frac{d^2z}{dh^2} + 6 \operatorname{cosec}^3 z \left(\frac{dz}{dh}\right)^2 \frac{d^2z}{dh^2} - \operatorname{cosec} z \cot^3 z \left(\frac{dz}{dh}\right)^4 - 5 \operatorname{cosec}^3 z \cot z \left(\frac{dz}{dh}\right)^4 = \frac{24}{7}\cos^7 p.$$

Denoting by brackets the values of these successive derivatives for $h=0$, remembering that functions of z become functions of p for $h=0$, we get. (For the necessary reductions see the reduction table, p. 88.)

$$[z] = p,$$

$$\left[\frac{dz}{dh}\right] = \sin p \cos p = \frac{1}{2}\sin 2p,$$

$$\left[\frac{d^2z}{dh^2}\right] = \frac{5}{3}\sin p \cos^3 p = \frac{5}{12}\sin 2p + \frac{5}{24}\sin 4p,$$

$$\left[\frac{d^3z}{dh^3}\right] = -\sin p \cos^3 p + \frac{26}{5}\sin p \cos^5 p = \frac{9}{16}\sin 2p + \frac{21}{40}\sin 4p + \frac{13}{80}\sin 6p,$$

$$\left[\frac{d^4z}{dh^4}\right] = -9 \sin p \cos^5 p + \frac{2474}{105}\sin p \cos^7 p = \frac{281}{240}\sin 2p + \frac{697}{480}\sin 4p + \frac{461}{560}\sin 6p + \frac{1237}{6720}\sin 8p.$$

By Maclaurin's series we have

$$z = [z] + \frac{\epsilon^2}{1!}\left[\frac{dz}{dh}\right] + \frac{\epsilon^4}{2!}\left[\frac{d^2z}{dh^2}\right] + \frac{\epsilon^6}{3!}\left[\frac{d^3z}{dh^3}\right] + \frac{\epsilon^8}{4!}\left[\frac{d^4z}{dh^4}\right] + \cdots,$$

in which h is replaced by its value ϵ^2.

By substituting the above values in this series and rearranging we get, as before, the approximation

$$z=p+\left(\frac{\epsilon^2}{2}+\frac{5\epsilon^4}{24}+\frac{3\epsilon^6}{32}+\frac{281\epsilon^8}{5760}\right)\sin 2p+\left(\frac{5\epsilon^4}{48}+\frac{7\epsilon^6}{80}+\frac{697\epsilon^8}{11520}\right)\sin 4p$$
$$+\left(\frac{13\epsilon^6}{480}+\frac{461\epsilon^8}{13440}\right)\sin 6p+\frac{1237\epsilon^8}{161280}\sin 8p,$$

or in φ and χ we get, as before, the approximation

$$\varphi-\chi=\left(\frac{\epsilon^2}{2}+\frac{5\epsilon^4}{24}+\frac{3\epsilon^6}{32}+\frac{281\epsilon^8}{5760}\right)\sin 2\varphi-\left(\frac{5\epsilon^4}{48}+\frac{7\epsilon^6}{80}+\frac{697\epsilon^8}{11520}\right)\sin 4\varphi$$
$$+\left(\frac{13\epsilon^6}{480}+\frac{461\epsilon^8}{13440}\right)\sin 6\varphi-\frac{1237\epsilon^8}{161280}\sin 8\varphi.$$

DEVELOPMENT OF $\varphi-\chi$ IN TERMS OF φ—FOURTH METHOD.

The isometric latitude could be developed by direct differentiation of the equation in the form

$$\tan\frac{z}{2}=\tan\frac{p}{2}\cdot\left(\frac{1+\epsilon\cos p}{1-\epsilon\cos p}\right)^{\epsilon/2},$$

or in the form

$$z=2\tan^{-1}\left[\tan\frac{p}{2}\cdot\left(\frac{1+\epsilon\cos p}{1-\epsilon\cos p}\right)^{\epsilon/2}\right].$$

In fact, Herz in his "Lehrbuch der Landkartenprojektionen" does differentiate the form

$$\frac{\pi}{2}+\chi=2\tan^{-1}\left[\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)\cdot\left(\frac{1-\epsilon\sin\varphi}{1+\epsilon\sin\varphi}\right)^{\epsilon/2}\right].$$

He obtains the development to include the term in ϵ^4. The difficulty lies in the fact that it is necessary to differentiate with respect to ϵ, although it is evident that χ, considered as a function of ϵ, is an even function. If one wishes to proceed in this manner, it is better to write the expression in the form

$$z=2\tan^{-1}\left[\tan\frac{p}{2}\cdot\exp^*\left(\epsilon^2\cos p+\frac{\epsilon^4}{3}\cos^3 p+\frac{\epsilon^6}{5}\cos^5 p\right.\right.$$
$$\left.\left.+\frac{\epsilon^8}{7}\cos^7 p+\cdots\right)\right],$$

* $\exp x = e^x$

or it is still better to develop the expotential to include all powers of ϵ to the eighth inclusive.

$$z=2\tan^{-1}\left\{\tan\frac{p}{2}\cdot\left[1+\epsilon^2\cos p+\epsilon^4\left(\frac{\cos^2 p}{2}+\frac{\cos^3 p}{3}\right)+\epsilon^6\left(\frac{\cos^3 p}{6}+\frac{\cos^4 p}{3}+\frac{\cos^5 p}{5}\right)+\epsilon^8\left(\frac{\cos^4 p}{24}+\frac{\cos^5 p}{6}+\frac{23\cos^6 p}{90}+\frac{\cos^7 p}{7}\right)\right]\right\}.$$

In this form the substitution $h=\epsilon^2$ can be made and the development attained by four differentiations instead of eight. Thus by careful consideration the formal work required can often be considerably reduced in amount. Using the expression as Herz did, it requires almost as much work to carry the development to ϵ^4 as it does in any of the above forms to carry it to include the term in ϵ^8. It is also more convenient to differentiate the expression not as an arc tangent, but in the form $\tan\frac{z}{2}$. As an illustration we shall make the development by differentiating successively the expression

$$\tan\frac{z}{2}=\tan\frac{p}{2}\left[1+h\cos p+h^2\left(\frac{\cos^2 p}{2}+\frac{\cos^3 p}{3}\right)+h^3\left(\frac{\cos^3 p}{6}+\frac{\cos^4 p}{3}+\frac{\cos^5 p}{5}\right)+h^4\left(\frac{\cos^4 p}{24}+\frac{\cos^5 p}{6}+\frac{23\cos^6 p}{90}+\frac{\cos^7 p}{7}\right)\right].$$

$$\frac{1}{2}\sec^2\frac{z}{2}\frac{dz}{dh}=\tan\frac{p}{2}\left[\cos p+2h\left(\frac{\cos^2 p}{2}+\frac{\cos^3 p}{3}\right)+3h^2\left(\frac{\cos^3 p}{6}+\frac{\cos^4 p}{3}+\frac{\cos^5 p}{5}\right)+4h^3\left(\frac{\cos^4 p}{24}+\frac{\cos^5 p}{6}+\frac{23\cos^6 p}{90}+\frac{\cos^7 p}{7}\right)\right],$$

$$\frac{1}{2}\sec^2\frac{z}{2}\frac{d^2z}{dh^2}+\frac{1}{2}\sec^2\frac{z}{2}\tan\frac{z}{2}\left(\frac{dz}{dh}\right)^2=\tan\frac{p}{2}\left[\cos^2 p+\frac{2\cos^3 p}{3}+6h\left(\frac{\cos^3 p}{6}+\frac{\cos^4 p}{3}+\frac{\cos^5 p}{5}\right)+12h^2\left(\frac{\cos^4 p}{24}+\frac{\cos^5 p}{6}+\frac{23\cos^6 p}{90}+\frac{\cos^7 p}{7}\right)\right],$$

$$\frac{1}{2}\sec^2\frac{z}{2}\frac{d^3z}{dh^3}+\frac{3}{2}\sec^2\frac{z}{2}\tan\frac{z}{2}\frac{dz}{dh}\frac{d^2z}{dh^2}+\left(\frac{1}{2}\sec^2\frac{z}{2}\tan^2\frac{z}{2}\right.$$

$$\left.+\frac{1}{4}\sec^4\frac{z}{2}\right)\left(\frac{dz}{dh}\right)^3=\tan\frac{p}{2}\left[\cos^3 p+2\cos^4 p\right.$$

$$+\frac{6\cos^5 p}{5}+24h\left(\frac{\cos^4 p}{24}+\frac{\cos^5 p}{6}+\frac{23\cos^6 p}{90}\right.$$

$$\left.\left.+\frac{\cos^7 p}{7}\right)\right],$$

$$\frac{1}{2}\sec^2\frac{z}{2}\frac{d^4z}{dh^4}+\frac{3}{2}\sec^2\frac{z}{2}\tan\frac{z}{2}\left(\frac{d^2z}{dh^2}\right)^2+2\sec^2\frac{z}{2}\tan\frac{z}{2}\frac{dz}{dh}\frac{d^3z}{dh^2}$$

$$+\left(3\sec^2\frac{z}{2}\tan^2\frac{z}{2}+\frac{3}{2}\sec^4\frac{z}{2}\right)\left(\frac{dz}{dh}\right)^2\frac{d^2z}{dh^2}$$

$$+\left(\frac{1}{2}\sec^2\frac{z}{2}\tan^3\frac{z}{2}+\sec^4\frac{z}{2}\tan\frac{z}{2}\right)\left(\frac{dz}{dh}\right)^4$$

$$=\tan\frac{p}{2}\left(\cos^4 p+4\cos^5 p+\frac{92\cos^6 p}{15}+\frac{24\cos^7 p}{7}\right).$$

Evaluating these derivatives for $h=0$, remembering that functions of z become functions of p for $h=0$, we obtain (for the necessary reductions see the reduction table, p. 88):

$$[z]=p,$$

$$\left[\frac{dz}{dh}\right]=\sin p\cos p=\frac{1}{2}\sin 2p,$$

$$\left[\frac{d^2z}{dh^2}\right]=\frac{5}{3}\sin p\cos^3 p=\frac{5}{12}\sin 2p+\frac{5}{24}\sin 4p,$$

$$\left[\frac{d^3z}{dh^3}\right]=-\sin p\cos^3 p+\frac{26}{5}\sin p\cos^5 p$$

$$=\frac{9}{16}\sin 2p+\frac{21}{40}\sin 4p+\frac{13}{80}\sin 6p,$$

$$\left[\frac{d^4z}{dh^4}\right]=-9\sin p\cos^5 p+\frac{2474}{105}\sin p\cos^7 p$$

$$=\frac{281}{240}\sin 2p+\frac{697}{480}\sin 4p+\frac{461}{560}\sin 6p+\frac{1237}{6720}\sin 8p.$$

Substituting these values in Maclaurin's series (see p. 22), we get, as before, the approximation:

$$z=p+\left(\frac{\epsilon^2}{2}+\frac{5\epsilon^4}{24}+\frac{3\epsilon^6}{32}+\frac{281\epsilon^8}{5760}\right)\sin 2p+\left(\frac{5\epsilon^4}{48}+\frac{7\epsilon^6}{80}+\frac{697\epsilon^8}{11520}\right)\sin 4p$$
$$+\left(\frac{13\epsilon^6}{480}+\frac{461\epsilon^8}{13440}\right)\sin 6p+\frac{1237\epsilon^8}{161280}\sin 8p,$$

or, in terms of φ and χ, the approximation,

$$\varphi-\chi=\left(\frac{\epsilon^2}{2}+\frac{5\epsilon^4}{24}+\frac{3\epsilon^6}{32}+\frac{281\epsilon^8}{5760}\right)\sin 2\varphi-\left(\frac{5\epsilon^4}{48}+\frac{7\epsilon^4}{80}+\frac{697\epsilon^8}{11520}\right)\sin 4\varphi$$
$$+\left(\frac{13\epsilon^6}{480}+\frac{461\epsilon^8}{13440}\right)\sin 6\varphi-\frac{1237\epsilon^8}{161280}\sin 8\varphi.$$

DEVELOPMENT OF $\varphi-\chi$ IN TERMS OF φ—FIFTH METHOD.

This difference can be expressed first in terms of φ directly from the equation of definition by the third, fourth, fifth, or sixth method given later under those developments (see pp. 38–55), and then the development can be inverted into terms of φ by Lagrange's theorem. We start with the approximation:

$$\chi=\varphi-\left(\frac{\epsilon^2}{2}+\frac{5\epsilon^4}{24}+\frac{\epsilon^6}{12}+\frac{13\epsilon^8}{360}\right)\sin 2\chi-\left(\frac{7\epsilon^4}{48}+\frac{29\epsilon^6}{240}+\frac{811\epsilon^8}{11520}\right)\sin 4\chi$$
$$-\left(\frac{7\epsilon^6}{120}+\frac{81\epsilon^8}{1120}\right)\sin 6\chi-\frac{4279\epsilon^8}{161280}\sin 8\chi.$$

In this case Lagrange's series becomes:

$$\chi=\varphi+\frac{1}{1!}g(\varphi)+\frac{1}{2!}\frac{d}{d\varphi}[g(\varphi)]^2+\frac{1}{3!}\frac{d^2}{d\varphi^2}[g(\varphi)]^3$$
$$+\frac{1}{4!}\frac{d^3}{d\varphi^3}[g(\varphi)]^4+\cdots,$$

in which

$$g(\varphi)=-\left(\frac{\epsilon^2}{2}+\frac{5\epsilon^4}{24}+\frac{\epsilon^6}{12}+\frac{13\epsilon^8}{360}\right)\sin 2\varphi-\left(\frac{7\epsilon^4}{48}+\frac{29\epsilon^6}{240}+\frac{811\epsilon^8}{11520}\right)\sin 4\varphi$$
$$-\left(\frac{7\epsilon^6}{120}+\frac{81\epsilon^8}{1120}\right)\sin 6\varphi-\frac{4279\epsilon^8}{161280}\sin 8\varphi.$$

In the powers of this function we must retain all powers of ϵ to the eighth, inclusive.

$$[g(\varphi)]^2=\left(\frac{\epsilon^4}{4}+\frac{5\epsilon^6}{24}+\frac{73\epsilon^8}{576}\right)\sin^2 2\varphi+\frac{49\epsilon^8}{2304}\sin^2 4\varphi+\left(\frac{7\epsilon^6}{48}+\frac{523\epsilon^8}{2880}\right)\sin 2\varphi\sin 4\varphi+\frac{7\epsilon^8}{120}\sin 2\varphi\sin 6\varphi$$

$$[g(\varphi)]^3=-\left(\frac{\epsilon^6}{8}+\frac{5\epsilon^8}{32}\right)\sin^3 2\varphi-\frac{7\epsilon^8}{64}\sin^2 2\varphi\sin 4\varphi$$

$$[g(\varphi)]^4=\frac{\epsilon^8}{16}\sin^4 2\varphi.$$

Differentiating and reducing by the table on p. 88, we get

$$\frac{d}{d\varphi}[g(\varphi)]^2=\left(\epsilon^4+\frac{5\epsilon^6}{6}+\frac{73\epsilon^8}{144}\right)\sin 2\varphi\cos 2\varphi+\frac{49\epsilon^8}{288}\sin 4\varphi\cos 4\varphi$$
$$+\left(\frac{7\epsilon^6}{24}+\frac{523\epsilon^8}{1440}\right)\cos 2\varphi\sin 4\varphi$$
$$+\left(\frac{7\epsilon^6}{12}+\frac{523\epsilon^8}{720}\right)\sin 2\varphi\cos 4\varphi+\frac{7\epsilon^8}{60}\cos 2\varphi\sin 6\varphi$$
$$+\frac{7\epsilon^8}{20}\sin 2\varphi\cos 6\varphi$$
$$=-\left(\frac{7\epsilon^6}{48}+\frac{523\epsilon^8}{2880}\right)\sin 2\varphi+\left(\frac{\epsilon^4}{2}+\frac{5\epsilon^6}{12}+\frac{197\epsilon^8}{1440}\right)\sin 4\varphi$$
$$+\left(\frac{7\epsilon^6}{16}+\frac{523\epsilon^8}{960}\right)\sin 6\varphi+\frac{917\epsilon^8}{2880}\sin 8\varphi,$$

$$\frac{d^2}{d\varphi^2}[g(\varphi)]^3=-\left(3\epsilon^6+\frac{15\epsilon^8}{4}\right)\sin 2\varphi\cos^2 2\varphi+\left(\frac{3\epsilon^6}{2}+\frac{15\epsilon^8}{8}\right)\sin^3 2\varphi$$
$$-\frac{21\epsilon^8}{16}\sin 8\varphi+\frac{7\epsilon^8}{4}\sin^2 2\varphi\sin 4\varphi$$
$$=\left(\frac{3\epsilon^6}{8}+\frac{15\epsilon^8}{32}\right)\sin 2\varphi+\frac{7\epsilon^8}{8}\sin 4\varphi-\left(\frac{9\epsilon^6}{8}+\frac{45\epsilon^8}{32}\right)\sin 6\varphi$$
$$-\frac{7\epsilon^8}{4}\sin 8\varphi,$$

$$\frac{d^3}{d\varphi^3}[g(\varphi)]^4=-2\epsilon^8\sin 4\varphi+4\epsilon^8\sin 8\varphi.$$

Substituting these values in Lagrange's series, we get the approximation

$$\chi=\varphi-\left(\frac{\epsilon^2}{2}+\frac{5\epsilon^4}{24}+\frac{\epsilon^6}{12}+\frac{13\epsilon^8}{360}\right)\sin 2\varphi-\left(\frac{7\epsilon^4}{48}+\frac{29\epsilon^6}{240}+\frac{811\epsilon^8}{11520}\right)\sin 4\varphi$$
$$-\left(\frac{7\epsilon^6}{120}+\frac{81\epsilon^8}{1120}\right)\sin 6\varphi-\frac{4279\epsilon^8}{161280}\sin 8\varphi$$
$$-\left(\frac{7\epsilon^6}{96}+\frac{523\epsilon^8}{5760}\right)\sin 2\varphi+\left(\frac{\epsilon^4}{4}+\frac{5\epsilon^6}{24}+\frac{197\epsilon^8}{2880}\right)\sin 4\varphi+\left(\frac{7\epsilon^6}{32}\right.$$
$$\left.+\frac{523\epsilon^8}{1920}\right)\sin 6\varphi+\frac{917\epsilon^8}{5760}\sin 8\varphi+\left(\frac{\epsilon^6}{16}+\frac{5\epsilon^8}{64}\right)\sin 2\varphi$$
$$+\frac{7\epsilon^8}{48}\sin 4\varphi-\left(\frac{3\epsilon^6}{16}+\frac{15\epsilon^8}{64}\right)\sin 6\varphi-\frac{7\epsilon^8}{24}\sin 8\varphi$$
$$-\frac{\epsilon^8}{12}\sin 4\varphi+\frac{\epsilon^8}{6}\sin 8\varphi.$$

By collecting and rearranging we get, as before, the approximation

$$\varphi-\chi=\left(\frac{\epsilon^2}{2}+\frac{5\epsilon^4}{24}+\frac{3\epsilon^6}{32}+\frac{281\epsilon^8}{5760}\right)\sin 2\varphi-\left(\frac{5\epsilon^4}{48}+\frac{7\epsilon^6}{80}+\frac{697\epsilon^8}{11520}\right)\sin 4\varphi$$
$$+\left(\frac{13\epsilon^6}{480}+\frac{461\epsilon^8}{13440}\right)\sin 6\varphi-\frac{1237\epsilon^8}{161280}\sin 8\varphi.$$

DEVELOPMENT OF $\varphi-\chi$ IN TERMS OF φ—SIXTH METHOD.

$\varphi-\chi$ can be developed in terms of φ by Arbogast's rule. If the symbol f denotes an arbitrary analytic function, we can expand

$$f\left(a_0+a_1\frac{x}{1!}+a_2\frac{x^2}{2!}+a_3\frac{x^3}{3!}+a_4\frac{x^4}{4!}+\cdots\right)$$

in terms of x at once by Taylor's theorem in the form

$$f\left(a_0+a_1\frac{x}{1!}+a_2\frac{x^2}{2!}+a_3\frac{x^3}{3!}+a_4\frac{x^4}{4!}+\cdots\right)=f(a_0)$$
$$+\frac{1}{1!}f^1(a_0)\left(a_1\frac{x}{1!}+a_2\frac{x^2}{2!}+a_3\frac{x^3}{3!}+a_4\frac{x^4}{4!}+\cdots\right)$$
$$+\frac{1}{2!}f^2(a_0)\left(a_1\frac{x}{1!}+a_2\frac{x^2}{2!}+a_3\frac{x^3}{3!}+a_4\frac{x^4}{4!}+\cdots\right)^2$$
$$+\frac{1}{3!}f^3(a_0)\left(a_1\frac{x}{1!}+a_2\frac{x^2}{2!}+a_3\frac{x^3}{3!}+a_4\frac{x^4}{4!}+\cdots\right)^3$$
$$+\frac{1}{4!}f^4(a_0)\left(a_1\frac{x}{1!}+a_2\frac{x^2}{2!}+a_3\frac{x^3}{3!}+a_4\frac{x^4}{4!}+\cdots\right)^4+\cdots$$

The exponents of f denote the values of the various successive derivatives of $f\ (a_0+g)$ with respect to g for $g=0$, or, what amounts to the same thing, the values of the various successive derivatives of $f\ (a_0)$ with respect to a_0. By expanding the various powers of the polynomial in x in the above expression and by rearranging in powers of x, we may determine the coefficients of the various $\frac{x_n}{n!}$ in the expansion in a series of terms of x. A more expeditious way of determining these coefficients is devised in the following manner:

Let us define a partial differential operator in the form

$$\Delta=a_1\frac{\partial}{\partial a_0}+a_2\frac{\partial}{\partial a_1}+a_3\frac{\partial}{\partial a_2}+a_4\frac{\partial}{\partial a_3}+\cdot\ \cdot\ \cdot\ \cdot\ ;$$

then

$$\Delta f\left(a_0+a_1\frac{x}{1!}+a_2\frac{x_2}{2!}+a_3\frac{x^3}{3!}+a_4\frac{x^4}{4!}+\cdot\ \cdot\ \cdot\ \cdot\right)=\left(a_1+a_2\frac{x}{1!}\right.$$

$$\left.+a_3\frac{x^2}{2!}+a_4\frac{x^3}{3!}+a_5\frac{x^4}{4!}+\cdot\ \cdot\ \cdot\ \cdot\right)f^1\left(a_0+a_1\frac{x}{1!}+a_2\frac{x^2}{2!}\right.$$

$$\left.+a_3\frac{x^3}{3!}+a_4\frac{x^4}{4!}+\cdot\ \cdot\ \cdot\ \cdot\right)=\frac{d}{dx}f\left(a_0+a_1\frac{x}{1!}+a_2\frac{x^2}{2!}\right.$$

$$\left.+a_3\frac{x^3}{3!}+a_4\frac{x^4}{4!}+\cdot\ \cdot\ \cdot\ \cdot\right).$$

Therefore, if A_n denotes the nth coefficient in the expansion of the given function, we have

$$\Delta A_0+\Delta A_1\frac{x}{1!}+\Delta A_2\frac{x^2}{2!}+\Delta A_3\frac{x^3}{3!}+\Delta A_4\frac{x^4}{4!}+\cdot\ \cdot\ \cdot\ \cdot$$

$$=A_1+A_2\frac{x}{1!}+A_3\frac{x^2}{2!}+A_4\frac{x^3}{3!}+A_5\frac{x^4}{4!}+\cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

Equating the coefficients of the like powers of x in this identity, we obtain the recurrence formula

$$A_{n+1}=\Delta A_n.$$

Now in the Taylor development we see that

$$A_0=f\ (a_0).$$

By applying the recurrence formula in succession, we may write down the coefficients.

$$A_1 = a_1 f^1(a_0),$$
$$A_2 = a_1^2 f^2(a_0) + a_2 f^1(a_0),$$
$$A_3 = a_1^3 f^3(a_0) + 3a_1 a_2 f^2(a_0) + a_3 f^1(a_0),$$
$$A_4 = a_1^4 f^4(a_0) + 6a_1^2 a_2 f^3(a_0) + 4a_1 a_3 f^2(a_0) + 3a_2^2 f^2(a_0) + a_4 f^1(a_0),$$

etc.

We shall now apply these principles in the development of the function

$$z = 2 \tan^{-1} \exp^* \left(\log_e \tan\frac{p}{2} + \frac{h}{1!}\cos p + \frac{h^2}{2!}\frac{2}{3}\cos^3 p + \frac{h^3}{3!}\frac{6}{5}\cos^5 p + \frac{h^4}{4!}\frac{24}{7}\cos^7 p + \cdots\right) = A_0 + A_1\frac{h}{1!} + A_2\frac{h^2}{2!} + A_3\frac{h^3}{3!} + A_4\frac{h^4}{4!} + \cdots,$$

in which $h = \epsilon^2$.

In this case

$$a_0 = \log_e \tan\frac{p}{2},$$

$$\frac{da_0}{dp} = \frac{\sec^2\frac{p}{2}}{2\tan\frac{p}{2}} = \frac{1}{\sin p},$$

or

$$\frac{dp}{da_0} = \sin p,$$

$$f(a_0) = 2\tan^{-1} e^{a_0} = p,$$

$$f^1(a_0) = \frac{dp}{da_0} = \sin p,$$

$$f^2(a_0) = \cos p\,\frac{dp}{da_0} = \sin p \cos p = \frac{1}{2}\sin 2p,$$

$$f^3(a_0) = \cos 2p\,\frac{dp}{da_0} = \sin p \cos 2p = \frac{1}{2}\sin 3p - \frac{1}{2}\sin p,$$

$$f^4(a_0) = \left(\frac{3}{2}\cos 3p - \frac{1}{2}\cos p\right)\frac{dp}{da_0} = \left(\frac{3}{2}\cos 3p - \frac{1}{2}\cos p\right)\sin p = \frac{3}{4}\sin 4p - \sin 2p.$$

---

* $\exp x = e^x$

In the function to be expanded, we are given

$$a_0=\log_e \tan \frac{p}{2},$$

$$a_1=\cos p,$$

$$a_2=\frac{2}{3}\cos^3 p,$$

$$a_3=\frac{6}{5}\cos^5 p,$$

$$a_4=\frac{24}{7}\cos^7 p.$$

With these values we may compute the various A_n and reduce them by aid of the reduction table on page 88.

$$A_0=p,$$

$$A_1=\sin p\cos p=\frac{1}{2}\sin 2p,$$

$$A_2=\frac{1}{2}\cos^2 p\sin 2p+\frac{2}{3}\cos^3 p\sin p=\frac{5}{3}\sin p\cos^3 p$$
$$=\frac{5}{12}\sin 2p+\frac{5}{24}\sin 4p,$$

$$A_3=\cos^3 p\left(\frac{1}{2}\sin 3p-\frac{1}{2}\sin p\right)+\cos^4 p\sin 2p+\frac{6}{5}\cos^5 p\sin p$$
$$=-\sin p\cos^3 p+\frac{26}{5}\sin p\cos^5 p$$
$$=\frac{9}{16}\sin 2p+\frac{21}{40}\sin 4p+\frac{13}{80}\sin 6p,$$

$$A_4=\cos^4 p\left(\frac{3}{4}\sin 4p-\sin 2p\right)+4\cos^5 p\sin p\cos 2p$$
$$+\frac{24}{5}\cos^6 p\sin p\cos p+\frac{4}{3}\cos^6 p\sin p\cos p$$
$$+\frac{24}{7}\cos^7 p\sin p=-9\sin p\cos^5 p+\frac{2474}{105}\sin p\cos^7 p$$
$$=\frac{281}{240}\sin 2p+\frac{697}{480}\sin 4p+\frac{461}{560}\sin 6p+\frac{1237}{6720}\sin 8p.$$

Substituting these values in the expansion and remembering that $h=\epsilon^2$, we get, after rearrangement as the desired approximation,

$$z=p+\left(\frac{\epsilon^2}{2}+\frac{5\epsilon^4}{24}+\frac{3\epsilon^6}{32}+\frac{281\epsilon^8}{5760}\right)\sin 2p+\left(\frac{5\epsilon^4}{48}+\frac{7\epsilon^6}{80}+\frac{697\epsilon^8}{11520}\right)\sin 4p$$
$$+\left(\frac{13\epsilon^6}{480}+\frac{461\epsilon^8}{13440}\right)\sin 6p+\frac{1237\epsilon^8}{161280}\sin 8p,$$

or, in terms of φ and χ, we get, as before, the approximation:

$$\varphi-\chi=\left(\frac{\epsilon^2}{2}+\frac{5\epsilon^4}{24}+\frac{3\epsilon^6}{32}+\frac{281\epsilon^8}{5760}\right)\sin 2\varphi-\left(\frac{5\epsilon^4}{48}+\frac{7\epsilon^6}{80}+\frac{697\epsilon^8}{11520}\right)\sin 4\varphi$$
$$+\left(\frac{13\epsilon^6}{480}+\frac{461\epsilon^8}{13440}\right)\sin 6\varphi-\frac{1237\epsilon^8}{161280}\sin 8\varphi.$$

DEVELOPMENT OF $\varphi-\chi$ IN TERMS OF χ—FIRST METHOD.

The quantity $\varphi-\chi$ can be expressed in terms of χ by the application of Lagrange's series. We have given

$$\varphi=\chi+f(\varphi).$$

Since $f(\varphi)$ is a small quantity, Lagrange's series becomes

$$\varphi=\chi+\frac{1}{1!}f(\chi)+\frac{1}{2!}\frac{d}{d\chi}[f(\chi)]^2+\frac{1}{3!}\frac{d^2}{d\chi^2}[f(\chi)]^3$$
$$+\frac{1}{4!}\frac{d^3}{d\chi^3}[f(\chi)]^4+\cdots.$$

But in the series for $\varphi-\chi$ given above, we see that

$$f(\varphi)=\left(\frac{\epsilon^2}{2}+\frac{5\epsilon^4}{24}+\frac{3\epsilon^6}{32}+\frac{281\epsilon^8}{5760}\right)\sin 2\varphi-\left(\frac{5\epsilon^4}{48}+\frac{7\epsilon^6}{80}+\frac{697\epsilon^8}{11520}\right)\sin 4\varphi$$
$$+\left(\frac{13\epsilon^6}{480}+\frac{461\epsilon^8}{13440}\right)\sin 6\varphi-\frac{1237\epsilon^8}{161280}\sin 8\varphi.$$

Squaring this expression and reducing by aid of the table on p. 88, we get

$$[f(\varphi)]^2=\frac{\epsilon^4}{8}+\frac{5\epsilon^6}{48}+\frac{341\epsilon^8}{4608}-\left(\frac{5\epsilon^6}{96}+\frac{377\epsilon^8}{5760}\right)\cos 2\varphi-\left(\frac{\epsilon^4}{8}+\frac{5\epsilon^6}{48}\right.$$
$$\left.+\frac{317\epsilon^8}{5760}\right)\cos 4\varphi+\left(\frac{5\epsilon^6}{96}+\frac{377\epsilon^8}{5760}\right)\cos 6\varphi-\frac{437\epsilon^8}{23040}\cos 8\varphi,$$

and, by cubing and reducing, we get

$$[f(\varphi)]^3=\left(\frac{3\epsilon^6}{32}+\frac{15\epsilon^8}{128}\right)\sin 2\varphi-\frac{5\epsilon^8}{128}\sin 4\varphi-\left(\frac{\epsilon^6}{32}+\frac{5\epsilon^8}{128}\right)\sin 6\varphi+\frac{5\epsilon^8}{256}\sin 8\varphi,$$

and for the fourth power, by a similar process, we obtain

$$[f(\varphi)]^4=\frac{3\epsilon^8}{128}-\frac{\epsilon^8}{32}\cos 4\varphi+\frac{\epsilon^8}{128}\cos 8\varphi.$$

$$\frac{d}{d\chi}[f(\chi)]^2=\left(\frac{5\epsilon^6}{48}+\frac{377\epsilon^8}{2880}\right)\sin 2\chi+\left(\frac{\epsilon^4}{2}+\frac{5\epsilon^6}{12}+\frac{317\epsilon^8}{1440}\right)\sin 4\chi-\left(\frac{5\epsilon^6}{16}+\frac{377\epsilon^8}{960}\right)\sin 6\chi+\frac{437\epsilon^8}{2880}\sin 8\chi,$$

$$\frac{d^2}{d\chi^2}[f(\chi)]^3=-\left(\frac{3\epsilon^6}{8}+\frac{15\epsilon^8}{32}\right)\sin 2\chi+\frac{5\epsilon^8}{8}\sin 4\chi+\left(\frac{9\epsilon^6}{8}+\frac{45\epsilon^8}{32}\right)\sin 6\chi-\frac{5\epsilon^8}{4}\sin 8\chi,$$

$$\frac{d^3}{d\chi^3}[f(\chi)]^4=-2\epsilon^8\sin 4\chi+4\epsilon^8\sin 8\chi.$$

Substituting these values in Lagrange's series, we get the approximation

$$\begin{aligned}\varphi-\chi=&\left(\frac{\epsilon^2}{2}+\frac{5\epsilon^4}{24}+\frac{3\epsilon^6}{32}+\frac{281\epsilon^8}{5760}\right)\sin 2\chi-\left(\frac{5\epsilon^4}{48}+\frac{7\epsilon^6}{80}+\frac{697\epsilon^8}{11520}\right)\sin 4\chi\\&+\left(\frac{13\epsilon^6}{480}+\frac{461\epsilon^8}{13440}\right)\sin 6\chi-\frac{1237\epsilon^8}{161280}\sin 8\chi\\&+\left(\frac{5\epsilon^6}{96}+\frac{377\epsilon^8}{5760}\right)\sin 2\chi+\left(\frac{\epsilon^4}{4}+\frac{5\epsilon^6}{24}+\frac{317\epsilon^8}{2880}\right)\sin 4\chi\\&-\left(\frac{5\epsilon^6}{32}+\frac{377\epsilon^8}{1920}\right)\sin 6\chi+\frac{437\epsilon^8}{5760}\sin 8\chi-\left(\frac{\epsilon^6}{16}+\frac{5\epsilon^8}{64}\right)\sin 2\chi\\&+\frac{5\epsilon^8}{48}\sin 4\chi+\left(\frac{3\epsilon^6}{16}+\frac{15\epsilon^8}{64}\right)\sin 6\chi-\frac{5\epsilon^8}{24}\sin 8\chi\\&-\frac{\epsilon^8}{12}\sin 4\chi+\frac{\epsilon^8}{6}\sin 8\chi.\end{aligned}$$

When similar terms are collected, this approximation becomes

$$\varphi-\chi=\left(\frac{\epsilon^2}{2}+\frac{5\epsilon^4}{24}+\frac{\epsilon^6}{12}+\frac{13\epsilon^8}{360}\right)\sin 2\chi+\left(\frac{7\epsilon^4}{48}+\frac{29\epsilon^6}{240}+\frac{811\epsilon^8}{11520}\right)\sin 4\chi$$
$$+\left(\frac{7\epsilon^6}{120}+\frac{81\epsilon^8}{1120}\right)\sin 6\chi+\frac{4279\epsilon^8}{161280}\sin 8\chi.$$

DEVELOPMENT OF $\varphi-\chi$ IN TERMS OF χ—SECOND METHOD

The series for $\varphi-\chi$ in terms of φ can be expressed in terms of the second argument by the method of successive approximations. In the series for $\varphi-\chi$, let

$$\varphi=\chi+a.$$

Then a will be equal to a small quantity since we have (see above)

$$a=\left(\frac{\epsilon^2}{2}+\frac{5\epsilon^4}{24}+\frac{3\epsilon^6}{32}+\frac{281\epsilon^8}{5760}\right)\sin 2\varphi-\left(\frac{5\epsilon^4}{48}+\frac{7\epsilon^6}{80}+\frac{697\epsilon^8}{11520}\right)\sin 4\varphi$$
$$+\left(\frac{13\epsilon^6}{480}+\frac{461\epsilon^8}{13440}\right)\sin 6\varphi-\frac{1237\epsilon^8}{161280}\sin 8\varphi.$$

In the series for $\varphi-\chi$, φ must first be replaced by $\chi+a$, and this developed far enough to include terms in ϵ^8. The value of a is then substituted, and this introduces terms in φ again. These terms are then developed in χ and a, and the process repeated until all terms in ϵ^8 have been included. To shorten the work assume

$$\varphi=\chi+A\sin 2\varphi-B\sin 4\varphi+C\sin 6\varphi-D\sin 8\varphi.$$

Then the lowest power of ϵ in A is ϵ^2; in B is ϵ^4; in C is ϵ^6; and in D is ϵ^8.

(For the necessary reductions see the reduction table, p. 88.)

Substitute for φ in the above series $\chi+a$ and we have

$$\varphi=\chi+A\sin(2\chi+2a)-B\sin(4\chi+4a)+C\sin(6\chi+6a)$$
$$-D\sin(8\chi+8a),$$

or

$$\varphi=\chi+A\sin 2\chi\cos 2a+A\cos 2\chi\sin 2a-B\sin 4\chi\cos 4a$$
$$-B\cos 4\chi\sin 4a+C\sin 6\chi\cos 6a+C\cos 6\chi\sin 6a$$
$$-D\sin 8\chi\cos 8a-D\cos 8\chi\sin 8a.$$

Developing the functions in a far enough to include all terms in ϵ^8, we get

$$\varphi=\chi+A\left(1-\frac{4a^2}{2}\right)\sin 2\chi+A\left(2a-\frac{8a^3}{6}\right)\cos 2\chi$$
$$-B\left(1-\frac{16a^2}{2}\right)\sin 4\chi-B\,4a\cos 4\chi+C\sin 6\chi$$
$$+C\,6a\cos 6\chi-D\sin 8\chi.$$

These coefficients must now be evaluated and second and third approximations applied wherever necessary to get the required exactness.

$$A\ (1-2a^2)=A\ [1-2\left(\frac{\epsilon^4}{4}+\frac{5\epsilon^6}{24}\right)\sin^2 2\varphi+\frac{5\epsilon^6}{24}\sin 2\varphi\sin 4\varphi]$$
$$=A\ [1-\left(\frac{\epsilon^4}{4}+\frac{5\epsilon^6}{24}\right)+\left(\frac{\epsilon^4}{4}+\frac{5\epsilon^6}{24}\right)\cos 4\varphi$$
$$+\frac{5\epsilon^6}{24}\sin 2\chi\sin 4\chi]$$

In this expression, since the lowest power of ϵ in A is ϵ^2, we do not have to carry the development farther than to include terms in ϵ^6.

But

$$\cos 4\varphi=\cos\ (4\chi+4a)=\cos 4\chi\cos 4a-\sin 4\chi\sin 4a.$$

No term beyond ϵ^2 is needed in this approximation. Hence for the exactness required

$$\cos 4\varphi=\cos 4\chi-2\epsilon^2\sin 2\chi\sin 4\chi.$$

Therefore

$$A\ (1-2a^2)=A\ [1-\left(\frac{\epsilon^4}{4}+\frac{5\epsilon^6}{24}\right)+\left(\frac{\epsilon^4}{4}+\frac{5\epsilon^6}{24}\right)\cos 4\chi$$
$$-\frac{\epsilon^6}{2}\sin 2\chi\sin 4\chi+\frac{5\epsilon^6}{24}\sin 2\chi\sin 4\chi].$$

On multiplying the two factors and reducing to a series of cosines of the multiple arcs, we get

$$A\ (1-2a^2)=\frac{\epsilon^2}{2}+\frac{5\epsilon^4}{24}-\frac{\epsilon^6}{32}-\frac{619\epsilon^8}{5760}-\frac{7\epsilon^8}{96}\cos 2\chi$$
$$+\left(\frac{\epsilon^6}{8}+\frac{5\epsilon^8}{32}\right)\cos 4\chi+\frac{7\epsilon^8}{96}\cos 6\chi.$$

By a similar procedure, we get

$$A\left(2a-\frac{4a^3}{3}\right)=A\left[\left(\epsilon^2+\frac{5\epsilon^4}{12}+\frac{3\epsilon^6}{16}\right)\sin\ 2\varphi-\left(\frac{5\epsilon^4}{24}+\frac{7\epsilon^6}{40}\right)\sin\ 4\varphi\right.$$
$$\left.+\frac{13\epsilon^6}{240}\sin\ 6\varphi-\frac{4}{3}\left(\frac{\epsilon^6}{8}\right)\sin^3\ 2\varphi\right],$$

or

$$A\left(2a-\frac{4a^3}{3}\right)=A\left[\left(\epsilon^2+\frac{5\epsilon^4}{12}+\frac{3\epsilon^6}{16}\right)(\sin\ 2\chi\cos\ 2a\right.$$
$$+\cos\ 2\chi\sin\ 2a)-\left(\frac{5\epsilon^4}{24}+\frac{7\epsilon^6}{40}\right)(\sin\ 4\chi\cos\ 4a$$
$$\left.+\cos\ 4\chi\sin\ 4a)+\frac{13\epsilon^6}{240}\sin\ 6\chi-\frac{\epsilon^6}{6}\sin^3\ 2\chi\right],$$

$$A\left(2a-\frac{4a^3}{3}\right)=A\left[\left(\epsilon^2+\frac{5\epsilon^4}{12}+\frac{3\epsilon^6}{16}\right)\left(1-\frac{4a^2}{2}\right)\sin\ 2\chi\right.$$
$$+2a\left(\epsilon^2+\frac{5\epsilon^4}{12}\right)\cos\ 2\chi-\left(\frac{5\epsilon^4}{24}+\frac{7\epsilon^6}{40}\right)\sin\ 4\chi$$
$$\left.-4a\left(\frac{5\epsilon^4}{24}\right)\cos\ 4\chi+\frac{13\epsilon^6}{240}\sin\ 6\chi-\frac{\epsilon^6}{6}\sin^3\ 2\chi\right],$$

$$A\left(2a-\frac{4a^3}{3}\right)=A\left[\left(\epsilon^2+\frac{5\epsilon^4}{12}+\frac{3\epsilon^6}{16}\right)\sin\ 2\chi-\frac{\epsilon^6}{2}\sin^3\ 2\chi\right.$$
$$+\left(\epsilon^4+\frac{5\epsilon^6}{6}\right)\sin\ 2\varphi\cos\ 2\chi-\frac{5\epsilon^6}{24}\sin\ 4\chi\cos\ 2\chi$$
$$-\left(\frac{5\epsilon^4}{24}+\frac{7\epsilon^6}{40}\right)\sin\ 4\chi-\frac{5\epsilon^6}{12}\sin\ 2\chi\cos\ 4\chi$$
$$\left.+\frac{13\epsilon^6}{240}\sin\ 6\chi-\frac{\epsilon^6}{6}\sin^3\ 2\chi\right].$$

Finally, on developing $\sin 2\varphi$ in the above expression, we get

$$A\left(2a-\frac{4a^3}{3}\right)=A\left[\left(\epsilon^2+\frac{5\epsilon^4}{12}+\frac{3\epsilon^6}{16}\right)\sin\ 2\chi-\frac{\epsilon^6}{2}\sin^3 2\chi\right.$$
$$+\left(\epsilon^4+\frac{5\epsilon^6}{6}\right)\sin 2\chi\cos 2\chi+\epsilon^6\sin 2\chi\cos^2 2\chi$$
$$-\frac{5\epsilon^6}{24}\sin\ 4\chi\cos\ 2\chi-\left(\frac{5\epsilon^4}{24}+\frac{7\epsilon^6}{40}\right)\sin 4\chi-\frac{5\epsilon^6}{12}\sin\ 2\chi\cos\ 4\chi$$
$$\left.+\frac{13\epsilon^6}{240}\sin\ 6\chi-\frac{\epsilon^6}{6}\sin^3\ 2\chi\right],$$

$$A\left(2a-\frac{4a^3}{3}\right)=\left(\frac{\epsilon^4}{2}+\frac{5\epsilon^6}{12}+\frac{79\epsilon^8}{288}\right)\sin 2\chi-\frac{\epsilon^8}{4}\sin^3 2\chi$$
$$+\left(\frac{\epsilon^6}{2}+\frac{5\epsilon^8}{8}\right)\sin 2\chi\cos 2\chi+\frac{\epsilon^8}{2}\sin 2\chi\cos^2 2\chi$$
$$-\frac{5\epsilon^8}{48}\sin 4\chi\cos 2\chi-\left(\frac{5\epsilon^6}{48}+\frac{377\epsilon^8}{2880}\right)\sin 4\chi$$
$$-\frac{5\epsilon^8}{24}\sin 2\chi\cos 4\chi+\frac{13\epsilon^8}{480}\sin 6\chi-\frac{\epsilon^8}{12}\sin^3 2\chi.$$

Reducing this to a series in the sines of multiple arcs by aid of the reduction table on p. 88, we obtain

$$A\left(2a-\frac{4a^3}{3}\right)=\left(\frac{\epsilon^4}{2}+\frac{5\epsilon^6}{12}+\frac{29\epsilon^8}{144}\right)\sin 2\chi+\left(\frac{7\epsilon^6}{48}+\frac{523\epsilon^8}{2880}\right)\sin 4\chi$$
$$+\frac{19\epsilon^8}{240}\sin 6\chi.$$

$$B(1-8a^2)=B(1-2\epsilon^4\sin^2 2\chi)=\frac{5\epsilon^4}{48}+\frac{7\epsilon^6}{80}-\frac{503\epsilon^8}{11520}+\frac{5\epsilon^8}{48}\cos 4\chi.$$

$$4Ba=4B\left[\left(\frac{\epsilon^2}{2}+\frac{5\epsilon^4}{24}\right)\sin 2\varphi-\frac{5\epsilon^4}{48}\sin 4\chi\right]$$
$$=4B\left[\left(\frac{\epsilon^2}{2}+\frac{5\epsilon^4}{24}\right)\sin 2\chi+\frac{\epsilon^4}{2}\sin 2\chi\cos 2\chi-\frac{5\epsilon^4}{48}\sin 4\chi\right]$$
$$=\left(\frac{5\epsilon^6}{24}+\frac{377\epsilon^8}{1440}\right)\sin 2\chi+\frac{35\epsilon^8}{576}\sin 4\chi.$$
$$6Ca=3C\epsilon^2\sin 2\chi=\frac{13\epsilon^8}{160}\sin 2\chi.$$

On substituting these values in the original expression, we get the approximation

$$\varphi-\chi=\left(\frac{\epsilon^2}{2}+\frac{5\epsilon^4}{24}-\frac{\epsilon^6}{32}-\frac{619\epsilon^8}{5760}\right)\sin 2\chi-\frac{7\epsilon^8}{96}\sin 2\chi\cos 2\chi$$
$$+\left(\frac{\epsilon^6}{8}+\frac{5\epsilon^8}{32}\right)\sin 2\chi\cos 4\chi+\frac{7\epsilon^8}{96}\sin 2\chi\cos 6\chi$$
$$+\left(\frac{\epsilon^4}{2}+\frac{5\epsilon^6}{12}+\frac{29\epsilon^8}{144}\right)\sin 2\chi\cos 2\chi+\left(\frac{7\epsilon^6}{48}+\frac{523\epsilon^8}{2880}\right)\sin 4\chi\cos 2\chi$$
$$+\frac{19\epsilon^8}{240}\sin 6\chi\cos 2\chi-\left(\frac{5\epsilon^4}{48}+\frac{7\epsilon^6}{80}-\frac{503\epsilon^8}{11520}\right)\sin 4\chi$$
$$-\frac{5\epsilon^8}{48}\sin 4\chi\cos 4\chi-\left(\frac{5\epsilon^6}{24}+\frac{377\epsilon^8}{1440}\right)\sin 2\chi\cos 4\chi$$
$$-\frac{35\epsilon^8}{576}\sin 4\chi\cos 4\chi+\left(\frac{13\epsilon^6}{480}+\frac{461\epsilon^8}{13440}\right)\sin 6\chi$$
$$+\frac{13\epsilon^8}{160}\sin 2\chi\cos 6\chi-\frac{1237\epsilon^8}{161280}\sin 8\chi,$$

or, on reduction and rearrangement, this becomes

$$\varphi-\chi=\left(\frac{\epsilon^2}{2}+\frac{5\epsilon^4}{24}+\frac{\epsilon^6}{12}+\frac{13\epsilon^8}{360}\right)\sin 2\chi+\left(\frac{7\epsilon^4}{48}+\frac{29\epsilon^6}{240}+\frac{811\epsilon^8}{11520}\right)\sin 4\chi$$
$$+\left(\frac{7\epsilon^6}{120}+\frac{81\epsilon^8}{1120}\right)\sin 6\chi+\frac{4279\epsilon^8}{161280}\sin 8\chi.$$

This approximation agrees with the expression determined by Lagrange's series.

DEVELOPMENT OF $\varphi-\chi$ IN TERMS OF χ—THIRD METHOD.

We can develop $\varphi-\chi$ in terms of χ by a method similar to the second method of developing the same in terms of φ. (See p. 18.)

Let

$$\frac{p}{2}=\frac{z}{2}-(a\epsilon^2+b\epsilon^4+c\epsilon^6+d\epsilon^8+\cdot\cdot\cdot\cdot).$$

But

$$\log_e \tan\frac{p}{2}=\log_e \tan\frac{z}{2}-\epsilon^2\cos p-\frac{\epsilon^4}{3}\cos^3 p-\frac{\epsilon^6}{5}\cos^5 p$$
$$-\frac{\epsilon^8}{7}\cos^7 p-\cdot\cdot\cdot\cdot$$

From the Taylor development on p. 19, by changing the sign of h, and by interchanging p and z, we get

$$\log_e \tan\left(\frac{z}{2}-h\right)=\log_e \tan\frac{z}{2}-\frac{2h}{\sin z}-\frac{2\cos z}{\sin^2 z}h^2-\left(\frac{4\cos^2 z}{3\sin^3 z}\right.$$
$$\left.+\frac{4}{3\sin^3 z}\right)h^3-\left(\frac{2\cos^3 z}{3\sin^4 z}+\frac{10\cos z}{3\sin^4 z}\right)h^4-\cdot\cdot\cdot\cdot$$

By substituting the value of h and by retaining terms to the eighth power of ϵ inclusive, we obtain

$$\log_e \tan\left(\frac{z}{2}-h\right)=\log_e \tan\frac{z}{2}-\frac{2}{\sin z}(a\epsilon^2+b\epsilon^4+c\epsilon^6+d\epsilon^8)$$
$$-\frac{2\cos z}{3\sin^2 z}[a^2\epsilon^4+2\,ab\epsilon^6+(b^2+2\,ac)\,\epsilon^8]-\left(\frac{4\cos^2 z}{3\sin^3 z}\right.$$
$$\left.+\frac{4}{3\sin^3 z}\right)(a^3\epsilon^6+3\,a^2\,b\epsilon^8)-\left(\frac{2\cos^3 z}{3\sin^4 z}+\frac{10\cos z}{3\sin^4 z}\right)a^4\epsilon^8.$$

The powers of $\cos p$ must now be approximated in terms of functions of z to include all eighth powers of ϵ. We have assumed

$$\frac{p}{2}=\frac{z}{2}-h,$$

or

$$p=z-2h.$$

$$\cos(z-2h)=\cos z\cos 2h+\sin z\sin 2h$$
$$=(1-2h^2)\cos z+\left(2h-\frac{4h^3}{3}\right)\sin z.$$

The approximation does not need to be carried further, since powers of ϵ are not required above the sixth, because we are approximating for $\epsilon^2\cos p$.

Substituting the value of h, we get

$$\cos(z-2h)=\cos p=(1-2a^2\epsilon^4-4ab\epsilon^6)\cos z+\left(2a\epsilon^2+2b\epsilon^4+2c\epsilon^6-\frac{4a^3}{3}\epsilon^6\right)\sin z,$$

$$\cos^3 p=(1-6a^2\epsilon^4)\cos^3 z+3(2a\epsilon^2+2b\epsilon^4)\sin z\cos^2 z+12a^2\epsilon^4\sin^2 z\cos z,$$

$$\cos^5 p=\cos^5 z+10a\epsilon^2\sin z\cos^4 z,$$

$$\cos^7 p=\cos^7 z.$$

Substituting these values in the series for

$$\log_e\tan\frac{p}{2},$$

we get

$$\log_e\tan\frac{p}{2}=\log_e\tan\frac{z}{2}-(\epsilon^2-2a^2\epsilon^6-4ab\epsilon^8)\cos z$$
$$-\left(2a\epsilon^4+2b\epsilon^6+2c\epsilon^8-\frac{4a^3}{3}\epsilon^8\right)\sin z-\left(\frac{\epsilon^4}{3}-2a^2\epsilon^8\right)\cos^3 z$$
$$-(2a\epsilon^6+2b\epsilon^8)\sin z\cos^2 z-4a^2\epsilon^8\sin^2 z\cos z-\frac{\epsilon^6}{5}\cos^5 z$$
$$-2a\epsilon^8\sin z\cos^4 z-\frac{\epsilon^8}{7}\cos^7 z.$$

This series must be identically equal to the series obtained above by the Taylor development and hence the coefficients of similar powers of ϵ must be equal in the two series.

Equating these coefficients, we get:

$$-\frac{2a}{\sin z}=-\cos z,$$

$$-\frac{2b}{\sin z}-\frac{2a^2\cos z}{\sin^2 z}=-2a\sin z-\frac{\cos^3 z}{3},$$

$$-\frac{2c}{\sin z}-\frac{4ab\cos z}{\sin^2 z}-\frac{4a^3\cos^2 z}{3\sin^3 z}-\frac{4a^3}{3\sin^3 z}=2a^2\cos z$$
$$-2b\sin z-2a\sin z\cos^2 z-\frac{\cos^5 z}{5},$$

$$-\frac{2d}{\sin z}-\frac{2b^2\cos z}{\sin^2 z}-\frac{4ac\cos z}{\sin^2 z}-\frac{4a^2b\cos^2 z}{\sin^3 z}-\frac{4a^2b}{\sin^3 z}$$
$$-\frac{2a^4\cos^3 z}{3\sin^4 z}-\frac{10a^4\cos z}{3\sin^4 z}=4ab\cos z-2c\sin z+\frac{4a^3\sin z}{3}$$
$$+2a^2\cos^3 z-2b\sin z\cos^2 z-4a^2\sin^2 z\cos z$$
$$-2a\sin z\cos^4 z-\frac{\cos^7 z}{7}.$$

From these equations in succession we obtain the values of a, b, c, and d.

(For the necessary reductions see the reduction table, p. 88.)

$$a=\frac{1}{2}\sin z\cos z=\frac{1}{4}\sin 2z,$$

$$b=\frac{1}{2}\sin z\cos z-\frac{7}{12}\sin z\cos^3 z=\frac{5}{48}\sin 2z-\frac{7}{96}\sin 4z,$$

$$c=\frac{1}{2}\sin z\cos z-\frac{17}{12}\sin z\cos^3 z+\frac{14}{15}\sin z\cos^5 z$$
$$=\frac{1}{24}\sin 2z-\frac{29}{480}\sin 4z+\frac{7}{240}\sin 6z,$$

$$d=\frac{1}{2}\sin z\cos z-\frac{5}{2}\sin z\cos^3 z+\frac{889}{240}\sin z\cos^5 z$$
$$-\frac{4279}{2520}\sin z\cos^7 z=\frac{13}{720}\sin 2z-\frac{811}{23040}\sin 4z$$
$$+\frac{81}{2240}\sin 6z-\frac{4279}{322560}\sin 8z.$$

With these values the series becomes after rearrangement the desired approximation

$$p=z-\left(\frac{\epsilon^2}{2}+\frac{5\epsilon^4}{24}+\frac{\epsilon^6}{12}+\frac{13\epsilon^8}{360}\right)\sin 2z+\left(\frac{7\epsilon^4}{48}+\frac{29\epsilon^6}{240}+\frac{811\epsilon^8}{11520}\right)\sin 4z$$
$$-\left(\frac{7\epsilon^6}{120}+\frac{81\epsilon^8}{1120}\right)\sin 6z+\frac{4279\epsilon^8}{161280}\sin 8z;$$

or, in terms of φ and χ, we get, as before, the approximation

$$\varphi-\chi=\left(\frac{\epsilon^2}{2}+\frac{5\epsilon^4}{24}+\frac{\epsilon^6}{12}+\frac{13\epsilon^8}{360}\right)\sin 2\chi+\left(\frac{7\epsilon^4}{48}+\frac{29\epsilon^6}{240}+\frac{811\epsilon^8}{11520}\right)\sin 4\chi$$
$$+\left(\frac{7\epsilon^6}{120}+\frac{81\epsilon^8}{1120}\right)\sin 6\chi+\frac{4279\epsilon^8}{161280}\sin 8\chi.$$

DEVELOPMENT OF $\varphi-\chi$ IN TERMS OF χ—FOURTH METHOD.

$\varphi-\chi$ may be developed in terms of χ by differentiating the equation of definition considering φ as a function of ϵ^2 or of h, or by using the more convenient form containing p and by considering p as a function of ϵ^2 or of h. For convenience we write the expression in the form

$$\log_e \tan\frac{z}{2}=\log_e \tan\frac{p}{2}+\left(h+\frac{h^2}{4}+\frac{h^3}{8}+\frac{5h^4}{64}\right)\cos p$$
$$+\left(\frac{h^2}{12}+\frac{h^3}{16}+\frac{3h^4}{64}\right)\cos 3p+\left(\frac{h^3}{80}+\frac{h^4}{64}\right)\cos 5p$$
$$+\frac{h^4}{448}\cos 7p.$$

Differentiating this expression, considering p as a function of h, we get

$$\Big[\operatorname{cosec} p-\left(h+\frac{h^2}{4}+\frac{h^3}{8}+\frac{5h^4}{64}\right)\sin p-\left(\frac{h^2}{4}+\frac{3h^3}{16}+\frac{9h^4}{64}\right)\sin 3p$$
$$-\left(\frac{h^3}{16}+\frac{5h^4}{64}\right)\sin 5p-\frac{h^4}{64}\sin 7p\Big]\frac{dp}{dh}$$
$$+\left(1+\frac{h}{2}+\frac{3h^2}{8}+\frac{5h^3}{16}\right)\cos p+\left(\frac{h}{6}+\frac{3h^2}{16}+\frac{3h^3}{16}\right)\cos 3p$$
$$+\left(\frac{3h^2}{80}+\frac{h^3}{16}\right)\cos 5p+\frac{h^3}{112}\cos 7p=0,$$

$$\left[\operatorname{cosec} p-\left(h+\frac{h^2}{4}+\frac{h^3}{8}+\frac{5h^4}{64}\right)\sin p-\left(\frac{h^2}{4}+\frac{3h^3}{16}+\frac{9h^4}{64}\right)\sin 3p\right.$$

$$\left.-\left(\frac{h^3}{16}+\frac{5h^4}{64}\right)\sin 5p-\frac{h^4}{64}\sin 7p\right]\frac{d^2p}{dh^2}-\left[\operatorname{cosec} p\cot p\right.$$

$$+\left(h+\frac{h^2}{4}+\frac{h^3}{8}+\frac{5h^4}{64}\right)\cos p+\left(\frac{3h^2}{4}+\frac{9h^3}{16}+\frac{27h^4}{64}\right)\cos 3p$$

$$\left.+\left(\frac{5h^3}{16}+\frac{25h^4}{64}\right)\cos 5p+\frac{7h^4}{64}\cos 7p\right]\left(\frac{dp}{dh}\right)^2$$

$$-2\left[\left(1+\frac{h}{2}+\frac{3h^2}{8}+\frac{5h^3}{16}\right)\sin p+\left(\frac{h}{2}+\frac{9h^2}{16}+\frac{9h^3}{16}\right)\sin 3p\right.$$

$$\left.+\left(\frac{3h^2}{16}+\frac{5h^3}{16}\right)\sin 5p+\frac{h^3}{16}\sin 7p\right]\frac{dp}{dh}$$

$$+\left(\frac{1}{2}+\frac{3h}{4}+\frac{15h^2}{16}\right)\cos p+\left(\frac{1}{6}+\frac{3h}{8}+\frac{9h^2}{16}\right)\cos 3p$$

$$+\left(\frac{3h}{40}+\frac{3h^2}{16}\right)\cos 5p+\frac{3h^2}{112}\cos 7p=0,$$

$$\left[\operatorname{cosec} p-\left(h+\frac{h^2}{4}+\frac{h^3}{8}+\frac{5h^4}{64}\right)\sin p-\left(\frac{h^2}{4}+\frac{3h^3}{16}+\frac{9h^4}{64}\right)\sin 3p\right.$$

$$\left.-\left(\frac{h^3}{16}+\frac{5h^4}{64}\right)\sin 5p-\frac{h^4}{64}\sin 7p\right]\frac{d^3p}{dh^3}+\left[\operatorname{cosec} p\cot^2 p\right.$$

$$+\operatorname{cosec}^3 p+\left(h+\frac{h^2}{4}+\frac{h^3}{8}+\frac{5h^4}{64}\right)\sin p$$

$$+\left(\frac{9h^2}{4}+\frac{27h^3}{16}+\frac{81h^4}{64}\right)\sin 3p+\left(\frac{25h^3}{16}+\frac{125h^4}{64}\right)\sin 5p$$

$$\left.+\frac{49h^4}{64}\sin 7p\right]\left(\frac{dp}{dh}\right)^3-3\left[\operatorname{cosec} p\cot p\right.$$

$$+\left(h+\frac{h^2}{4}+\frac{h^3}{8}+\frac{5h^4}{64}\right)\cos p+\left(\frac{3h^2}{4}+\frac{9h^3}{16}+\frac{27h^4}{64}\right)\cos 3p$$

$$\left.+\left(\frac{5h^3}{16}+\frac{25h^4}{64}\right)\cos 5p+\frac{7h^4}{64}\cos 7p\right]\frac{dp}{dh}\frac{d^2p}{dh^2}$$

$$-3\left[\left(1+\frac{h}{2}+\frac{3h^2}{8}+\frac{5h^3}{16}\right)\sin p+\left(\frac{h}{2}+\frac{9h^2}{16}+\frac{9h^3}{16}\right)\sin 3p\right.$$

$$\left.+\left(\frac{3h^2}{16}+\frac{5h^3}{16}\right)\sin 5p+\frac{h^3}{16}\sin 7p\right]\frac{d^2p}{dh^2}$$

$$-3\left[\left(1+\frac{h}{2}+\frac{3h^2}{8}+\frac{5h^3}{16}\right)\cos p+\left(\frac{3h}{2}+\frac{27h^2}{16}+\frac{27h^3}{16}\right)\cos 3p\right.$$

$$\left.+\left(\frac{15h^2}{16}+\frac{25h^3}{16}\right)\cos 5p+\frac{7h^3}{16}\cos 7p\right]\left(\frac{dp}{dh}\right)^2$$

$$-3\Big[\Big(\frac{1}{2}+\frac{3h}{4}+\frac{15h^2}{16}\Big)\sin p+\Big(\frac{1}{2}+\frac{9h}{8}+\frac{27h^2}{16}\Big)\sin 3p$$
$$+\Big(\frac{3h}{8}+\frac{15h^2}{16}\Big)\sin 5p+\frac{3h^2}{16}\sin 7p\Big]\frac{dp}{dh}+\Big(\frac{3}{4}+\frac{15h}{8}\Big)\cos p$$
$$+\Big(\frac{3}{8}+\frac{9h}{8}\Big)\cos 3p+\Big(\frac{3}{40}+\frac{3h}{8}\Big)\cos 5p+\frac{3h}{56}\cos 7p=0,$$

$$\Big[\text{cosec}\,p-\Big(h+\frac{h^2}{4}+\frac{h^3}{8}+\frac{5h^4}{64}\Big)\sin p-\Big(\frac{h^2}{4}+\frac{3h^3}{16}+\frac{9h^4}{64}\Big)\sin 3p$$
$$-\Big(\frac{h^3}{16}+\frac{5h^4}{64}\Big)\sin 5p-\frac{h^4}{64}\sin 7p\Big]\frac{d^4p}{dh^4}+\Big[-\text{cosec}\,p\cot^3 p$$
$$-5\,\text{cosec}^3 p\cot p+\Big(h+\frac{h^2}{4}+\frac{h^3}{8}+\frac{5h^4}{64}\Big)\cos p$$
$$+\Big(\frac{27h^2}{4}+\frac{81h^3}{16}+\frac{243h^4}{64}\Big)\cos 3p+\Big(\frac{125h^3}{16}+\frac{625h^4}{64}\Big)\cos 5p$$
$$+\frac{343h^4}{64}\cos 7p\Big]\Big(\frac{dp}{dh}\Big)^4-4\Big[\text{cosec}\,p\cot p$$
$$+\Big(h+\frac{h^2}{4}+\frac{h^3}{8}+\frac{5h^4}{64}\Big)\cos p+\Big(\frac{3h^2}{4}+\frac{9h^3}{16}+\frac{27h^4}{64}\Big)\cos 3p$$
$$+\Big(\frac{5h^3}{16}+\frac{25h^4}{64}\Big)\cos 5p+\frac{7h^4}{64}\cos 7p\Big]\frac{dp}{dh}\frac{d^3p}{dh^3}$$
$$-4\Big[\Big(1+\frac{h}{2}+\frac{3h^2}{8}+\frac{5h^3}{16}\Big)\sin p+\Big(\frac{h}{2}+\frac{9h^2}{16}+\frac{9h^3}{16}\Big)\sin 3p$$
$$+\Big(\frac{3h^2}{16}+\frac{5h^3}{16}\Big)\sin 5p+\frac{h^3}{16}\sin 7p\Big]\frac{d^3p}{dh^3}$$
$$+4\Big[\Big(1+\frac{h}{2}+\frac{3h^2}{8}+\frac{5h^3}{16}\Big)\sin p+\Big(\frac{9h}{2}+\frac{81h^2}{16}+\frac{81h^3}{16}\Big)\sin 3p$$
$$+\Big(\frac{75h^2}{16}+\frac{125h^3}{16}\Big)\sin 5p+\frac{49h^3}{16}\sin 7p\Big]\Big(\frac{dp}{dh}\Big)^3$$
$$+6\Big[\text{cosec}\,p\cot^2 p+\text{cosec}^3 p+\Big(h+\frac{h^2}{4}+\frac{h^3}{8}+\frac{5h^4}{64}\Big)\sin p$$
$$+\Big(\frac{9h^2}{4}+\frac{27h^3}{16}+\frac{81h^4}{64}\Big)\sin 3p+\Big(\frac{25h^3}{16}+\frac{125h^4}{64}\Big)\sin 5p$$
$$+\frac{49h^4}{64}\sin 7p\Big]\Big(\frac{dp}{dh}\Big)^2\frac{d^2p}{dh^2}-3\Big[\text{cosec}\,p\cot p$$
$$+\Big(h+\frac{h^2}{4}+\frac{h^3}{8}+\frac{5h^4}{64}\Big)\cos p+\Big(\frac{3h^2}{4}+\frac{9h^3}{16}+\frac{27h^4}{64}\Big)\cos 3p$$
$$+\Big(\frac{5h^3}{16}+\frac{25h^4}{64}\Big)\cos 5p+\frac{7h^4}{64}\cos 7p\Big]\Big(\frac{d^2p}{dh^2}\Big)^2$$
$$-12\Big[\Big(1+\frac{h}{2}+\frac{3h^2}{8}+\frac{5h^3}{16}\Big)\cos p+\Big(\frac{3h}{2}+\frac{27h^2}{16}+\frac{27h^3}{16}\Big)\cos 3p$$

$$+\left(\frac{15h^2}{16}+\frac{25h^3}{16}\right)\cos 5p+\frac{7h^3}{16}\cos 7p\Bigg]\frac{dp}{dh}\frac{d^2p}{dh^2}$$

$$-6\Bigg[\left(\frac{1}{2}+\frac{3h}{4}+\frac{15h^2}{16}\right)\sin p+\left(\frac{1}{2}+\frac{9h}{8}+\frac{27h^2}{16}\right)\sin 3p$$

$$+\left(\frac{3h}{8}+\frac{15h^2}{16}\right)\sin 5p+\frac{3h^2}{16}\sin 7p\Bigg]\frac{d^2p}{dh^2}$$

$$-6\Bigg[\left(\frac{1}{2}+\frac{3h}{4}+\frac{15h^2}{16}\right)\cos p+\left(\frac{3}{2}+\frac{27h}{8}+\frac{81h^2}{16}\right)\cos 3p$$

$$+\left(\frac{15h}{8}+\frac{75h^2}{16}\right)\cos 5p+\frac{21h^2}{16}\cos 7p\Bigg]\left(\frac{dp}{dh}\right)^2$$

$$-4\Bigg[\left(\frac{3}{4}+\frac{15h}{8}\right)\sin p+\left(\frac{9}{8}+\frac{27h}{8}\right)\sin 3p$$

$$+\left(\frac{3}{8}+\frac{15h}{8}\right)\sin 5p+\frac{3h}{8}\sin 7p\Bigg]\frac{dp}{dh}+\frac{15}{8}\cos p$$

$$+\frac{9}{8}\cos 3p+\frac{3}{8}\cos 5p+\frac{3}{56}\cos 7p=0.$$

Denoting by brackets the values of these derivatives for $h=0$, and remembering that functions of p become functions of z for $h=0$, we obtain in succession by substitution and reduction (for the necessary reductions see the reduction table, p. 88):

$$[p]=z,$$

$$\left[\frac{dp}{dh}\right]=-\sin z\cos z=-\frac{1}{2}\sin 2z,$$

$$\left[\frac{d^2p}{dh^2}\right]=-2\sin z\cos z+\frac{7}{3}\sin z\cos^3 z=-\frac{5}{12}\sin 2z+\frac{7}{24}\sin 4z,$$

$$\left[\frac{d^3p}{dh^3}\right]=-6\sin z\cos z+17\sin z\cos^3 z-\frac{56}{5}\sin z\cos^5 z$$

$$=-\frac{1}{2}\sin 2z+\frac{29}{40}\sin 4z-\frac{7}{20}\sin 6z,$$

$$\left[\frac{d^4p}{dh^4}\right]=-24\sin z\cos z+120\sin z\cos^3 z-\frac{889}{5}\sin z\cos^5 z$$

$$+\frac{8558}{105}\sin z\cos^7 z=-\frac{13}{15}\sin 2z+\frac{811}{480}\sin 4z$$

$$-\frac{243}{140}\sin 6z+\frac{4279}{6720}\sin 8z.$$

But from Maclauren's theorem, we have:

$$p=\left[p\right]+\frac{\epsilon^2}{1!}\left[\frac{dp}{dh}\right]+\frac{\epsilon^4}{2!}\left[\frac{d^2p}{dh^2}\right]+\frac{\epsilon^6}{3!}\left[\frac{d^3p}{dh^3}\right]+\frac{\epsilon^8}{4!}\left[\frac{d^4p}{dh^4}\right]+\cdots\cdots$$

Substituting the above values in this series and rearranging, we obtain the approximation sought:

$$p=z-\left(\frac{\epsilon^2}{2}+\frac{5\epsilon^4}{24}+\frac{\epsilon^6}{12}+\frac{13\epsilon^8}{360}\right)\sin 2z+\left(\frac{7\epsilon^4}{48}+\frac{29\epsilon^6}{240}+\frac{811\epsilon^8}{11520}\right)\sin 4z$$
$$-\left(\frac{7\epsilon^6}{120}+\frac{81\epsilon^8}{1120}\right)\sin 6z+\frac{4279\epsilon^8}{161280}\sin 8z,$$

or, in terms of φ and χ, we obtain, as before, the approximation:

$$\varphi-\chi=\left(\frac{\epsilon^2}{2}+\frac{5\epsilon^4}{24}+\frac{\epsilon^6}{12}+\frac{13\epsilon^8}{360}\right)\sin 2\chi+\left(\frac{7\epsilon^4}{48}+\frac{29\epsilon^6}{240}+\frac{811\epsilon^8}{11520}\right)\sin 4\chi$$
$$+\left(\frac{7\epsilon^6}{120}+\frac{81\epsilon^8}{1120}\right)\sin 6\chi+\frac{4279\epsilon^8}{161280}\sin 8\chi.$$

DEVELOPMENT OF $\varphi-\chi$ IN TERMS OF χ—FIFTH METHOD.

If we wish to develop $\varphi-\chi$ in terms of χ, starting with the expression:

$$\tan\frac{z}{2}=\tan\frac{p}{2}\Bigg[1+h\cos p+h^2\left(\frac{\cos^2 p}{2}+\frac{\cos^3 p}{3}\right)+h^3\bigg(\frac{\cos^3 p}{6}$$
$$+\frac{\cos^4 p}{3}+\frac{\cos^5 p}{5}\bigg)+h^4\left(\frac{\cos^4 p}{24}+\frac{\cos^5 p}{6}+\frac{23\cos^6 p}{90}+\frac{\cos^7 p}{7}\right)\Bigg]$$

it is more convenient to make the substitution

$$\tan\frac{p}{2}=\frac{\sin p}{1+\cos p}$$

and write it in the form (for the reductions, see table, p. 88.)

$$-(1+\cos p)\tan\frac{z}{2}+\left(1+\frac{h^2}{8}+\frac{h^3}{24}+\frac{29h^4}{1152}\right)\sin p$$
$$+\left(\frac{h}{2}+\frac{h^2}{12}+\frac{7h^3}{96}+\frac{h^4}{24}\right)\sin 2p+\left(\frac{h^2}{8}+\frac{h^3}{16}+\frac{7h^4}{160}\right)\sin 3p$$
$$+\left(\frac{h^2}{24}+\frac{11h^3}{240}+\frac{7h^4}{192}\right)\sin 4p+\left(\frac{h^3}{48}+\frac{13h^4}{576}\right)\sin 5p$$
$$+\left(\frac{h^3}{160}+\frac{h^4}{84}\right)\sin 6p+\frac{23h^4}{5760}\sin 7p+\frac{h^4}{896}\sin 8p=0,$$

We shall now differentiate this expression considering p as a function of h or of ϵ^2.

$$\Big[\sin p \tan \frac{z}{2}+\left(1+\frac{h^2}{8}+\frac{h^3}{24}+\frac{29h^4}{1152}\right)\cos p$$
$$+\left(h+\frac{h^2}{6}+\frac{7h^3}{48}+\frac{h^4}{12}\right)\cos 2p+\left(\frac{3h^2}{8}+\frac{3h^3}{16}+\frac{21h^4}{160}\right)\cos 3p$$
$$+\left(\frac{h^2}{6}+\frac{11h^3}{60}+\frac{7h^4}{48}\right)\cos 4p+\left(\frac{5h^3}{48}+\frac{65h^4}{576}\right)\cos 5p$$
$$+\left(\frac{3h^3}{80}+\frac{h^4}{14}\right)\cos 6p+\frac{161h^4}{5760}\cos 7p+\frac{h^4}{112}\cos 8p\Big]\frac{dp}{dh}$$
$$+\left(\frac{h}{4}+\frac{h^2}{8}+\frac{29h^3}{288}\right)\sin p+\left(\frac{1}{2}+\frac{h}{6}+\frac{7h^2}{32}+\frac{h^3}{6}\right)\sin 2p$$
$$+\left(\frac{h}{4}+\frac{3h^2}{16}+\frac{7h^3}{40}\right)\sin 3p+\left(\frac{h}{12}+\frac{11h^2}{80}+\frac{7h^3}{48}\right)\sin 4p$$
$$+\left(\frac{h^2}{16}+\frac{13h^3}{144}\right)\sin 5p+\left(\frac{3h^2}{160}+\frac{h^3}{21}\right)\sin 6p+\frac{23h^3}{1440}\sin 7p$$
$$+\frac{h^3}{224}\sin 8p=0,$$

$$\Big[\sin p \tan \frac{z}{2}+\left(1+\frac{h^2}{8}+\frac{h^3}{24}+\frac{29h^4}{1152}\right)\cos p$$
$$+\left(h+\frac{h^2}{6}+\frac{7h^3}{48}+\frac{h^4}{12}\right)\cos 2p+\left(\frac{3h^2}{8}+\frac{3h^3}{16}+\frac{21h^4}{160}\right)\cos 3p$$
$$+\left(\frac{h^2}{6}+\frac{11h^3}{60}+\frac{7h^4}{48}\right)\cos 4p+\left(\frac{5h^3}{48}+\frac{65h^4}{576}\right)\cos 5p$$
$$+\left(\frac{3h^3}{80}+\frac{h^4}{14}\right)\cos 6p+\frac{161h^4}{5760}\cos 7p$$
$$+\frac{h^4}{112}\cos 8p\Big]\frac{d^2p}{dh^2}+\Big[\cos p \tan \frac{z}{2}$$
$$-\left(1+\frac{h^2}{8}+\frac{h^3}{24}+\frac{29h^4}{1152}\right)\sin p-\left(2h+\frac{h^2}{3}+\frac{7h^3}{24}+\frac{h^4}{6}\right)\sin 2p$$
$$-\left(\frac{9h^2}{8}+\frac{9h^3}{16}+\frac{63h^4}{160}\right)\sin 3p-\left(\frac{2h^2}{3}+\frac{11h^3}{15}+\frac{7h^4}{12}\right)\sin 4p$$
$$-\left(\frac{25h^3}{48}+\frac{325h^4}{576}\right)\sin 5p-\left(\frac{9h^3}{40}+\frac{3h^4}{7}\right)\sin 6p$$
$$-\frac{1127h^4}{5760}\sin 7p-\frac{h^4}{14}\sin 8p\Big]\left(\frac{dp}{dh}\right)^2$$
$$+2\Big[\left(\frac{h}{4}+\frac{h^2}{8}+\frac{29h^3}{288}\right)\cos p+\left(1+\frac{h}{3}+\frac{7h^2}{16}+\frac{h^3}{3}\right)\cos 2p$$

$$+\left(\frac{3h}{4}+\frac{9h^2}{16}+\frac{21h^3}{40}\right)\cos 3p+\left(\frac{h}{3}+\frac{11h^2}{20}+\frac{7h^3}{12}\right)\cos 4p$$

$$+\left(\frac{5h^2}{16}+\frac{65h^3}{144}\right)\cos 5p+\left(\frac{9h^2}{80}+\frac{2h^3}{7}\right)\cos 6p$$

$$+\frac{161h^3}{1440}\cos 7p+\frac{h^3}{28}\cos 8p\Big]\frac{dp}{dh}+\left(\frac{1}{4}+\frac{h}{4}+\frac{29h^2}{96}\right)\sin p$$

$$+\left(\frac{1}{6}+\frac{7h}{16}+\frac{h^2}{2}\right)\sin 2p+\left(\frac{1}{4}+\frac{3h}{8}+\frac{21h^2}{40}\right)\sin 3p$$

$$+\left(\frac{1}{12}+\frac{11h}{40}+\frac{7h^2}{16}\right)\sin 4p+\left(\frac{h}{8}+\frac{13h^2}{48}\right)\sin 5p$$

$$+\left(\frac{3h}{80}+\frac{h^2}{7}\right)\sin 6p+\frac{23h^2}{480}\sin 7p+\frac{3h^2}{224}\sin 8p=0,$$

$$\Big[\sin p\tan\frac{z}{2}+\left(1+\frac{h^2}{8}+\frac{h^3}{24}+\frac{29h^4}{1152}\right)\cos p$$

$$+\left(h+\frac{h^2}{6}+\frac{7h^3}{48}+\frac{h^4}{12}\right)\cos 2p+\left(\frac{3h^2}{8}+\frac{3h^3}{16}+\frac{21h^4}{160}\right)\cos 3p$$

$$+\left(\frac{h^2}{6}+\frac{11h^3}{60}+\frac{7h^4}{48}\right)\cos 4p+\left(\frac{5h^3}{48}+\frac{65h^4}{576}\right)\cos 5p$$

$$+\left(\frac{3h^3}{80}+\frac{h^4}{14}\right)\cos 6p+\frac{161h^4}{5760}\cos 7p+\frac{h^4}{112}\cos 8p\Big]\frac{d^3p}{dh^3}$$

$$+\Big[-\sin p\tan\frac{z}{2}-\left(1+\frac{h^2}{8}+\frac{h^3}{24}+\frac{29h^4}{1152}\right)\cos p$$

$$-\left(4h+\frac{2h^2}{3}+\frac{7h^3}{12}+\frac{h^4}{3}\right)\cos 2p-\left(\frac{27h^2}{8}+\frac{27h^3}{16}+\frac{189h^4}{160}\right)\cos 3p$$

$$-\left(\frac{8h^2}{3}+\frac{44h^3}{15}+\frac{7h^4}{3}\right)\cos 4p-\left(\frac{125h^3}{48}+\frac{1625h^4}{576}\right)\cos 5p$$

$$-\left(\frac{27h^3}{20}+\frac{18h^4}{7}\right)\cos 6p-\frac{7889h^4}{5760}\cos 7p$$

$$-\frac{4h^4}{7}\cos 8p\Big]\left(\frac{dp}{dh}\right)^3+3\Big[\cos p\tan\frac{z}{2}$$

$$-\left(1+\frac{h^2}{8}+\frac{h^3}{24}+\frac{29h^4}{1152}\right)\sin p-\left(2h+\frac{h^2}{3}+\frac{7h^3}{24}+\frac{h^4}{6}\right)\sin 2p$$

$$-\left(\frac{9h^2}{8}+\frac{9h^3}{16}+\frac{63h^4}{160}\right)\sin 3p-\left(\frac{2h^2}{3}+\frac{11h^3}{15}+\frac{7h^4}{12}\right)\sin 4p$$

$$-\left(\frac{25h^3}{48}+\frac{325h^4}{576}\right)\sin 5p-\left(\frac{9h^3}{40}+\frac{3h^4}{7}\right)\sin 6p$$

$$-\frac{1127h^4}{5760}\sin 7p-\frac{h^4}{14}\sin 8p\Big]\frac{dp}{dh}\frac{d^2p}{dh^2}$$

$$+3\Big[\Big(\frac{h}{4}+\frac{h^2}{8}+\frac{29h^3}{288}\Big)\cos p+\Big(1+\frac{h}{3}+\frac{7h^2}{16}+\frac{h^3}{3}\Big)\cos 2p$$

$$+\Big(\frac{3h}{4}+\frac{9h^2}{16}+\frac{21h^3}{40}\Big)\cos 3p+\Big(\frac{h}{3}+\frac{11h^2}{20}+\frac{7h^3}{12}\Big)\cos 4p$$

$$+\Big(\frac{5h^2}{16}+\frac{65h^3}{144}\Big)\cos 5p+\Big(\frac{9h^2}{80}+\frac{2h^3}{7}\Big)\cos 6p$$

$$+\frac{161h^3}{1440}\cos 7p+\frac{h^3}{28}\cos 8p\Big]\frac{d^2p}{dh^2}+3\Big[-\Big(\frac{h}{4}+\frac{h^2}{8}+\frac{29h^3}{288}\Big)\sin p$$

$$-\Big(2+\frac{2h}{3}+\frac{7h^2}{8}+\frac{2h^3}{3}\Big)\sin 2p-\Big(\frac{9h}{4}+\frac{27h^2}{16}+\frac{63h^3}{40}\Big)\sin 3p$$

$$-\Big(\frac{4h}{3}+\frac{11h^2}{5}+\frac{7h^3}{3}\Big)\sin 4p-\Big(\frac{25h^2}{16}+\frac{325h^3}{144}\Big)\sin 5p$$

$$-\Big(\frac{27h^2}{40}+\frac{12h^3}{7}\Big)\sin 6p-\frac{1127h^3}{1440}\sin 7p$$

$$-\frac{2h^3}{7}\sin 8p\Big]\Big(\frac{dp}{dh}\Big)^2+3\Big[\Big(\frac{1}{4}+\frac{h}{4}+\frac{29h^2}{96}\Big)\cos p$$

$$+\Big(\frac{1}{3}+\frac{7h}{8}+h^2\Big)\cos 2p+\Big(\frac{3}{4}+\frac{9h}{8}+\frac{63h^2}{40}\Big)\cos 3p$$

$$+\Big(\frac{1}{3}+\frac{11h}{10}+\frac{7h^2}{4}\Big)\cos 4p+\Big(\frac{5h}{8}+\frac{65h^2}{48}\Big)\cos 5p$$

$$+\Big(\frac{9h}{40}+\frac{6h^2}{7}\Big)\cos 6p+\frac{161h^2}{480}\cos 7p$$

$$+\frac{3h^2}{28}\cos 8p\Big]\frac{dp}{dh}+\Big(\frac{1}{4}+\frac{29h}{48}\Big)\sin p+\Big(\frac{7}{16}+h\Big)\sin 2p$$

$$+\Big(\frac{3}{8}+\frac{21h}{20}\Big)\sin 3p+\Big(\frac{11}{40}+\frac{7h}{8}\Big)\sin 4p$$

$$+\Big(\frac{1}{8}+\frac{13h}{24}\Big)\sin 5p+\Big(\frac{3}{80}+\frac{2h}{7}\Big)\sin 6p+\frac{23h}{240}\sin 7p$$

$$+\frac{3h}{112}\sin 8p=0,$$

$$\Big[\sin p\tan\frac{z}{2}+\Big(1+\frac{h^2}{8}+\frac{h^3}{24}+\frac{29h^4}{1152}\Big)\cos p$$

$$+\Big(h+\frac{h^2}{6}+\frac{7h^3}{48}+\frac{h^4}{12}\Big)\cos 2p+\Big(\frac{3h^2}{8}+\frac{3h^3}{16}+\frac{21h^4}{160}\Big)\cos 3p$$

$$+\Big(\frac{h^2}{6}+\frac{11h^3}{60}+\frac{7h^4}{48}\Big)\cos 4p+\Big(\frac{5h^3}{48}+\frac{65h^4}{576}\Big)\cos 5p$$

$$+\Big(\frac{3h^3}{80}+\frac{h^4}{14}\Big)\cos 6p+\frac{161h^4}{5760}\cos 7p+\frac{h^4}{112}\cos 8p\Big]\frac{d^4p}{dh^4}$$

$$+\Big[-\cos p\tan\frac{z}{2}+\Big(1+\frac{h^2}{8}+\frac{h^3}{24}+\frac{29h^4}{1152}\Big)\sin p$$

$$+\left(8h+\frac{4h^2}{3}+\frac{7h^3}{6}+\frac{2h^4}{3}\right)\sin 2p$$

$$+\left(\frac{81h^2}{8}+\frac{81h^3}{16}+\frac{567h^4}{160}\right)\sin 3p$$

$$+\left(\frac{32h^2}{3}+\frac{176h^3}{15}+\frac{28h^4}{3}\right)\sin 4p$$

$$+\left(\frac{625h^3}{48}+\frac{8125h^4}{576}\right)\sin 5p+\left(\frac{81h^3}{10}+\frac{108h^4}{7}\right)\sin 6p$$

$$+\frac{55223h^4}{5760}\sin 7p+\frac{32h^4}{7}\sin 8p\Bigg]\left(\frac{dp}{dh}\right)^4$$

$$+4\Bigg[\cos p\tan\frac{z}{2}-\left(1+\frac{h^2}{8}+\frac{h^3}{24}+\frac{29h^4}{1152}\right)\sin p$$

$$-\left(2h+\frac{h^2}{3}+\frac{7h^3}{24}+\frac{h^4}{6}\right)\sin 2p-\left(\frac{9h^2}{8}+\frac{9h^3}{16}+\frac{63h^4}{160}\right)\sin 3p$$

$$-\left(\frac{2h^2}{3}+\frac{11h^3}{15}+\frac{7h^4}{12}\right)\sin 4p-\left(\frac{25h^3}{48}+\frac{325h^4}{576}\right)\sin 5p$$

$$-\left(\frac{9h^3}{40}+\frac{3h^4}{7}\right)\sin 6p-\frac{1127h^4}{5760}\sin 7p$$

$$-\frac{h^4}{14}\sin 8p\Bigg]\frac{dp}{dh}\frac{d^3p}{dh^3}+4\Bigg[\left(\frac{h}{4}+\frac{h^2}{8}+\frac{29h^3}{288}\right)\cos p$$

$$+\left(1+\frac{h}{3}+\frac{7h^2}{16}+\frac{h^3}{3}\right)\cos 2p+\left(\frac{3h}{4}+\frac{9h^2}{16}+\frac{21h^3}{40}\right)\cos 3p$$

$$+\left(\frac{h}{3}+\frac{11h^2}{20}+\frac{7h^3}{12}\right)\cos 4p+\left(\frac{5h^2}{16}+\frac{65h^3}{144}\right)\cos 5p$$

$$+\left(\frac{9h^2}{80}+\frac{2h^3}{7}\right)\cos 6p+\frac{161h^3}{1440}\cos 7p$$

$$+\frac{h^3}{28}\cos 8p\Bigg]\frac{d^3p}{dh^3}+4\Bigg[-\left(\frac{h}{4}+\frac{h^2}{8}+\frac{29h^3}{288}\right)\cos p$$

$$-\left(4+\frac{4h}{3}+\frac{7h^2}{4}+\frac{4h^3}{3}\right)\cos 2p-\left(\frac{27h}{4}+\frac{81h^2}{16}+\frac{189h^3}{40}\right)\cos 3p$$

$$-\left(\frac{16h}{3}+\frac{44h^2}{5}+\frac{28h^3}{3}\right)\cos 4p-\left(\frac{125h^2}{16}+\frac{1625h^3}{144}\right)\cos 5p$$

$$-\left(\frac{81h^2}{20}+\frac{72h^3}{7}\right)\cos 6p-\frac{7889h^3}{1440}\cos 7p$$

$$-\frac{16h^3}{7}\cos 8p\Bigg]\left(\frac{dp}{dh}\right)^3+6\Bigg[-\sin p\tan\frac{z}{2}$$

$$-\left(1+\frac{h^2}{8}+\frac{h^3}{24}+\frac{29h^4}{1152}\right)\cos p-\left(4h+\frac{2h^2}{3}+\frac{7h^3}{12}+\frac{h^4}{3}\right)\cos 2p$$

$$-\left(\frac{27h^2}{8}+\frac{27h^3}{16}+\frac{189h^4}{160}\right)\cos 3p-\left(\frac{8h^2}{3}+\frac{44h^3}{15}+\frac{7h^4}{3}\right)\cos 4p$$

$$-\left(\frac{125h^3}{48}+\frac{1625h^4}{576}\right)\cos 5p-\left(\frac{27h^3}{20}+\frac{18h^4}{7}\right)\cos 6p$$

$$-\frac{7889h^4}{5760}\cos 7p-\frac{4h^4}{7}\cos 8p\Bigg]\left(\frac{dp}{dh}\right)^2\frac{d^2p}{dh^2}$$

$$+3\Bigg[\cos p\tan\frac{z}{2}-\left(1+\frac{h^2}{8}+\frac{h^3}{24}+\frac{29h^4}{1152}\right)\sin p$$

$$-\left(2h+\frac{h^2}{3}+\frac{7h^3}{24}+\frac{h^4}{6}\right)\sin 2p-\left(\frac{9h^2}{8}+\frac{9h^3}{16}+\frac{63h^4}{160}\right)\sin 3p$$

$$-\left(\frac{2h^2}{3}+\frac{11h^3}{15}+\frac{7h^4}{12}\right)\sin 4p-\left(\frac{25h^3}{48}+\frac{325h^4}{576}\right)\sin 5p$$

$$-\left(\frac{9h^3}{40}+\frac{3h^4}{7}\right)\sin 6p-\frac{1127h^4}{5760}\sin 7p$$

$$-\frac{h^4}{14}\sin 8p\Bigg]\left(\frac{d^2p}{dh^2}\right)^2+12\Bigg[-\left(\frac{h}{4}+\frac{h^2}{8}+\frac{29h^3}{288}\right)\sin p$$

$$-\left(2+\frac{2h}{3}+\frac{7h^2}{8}+\frac{2h^3}{3}\right)\sin 2p-\left(\frac{9h}{4}+\frac{27h^2}{16}+\frac{63h^3}{40}\right)\sin 3p$$

$$-\left(\frac{4h}{3}+\frac{11h^2}{5}+\frac{7h^3}{3}\right)\sin 4p-\left(\frac{25h^2}{16}+\frac{325h^3}{144}\right)\sin 5p$$

$$-\left(\frac{27h^2}{40}+\frac{12h^3}{7}\right)\sin 6p-\frac{1127h^3}{1440}\sin 7p$$

$$-\frac{2h^3}{7}\sin 8p\Bigg]\frac{dp}{dh}\frac{d^2p}{dh^2}+6\Bigg[\left(\frac{1}{4}+\frac{h}{4}+\frac{29h^2}{96}\right)\cos p$$

$$+\left(\frac{1}{3}+\frac{7h}{8}+h^2\right)\cos 2p+\left(\frac{3}{4}+\frac{9h}{8}+\frac{63h^2}{40}\right)\cos 3p$$

$$+\left(\frac{1}{3}+\frac{11h}{10}+\frac{7h^2}{4}\right)\cos 4p+\left(\frac{5h}{8}+\frac{65h^2}{48}\right)\cos 5p$$

$$+\left(\frac{9h}{40}+\frac{6h^2}{7}\right)\cos 6p+\frac{161h^2}{480}\cos 7p+\frac{3h^2}{28}\cos 8p\Bigg]\frac{d^2p}{dh^2}$$

$$+6\Bigg[-\left(\frac{1}{4}+\frac{h}{4}+\frac{29h^2}{96}\right)\sin p-\left(\frac{2}{3}+\frac{7h}{4}+2h^2\right)\sin 2p$$

$$-\left(\frac{9}{4}+\frac{27h}{8}+\frac{189h^2}{40}\right)\sin 3p-\left(\frac{4}{3}+\frac{22h}{5}+7h^2\right)\sin 4p$$

$$-\left(\frac{25h}{8}+\frac{325h^2}{48}\right)\sin 5p-\left(\frac{27h}{20}+\frac{36h^2}{7}\right)\sin 6p$$

$$-\frac{1127h^2}{480}\sin 7p-\frac{6h^2}{7}\sin 8p\Bigg]\left(\frac{dp}{dh}\right)^2$$

$$+4\Bigg[\left(\frac{1}{4}+\frac{29h}{48}\right)\cos p+\left(\frac{7}{8}+2h\right)\cos 2p$$

$$+\left(\frac{9}{8}+\frac{63h}{20}\right)\cos 3p+\left(\frac{11}{10}+\frac{7h}{2}\right)\cos 4p$$

$$+\left(\frac{5}{8}+\frac{65h}{24}\right)\cos 5p+\left(\frac{9}{40}+\frac{12h}{7}\right)\cos 6p$$

$$+\frac{161h}{240}\cos 7p+\frac{3h}{14}\cos 8p\Big]\frac{dp}{dh}+\frac{29}{48}\sin p+\sin 2p$$

$$+\frac{21}{20}\sin 3p+\frac{7}{8}\sin 4p+\frac{13}{24}\sin 5p+\frac{2}{7}\sin 6p$$

$$+\frac{23}{240}\sin 7p+\frac{3}{112}\sin 8p=0.$$

Evaluating these derivatives for $h=0$, remembering that functions of p become functions of z for $h=0$, we get by successive substitution and reduction (for the necessary reductions see the reduction table, p. 88):

$$[p]=z,$$

$$\left[\frac{dp}{dh}\right]=-\sin z\cos z=-\frac{1}{2}\sin 2z,$$

$$\left[\frac{d^2p}{dh^2}\right]=-2\sin z\cos z+\frac{7}{3}\sin z\cos^3 z=-\frac{5}{12}\sin 2z+\frac{7}{24}\sin 4z,$$

$$\left[\frac{d^3p}{dh^3}\right]=-6\sin z\cos z+17\sin z\cos^3 z-\frac{56}{5}\sin z\cos^5 z$$

$$=-\frac{1}{2}\sin 2z+\frac{29}{40}\sin 4z-\frac{7}{20}\sin 6z,$$

$$\left[\frac{d^4p}{dh^4}\right]=-24\sin z\cos z+120\sin z\cos^3 z-\frac{889}{5}\sin z\cos^5 z$$

$$+\frac{8558}{105}\sin z\cos^7 z$$

$$=-\frac{13}{15}\sin 2z+\frac{811}{480}\sin 4z-\frac{243}{140}\sin 6z+\frac{4279}{6720}\sin 8z.$$

When these values are substituted in the Maclaurin development (see p. 45) and rearranged, we obtain, as before, the desired approximation:

$$z=p+\left(\frac{\epsilon^2}{2}+\frac{5\epsilon^4}{24}+\frac{\epsilon^6}{12}+\frac{13\epsilon^8}{360}\right)\sin 2z-\left(\frac{7\epsilon^4}{48}+\frac{29\epsilon^6}{240}+\frac{811\epsilon^8}{11520}\right)\sin 4z$$

$$+\left(\frac{7\epsilon^6}{120}+\frac{81\epsilon^8}{1120}\right)\sin 6z-\frac{4279\epsilon^8}{161280}\sin 8z,$$

or, in terms of φ and χ, the approximation:

$$\varphi-\chi=\left(\frac{\epsilon^2}{2}+\frac{5\epsilon^4}{24}+\frac{\epsilon^6}{12}+\frac{13\epsilon^8}{360}\right)\sin 2\chi+\left(\frac{7\epsilon^4}{48}+\frac{29\epsilon^6}{240}+\frac{811\epsilon^8}{11520}\right)\sin 4\chi$$
$$+\left(\frac{7\epsilon^6}{120}+\frac{81\epsilon^8}{1120}\right)\sin 6\chi+\frac{4279\epsilon^8}{161280}\sin 8\chi.$$

DEVELOPMENT OF $\varphi-\chi$ IN TERMS OF χ—SIXTH METHOD.

$\varphi-\chi$ can be developed in terms of χ directly from the equation of definition by the application of Lagrange's theorem. Let us take the form:

$$\log_e \tan\frac{p}{2}=\log_e \tan\frac{z}{2}-\epsilon^2\cos p-\frac{\epsilon^4}{3}\cos^3 p-\frac{\epsilon^6}{5}\cos^5 p$$
$$-\frac{\epsilon^8}{7}\cos^7 p-\cdots\cdot$$

In this expression, let

$$x=\log_e \tan\frac{p}{2}$$

$$y=\log_e \tan\frac{z}{2},$$

and it becomes:

$$x=y-\epsilon^2\cos p-\frac{\epsilon^4}{3}\cos^3 p-\frac{\epsilon^6}{5}\cos^5 p-\frac{\epsilon^8}{7}\cos^7 p-\cdots\cdot$$

The series in ϵ^2 is a function of x, since p is a function of x. We might replace $\cos p$ by its value in terms of x, but this is not necessary. The problem in hand is to develop the function $2\tan^{-1}e^x$ in terms of y or in terms of z through the functional relation between y and z. By Lagrange's theorem, since the series in ϵ^2 is a small quantity, the development may be expressed in general terms as follows:

$$f(x)=f(y)+\frac{1}{1!}g(y)f'(y)+\frac{1}{2!}\frac{d}{dy}\{[g(y)]^2f'(y)\}$$
$$+\frac{1}{3!}\frac{d^2}{dy^2}\{[g(y)]^3f'(y)\}+\frac{1}{4!}\frac{d^3}{dy^3}\{[g(y)]^4f'(y)\}+\cdots\cdot\cdot$$

in which $f(x)$ denotes the function of x to be developed and $g(y)$ denotes the series in ϵ^2 with z replacing p. The prime denotes differentiation with respect to y.

$$f(y)=2\tan^{-1}e^y$$

$$f'(y)=\frac{2e^y}{1+e^{2y}}=\frac{2\tan\frac{z}{2}}{\sec^2\frac{z}{2}}=2\sin\frac{z}{2}\cos\frac{z}{2}=\sin z.$$

Retaining all powers of ϵ up to the eighth inclusive, we get:

$$g\ (y) = -\left(\epsilon^2 \cos z + \frac{\epsilon^4}{3} \cos^3 z + \frac{\epsilon^6}{5} \cos^5 z + \frac{\epsilon^8}{7} \cos^7 z\right)$$

$$g\ (y)\, f'\ (y) = -\left(\epsilon^2 \sin z \cos z + \frac{\epsilon^4}{3} \sin z \cos^3 z + \frac{\epsilon^6}{5} \sin z \cos^5 z + \frac{\epsilon^8}{7} \sin z \cos^7 z\right)$$

$$[g\ (y)]^2 = \epsilon^4 \cos^2 z + \frac{2\epsilon^6}{3} \cos^4 z + \frac{23\epsilon^8}{45} \cos^6 z$$

$$[g(y)]^2 f'\ (y) = \epsilon^4 \sin z \cos^2 z + \frac{2\epsilon^6}{3} \sin z \cos^4 z + \frac{23\epsilon^8}{45} \sin z \cos^4 z$$

$$[g(y)]^3 = -\epsilon^6 \cos^3 z - \epsilon^8 \cos^5 z,$$

$$[g(y)]^3 f'\ (y) = -\epsilon^6 \sin z \cos^3 z - \epsilon^8 \sin z \cos^5 z,$$

$$[g(y)]^4 = \epsilon^8 \cos^4 z,$$

$$[g(y)]^4 f'\ (y) = \epsilon^8 \sin z \cos^4 z.$$

To differentiate these expressions with respect to y, we may differentiate with respect to z and multiply by $\frac{dz}{dy}$ or $\sin z$, since $f\ (y)$ is equal to z. For successive differentiations, we differentiate with respect to z and multiply by $\sin z$; then differentiate with respect to z again and multiply by $\sin z$ again, and so on for the remaining differentiations.

With this understanding, we get by differentiation and reduction:

$$\frac{d}{dy}\{[g(y)]^2 f'(y)\} = \epsilon^4 \sin z \cos^3 z - 2\epsilon^4 \sin^3 z \cos z + \frac{2\epsilon^6}{3} \sin z \cos^5 z$$
$$- \frac{8\epsilon^6}{3} \sin^3 z \cos^3 z + \frac{23\epsilon^8}{45} \sin z \cos^7 z - \frac{46\epsilon^8}{15} \sin^3 z \cos^5 z$$
$$= -2\epsilon^4 \sin z \cos z + \left(3\epsilon^4 - \frac{8\epsilon^6}{3}\right) \sin z \cos^3 z$$
$$+ \left(\frac{10\epsilon^6}{3} - \frac{46\epsilon^8}{15}\right) \sin z \cos^5 z + \frac{161\epsilon^8}{45} \sin z \cos^7 z,$$

$$\frac{d^2}{dy^2}\{[g(y)]^3 f'(y)\} = -\epsilon^6 \sin z \cos^5 z + 4\epsilon^6 \sin^3 z \cos^3 z$$
$$+ 9\epsilon^6 \sin^3 z \cos^3 z - 6\epsilon^6 \sin^5 z \cos z - \epsilon^8 \sin z \cos^7 z$$
$$+ 6\epsilon^8 \sin^3 z \cos^5 z + 15\epsilon^8 \sin^3 z \cos^5 z - 20\epsilon^8 \sin^5 z \cos^3 z$$
$$= -6\epsilon^6 \sin z \cos z + (25\epsilon^6 - 20\epsilon^8) \sin z \cos^3 z - (20\epsilon^6 - 61\epsilon^8)$$
$$\sin z \cos^5 z - 42\epsilon^8 \sin z \cos^7 z,$$

$$\frac{d^3}{dy^3}\{[g(y)]^4 f'(y)\} = \epsilon^8 \sin z \cos^7 z - 6\epsilon^8 \sin^3 z \cos^5 z$$
$$- 51\epsilon^8 \sin^3 z \cos^5 z + 68\epsilon^8 \sin^5 z \cos^3 z + 60\epsilon^8 \sin^5 z \cos^3 z$$
$$- 24\epsilon^8 \sin^7 z \cos z = -24\epsilon^8 \sin z \cos z + 200\epsilon^8 \sin z \cos^3 z$$
$$- 385\epsilon^8 \sin z \cos^5 z + 210\epsilon^8 \sin z \cos^7 z.$$

Substituting these values in Lagrange's development, we get

$$p = z - \epsilon^2 \sin z \cos z - \frac{\epsilon^4}{3} \sin z \cos^3 z - \frac{\epsilon^6}{5} \sin z \cos^5 z - \frac{\epsilon^8}{7} \sin z \cos^7 z$$
$$- \epsilon^4 \sin z \cos z + \left(\frac{3\epsilon^4}{2} - \frac{4\epsilon^6}{3}\right) \sin z \cos^3 z$$
$$+ \left(\frac{5\epsilon^6}{3} - \frac{23\epsilon^8}{15}\right) \sin z \cos^5 z + \frac{161\epsilon^8}{90} \sin z \cos^7 z - \epsilon^6 \sin z \cos z$$
$$+ \left(\frac{25\epsilon^6}{6} - \frac{10\epsilon^8}{3}\right) \sin z \cos^3 z - \left(\frac{10\epsilon^6}{3} - \frac{61\epsilon^8}{6}\right) \sin z \cos^5 z$$
$$- 7\epsilon^8 \sin z \cos^7 z - \epsilon^8 \sin z \cos z + \frac{25\epsilon^8}{3} \sin z \cos^3 z$$
$$- \frac{385\epsilon^8}{24} \sin z \cos^5 z + \frac{35\epsilon^8}{4} \sin z \cos^7 z.$$

By collecting like terms, this becomes

$$p = z - (\epsilon^2 + \epsilon^4 + \epsilon^6 + \epsilon^8) \sin z \cos z + \left(\frac{7\epsilon^4}{6} + \frac{17\epsilon^6}{6} + 5\epsilon^8\right) \sin z \cos^3 z$$
$$- \left(\frac{28\epsilon^6}{15} + \frac{889\epsilon^8}{120}\right) \sin z \cos^5 z + \frac{4279\epsilon^8}{1260} \sin z \cos^7 z.$$

Substituting the reductions from the table on page 88 and rearranging, we obtain the desired approximation

$$p = z - \left(\frac{\epsilon^2}{2} + \frac{5\epsilon^4}{24} + \frac{\epsilon^6}{12} + \frac{13\epsilon^8}{360}\right) \sin 2z + \left(\frac{7\epsilon^4}{48} + \frac{29\epsilon^6}{240} + \frac{811\epsilon^8}{11520}\right) \sin 4z$$
$$- \left(\frac{7\epsilon^6}{120} + \frac{81\epsilon^8}{1120}\right) \sin 6z + \frac{4279\epsilon^8}{161280} \sin 8z.$$

In terms of φ and χ this becomes, as before, the approximation

$$\varphi-\chi=\left(\frac{\epsilon^2}{2}+\frac{5\epsilon^4}{24}+\frac{\epsilon^6}{12}+\frac{13\epsilon^8}{360}\right)\sin 2\chi+\left(\frac{7\epsilon^4}{48}+\frac{29\epsilon^6}{240}+\frac{811\epsilon^8}{11520}\right)\sin 4\chi$$
$$\left(\frac{7\epsilon^6}{120}+\frac{81\epsilon^8}{1120}\right)\sin 6\chi+\frac{4279\epsilon^8}{161280}\sin 8\chi.$$

DEVELOPMENT OF $\varphi-\chi$ IN TERMS OF χ—SEVENTH METHOD.

$\varphi-\chi$ can be developed in terms of χ by the application of Arbogast's rule (see p. 28). The function to be developed in this case is

$$p=2\tan^{-1}\exp^*\left(\log_e\tan\frac{z}{2}-\frac{h}{1!}\cos p-\frac{h^2}{2!}\frac{2}{3}\cos^3 p\right.$$
$$\left.-\frac{h^3}{3!}\frac{6}{5}\cos^5 p-\frac{h^4}{4!}\frac{24}{7}\cos^7 p-\cdots\right)=A_0+A_1\frac{h}{1!}$$
$$+A_2\frac{h^2}{2!}+A_3\frac{h^3}{3!}+A_4\frac{h^4}{4!}+\cdots$$

The A's are computed as before in the forms

$$A_0=f(a_0),$$
$$A_1=a_1 f^1(a_0),$$
$$A_2=a_1^2 f^2(a_0)+a_2 f^1(a_0),$$
$$A_3=a_1^3 f^3(a_0)+3a_1 a_2 f^2(a_0)+a_3 f^1(a_0),$$
$$A_4=a_1^4 f^4(a_0)+6a_1^2 a_2 f^3(a_0)+4a_1 a_3 f^2(a_0)$$
$$+3a_2^2 f^2(a_0)+a_4 f^1(a_0),$$

etc.

(For the necessary reductions in the following work consult the table on p. 88.)

$$f(a_0)=z,$$
$$f^1(a_0)=\frac{dz}{da_0};$$

but

$$a_0=\log_e\tan\frac{z}{2},$$

$$\frac{da_0}{dz}=\frac{\sec^2\frac{z}{2}}{2\tan\frac{z}{2}}=\frac{1}{\sin z},$$

* $\exp x=e^x$

or

$$\frac{dz}{da_0}=\sin z;$$

hence

$$f^1(a_0)=\sin z,$$

$$f^2(a_0)=\cos z\frac{dz}{da_0}=\sin z\cos z,$$

$$f^3(a_0)=\cos 2z\frac{dz}{da_0}=\sin z\cos 2z,$$

$$f^4(a_0)=(\cos z\cos 2z-2\sin z\sin 2z)\frac{dz}{da_0}$$
$$=\sin z\cos z\cos 2z-2\sin^2 z\sin 2z.$$

Also

$$a_1=\ -\cos p,$$

$$a_2=-\frac{2}{3}\cos^3 p,$$

$$a_3=-\frac{6}{5}\cos^5 p,$$

$$a_4=-\frac{24}{7}\cos^7 p.$$

The A's could now be computed, but they would contain a combination of functions of z and of p, since the a's are functions of p.

It is necessary, then, to approximate these functions in terms of z. a_1 must include terms in ϵ^6 or h^3; a_2, in h^2; and a_3, terms in h. a_4 needs only to have p replaced by z. In order to obtain this approximation we must develop the function

$$f(p)=\cos\left[2\tan^{-1}\exp\left(\log_e\tan\frac{z}{2}-\frac{h}{1!}\cos p-\frac{h^2}{2!}\frac{2}{3}\cos^3 p\right.\right.$$
$$\left.\left.-\frac{h^3}{3!}\frac{6}{5}\cos^5 p-\cdots\right)\right]=A_0+A_1\frac{h}{1!}+A_2\frac{h^2}{2!}+A_3\frac{h^3}{3!}+\cdots\cdots$$

We have now

$$a_0=\log_e\tan\frac{z}{2},$$

$$\frac{da_0}{dz}=\frac{1}{\sin z},$$

or,

$$\frac{dz}{da_0}=\sin z,$$

$$f(a_0)=\cos z,$$

$$f^1(a_0)=-\sin z\,\frac{dz}{da_0}=-\sin^2 z,$$

$$f^2(a_0)=-\sin 2z\,\frac{dz}{da_0}=-\sin z\sin 2z,$$

$$f^3(a_0)=(-\cos z\sin 2z-2\sin z\cos 2z)\,\frac{dz}{da_0}$$
$$=-\sin z\cos z\sin 2z-2\sin^2 z\cos 2z.$$

Also

$$a_1=-\cos p,$$

$$a_2=-\frac{2}{3}\cos^3 p,$$

$$a_3=-\frac{6}{5}\cos^5 p.$$

Therefore

$$A_0=\cos z,$$

$$A_1=\sin^2 z\cos p,$$

$$A_2=(\sin z\sin 2z\cos^2 p+\frac{2}{3}\sin^2 z)\cos^3 p,$$

$$A_3=(\sin z\cos z\sin 2z+2\sin^2 z\cos 2z)\cos^3 p$$
$$-2\sin z\sin 2z\cos^4 p+\frac{6}{5}\sin^2 z\cos^5 p.$$

By substituting these values, we obtain

$$\cos p=\cos z+\frac{h}{1!}\sin^2 z\cos p+\frac{h^2}{2!}(-\sin z\sin 2z\cos^2 p$$
$$+\frac{2}{3}\sin^2 z\cos^3 p)+\frac{h^3}{3!}[(\sin z\cos z\sin 2z$$
$$+2\sin^2 z\cos 2z)\cos^3 p-2\sin z\sin 2z\cos^4 p$$
$$+\frac{6}{5}\sin^2 z\cos^5 p].$$

As an approximation with the first power of h, we get

$$\cos p=\cos z+h\sin^2 z\cos z.$$

Substituting this in the above expression for cos p and retaining the second powers of h, we get

$$\cos p = \cos z + h \sin^2 z \cos z + h^2 \sin^4 z \cos z$$
$$+\frac{h^2}{2}\left(-\sin z \sin 2z \cos^2 z + \frac{2}{3}\sin^2 z \cos^3 z\right).$$

Finally, substituting this approximation in the expression for cos p and retaining all third powers of h, we get the required approximation

$$\cos p = \cos z + h \sin^2 z \cos z + h^2 \sin^4 z \cos z + h^3 \sin^6 z \cos z$$
$$+\frac{h^3}{2}\left(-\sin^3 z \sin 2z \cos^2 z + \frac{2}{3}\sin^4 z \cos^3 z\right)$$
$$+\frac{h^2}{2}\left(-\sin z \sin 2z \cos^2 z\right) + h^3\left(-\sin^3 z \sin 2z \cos^2 z\right)$$
$$+\frac{h^2}{3}\sin^2 z \cos^3 z + h^3 \sin^4 z \cos^3 z$$
$$+\frac{h^3}{6}\left(\sin z \cos^4 z \sin 2z + 2 \sin^2 z \cos^3 z \cos 2z\right.$$
$$\left.-2 \sin z \cos^4 z \sin 2z + \frac{6}{5}\sin^2 z \cos^5 z\right),$$

or

$$\cos p = \cos z + h \sin^2 z \cos z + h^2\left(\cos z - \frac{8}{3}\cos^3 z + \frac{5}{3}\cos^5 z\right)$$
$$+h^3\left(\cos z - 5\cos^3 z + \frac{36}{5}\cos^5 z - \frac{16}{5}\cos^7 z\right),$$

$$\cos^3 p = \cos^3 z + 3h \sin^2 z \cos^3 z + h^2\left(6\cos^3 z - 14\cos^5 z + 8\cos^7 z\right),$$

$$\cos^5 p = \cos^5 z + 5h \sin^2 z \cos^5 z,$$

$$\cos^7 p = \cos^7 z.$$

If these values are substituted in the expressions for the a's on page 56, we get

$$a_1 = -\left[\cos z + h \sin^2 z \cos z + h^2\left(\cos z - \frac{8}{3}\cos^3 z + \frac{5}{3}\cos^5 z\right)\right.$$
$$\left.+h^3\left(\cos z - 5\cos^3 z + \frac{36}{5}\cos^5 z - \frac{16}{5}\cos^7 z\right)\right],$$

$$a_2 = -\frac{2}{3}\left[\cos^3 z + 3h \sin^2 z \cos^3 z + h^2\left(6\cos^3 z - 14\cos^5 z + 8\cos^7 z\right)\right],$$

$$a_3 = -\frac{6}{5}\,(\cos^5 z + 5h \sin^2 z \cos^5 z),$$

$$a_4 = -\frac{24}{7}\cos^7 z.$$

With these values and the values of the derivatives of $f(a_0)$ on page 56, we get by retaining all requisite powers of h the following approximations

$$A_0 = z,$$
$$A_1 = -[\sin z \cos z + h \sin^3 z \cos z + h^2\,(\sin z \cos z - \frac{8}{3}\sin z \cos^3 z + \frac{5}{3}\sin z \cos^5 z) + h^3\,(\sin z \cos z - 5 \sin z \cos^3 z + \frac{36}{5}\sin z \cos^5 z - \frac{16}{5}\sin z \cos^7 z)],$$

$$A_2 = \sin z \cos^3 z + 2h \sin^3 z \cos^3 z + h^2\,(3 \sin z \cos^3 z - \frac{22}{3}\sin z \cos^5 z + \frac{13}{3}\sin z \cos^7 z) - \frac{2}{3}\,[\sin z \cos^3 z + 3h \sin^3 z \cos^3 z + h^2\,(6 \sin z \cos^3 z - 14 \sin z \cos^5 z + 8 \sin z \cos^7 z)],$$

$$A_3 = -(\sin z \cos^3 z \cos 2z + 3h \sin^3 z \cos^3 z \cos 2z) + 2\,(\sin z \cos^5 z + 4h \sin^3 z \cos^5 z) - \frac{6}{5}\,(\sin z \cos^5 z + 5h \sin^3 z \cos^5 z),$$

$$A_4 = \sin z \cos^5 z \cos 2z - 2 \sin^2 z \cos^4 z \sin 2z - 4 \sin z \cos^5 z \cos 2z + \frac{24}{5}\sin z \cos^7 z + \frac{4}{3}\sin z \cos^7 z - \frac{24}{7}\sin z \cos^7 z.$$

Substituting these values and reducing by use of the table on p. 88, we get the approximation

$$p = z - \frac{h}{1!}\Big[\sin z \cos z + h \sin z \cos z - h \sin z \cos^3 z + h^2\left(\sin z \cos z - \frac{8}{3}\sin z \cos^3 z + \frac{5}{3}\sin z \cos^5 z\right) + h^3\Big(\sin z \cos z - 5 \sin z \cos^3 z + \frac{36}{5}\sin z \cos^5 z$$

$$-\frac{16}{5}\sin z\cos^7 z\Big)\Big]+\frac{h^2}{2!}\Big[\frac{1}{3}\sin z\cos^3 z-h^2(\sin z\cos^3 z$$
$$-2\sin z\cos^5 z+\sin z\cos^7 z)\Big]+\frac{h^3}{3!}\Big[\sin z\cos^3 z$$
$$-\frac{6}{5}\sin z\cos^5 z+h(3\sin z\cos^3 z-7\sin z\cos^5 z$$
$$+4\sin z\cos^7 z)\Big]+\frac{h^4}{4!}\left(-\sin z\cos^5 z+\frac{74}{105}\sin z\cos^7 z\right).$$

Rearranging this in powers of h or ϵ^2, we get

$$p=z-\epsilon^2\sin z\cos z+\epsilon^4\left(-\sin z\cos z+\frac{7}{6}\sin z\cos^3 z\right)$$
$$+\epsilon^6\left(-\sin z\cos z+\frac{17}{6}\sin z\cos^3 z-\frac{28}{15}\sin z\cos^5 z\right)$$
$$+\epsilon^8\Big(-\sin z\cos z+5\sin z\cos^3 z-\frac{889}{120}\sin z\cos^5 z$$
$$+\frac{4279}{1260}\sin z\cos^7 z\Big).$$

Making the reductions by use of the table on p. 88 and rearranging, we get the approximation

$$p=z-\left(\frac{\epsilon^2}{2}+\frac{5\epsilon^4}{24}+\frac{\epsilon^6}{12}+\frac{13\epsilon^8}{360}\right)\sin 2z+\left(\frac{7\epsilon^4}{48}+\frac{29\epsilon^6}{240}+\frac{811\epsilon^8}{11520}\right)\sin 4z$$
$$-\left(\frac{7\epsilon^6}{120}+\frac{81\epsilon^8}{1120}\right)\sin 6z+\frac{4279\epsilon^8}{161280}\sin 8z.$$

In terms of φ and χ this becomes, as before, the approximation sought

$$\varphi-\chi=\left(\frac{\epsilon^2}{2}+\frac{5\epsilon^4}{24}+\frac{\epsilon^6}{12}+\frac{13\epsilon^8}{360}\right)\sin 2\chi+\left(\frac{7\epsilon^4}{48}+\frac{29\epsilon^6}{240}+\frac{811\epsilon^8}{11520}\right)\sin 4\chi$$
$$+\left(\frac{7\epsilon^6}{120}+\frac{81\epsilon^8}{1120}\right)\sin 6\chi+\frac{4279\epsilon^8}{161280}\sin 8\chi.$$

DEVELOPMENT OF $\varphi-\beta$ IN TERMS OF φ—FIRST METHOD.

In the equation of definition of authalic latitude (see p. 12), if the fractional part is divided out up to and including the term in ϵ^6, we get

$$\sin\beta=\sin\varphi\,[1-\frac{2\epsilon^2}{3}\cos^2\varphi-\left(\frac{7}{45}\cos^2\varphi+\frac{3}{5}\sin^2\varphi\cos^2\varphi\right)\epsilon^4$$
$$-\left(\frac{64}{945}\cos^2\varphi+\frac{6}{35}\sin^2\varphi\cos^2\varphi+\frac{4}{7}\sin^4\varphi\cos^2\varphi\right)\epsilon^6+\cdots].$$

Now let

$$\sin\beta = a\sin\varphi,$$

then

$$\tan\beta = \frac{a\tan\varphi}{[1+(1-a^2)\tan^2\varphi]^{1/2}}.$$

Assume

$$\tan\beta = (1-b)\tan\varphi,$$

then

$$\tan(\varphi-\beta) = \frac{b\sin 2\varphi}{2-b+b\cos 2\varphi}.$$

If

$$q = \frac{b}{2-b} = \frac{b}{2}+\frac{b^2}{4}+\frac{b^3}{8}+\frac{b^4}{16}+\cdots.$$

we get, as in former developments (see p. 13):

$$\varphi-\beta = q\sin 2\varphi - \frac{q^2}{2}\sin 4\varphi + \frac{q^3}{3}\sin 6\varphi - \cdots.$$

Now

$$a = 1-\frac{2\epsilon^2}{3}\cos^2\varphi - \left(\frac{7}{45}\cos^2\varphi + \frac{3}{5}\sin^2\varphi\cos^2\varphi\right)\epsilon^4$$

$$-\left(\frac{64}{945}\cos^2\varphi + \frac{6}{35}\sin^2\varphi\cos^2\varphi + \frac{4}{7}\sin^4\varphi\cos^2\varphi\right)\epsilon^6 + \cdots.$$

or arranged in powers of $\cos^2\varphi$, this becomes

$$a = 1-\left(\frac{2\epsilon^2}{3}+\frac{34\epsilon^4}{45}+\frac{766\epsilon^6}{945}\right)\cos^2\varphi + \left(\frac{3\epsilon^4}{5}+\frac{46\epsilon^6}{35}\right)\cos^4\varphi$$

$$-\frac{4\epsilon^6}{7}\cos^6\varphi.$$

$$1-b = \frac{a}{[1+(1-a^2)\tan^2\varphi]^{\frac{1}{2}}}$$

$$= a\,[1-\frac{1}{2}(1-a^2)\tan^2\varphi + \frac{3}{8}(1-a^2)^2\tan^4\varphi$$

$$-\frac{5}{16}(1-a^2)^3\tan^6\varphi + \cdots)$$

$$a^2 = 1-\left(\frac{4\epsilon^2}{3}+\frac{68\epsilon^4}{45}+\frac{1532\epsilon^6}{945}\right)\cos^2\varphi + \left(\frac{74\epsilon^4}{45}+\frac{3436\epsilon^6}{945}\right)\cos^4\varphi$$

$$-\frac{68\epsilon^6}{35}\cos^6\varphi$$

$$(1-a^2)\tan^2\varphi = \left(\frac{4\epsilon^2}{3}+\frac{68\epsilon^4}{45}+\frac{1532\epsilon^6}{945}\right)\sin^2\varphi$$

$$-\left(\frac{74\epsilon^4}{45}+\frac{3436\epsilon^6}{945}\right)\sin^2\varphi\cos^2\varphi + \frac{68\epsilon^6}{35}\sin^2\varphi\cos^4\varphi,$$

or, when arranged in powers of $\cos^2\varphi$,

$$(1-a^2)\tan^2\varphi=\left(\frac{4\epsilon^2}{3}+\frac{68\epsilon^4}{45}+\frac{1532\epsilon^6}{945}\right)-\left(\frac{4\epsilon^2}{3}+\frac{142\epsilon^4}{45}+\frac{184\epsilon^6}{35}\right)\cos^2\varphi+\left(\frac{74\epsilon^4}{45}+\frac{5272\epsilon^6}{945}\right)\cos^4\varphi-\frac{68\epsilon^6}{35}\cos^6\varphi,$$

$$[(1-a^2)\tan^2\varphi]^2=\frac{16\epsilon^4}{9}+\frac{544\epsilon^6}{135}-\left(\frac{32\epsilon^4}{9}+\frac{112\epsilon^6}{9}\right)\cos^2\varphi+\left(\frac{16\epsilon^4}{9}+\frac{64\epsilon^6}{5}\right)\cos^4\varphi-\frac{592\epsilon^6}{135}\cos^6\varphi,$$

$$[(1-a^2)\tan^2\varphi]^3=\frac{64\epsilon^6}{27}-\frac{64\epsilon^6}{9}\cos^2\varphi+\frac{64\epsilon^6}{9}\cos^4\varphi-\frac{64\epsilon^6}{27}\cos^6\varphi,$$

$$\frac{1}{[1+(1-a^2)\tan^2\varphi]^{\frac{1}{2}}}=1-\left(\frac{2\epsilon^2}{3}+\frac{4\epsilon^4}{45}+\frac{38\epsilon^6}{945}\right)+\left(\frac{2\epsilon^2}{3}+\frac{11\epsilon^4}{45}+\frac{58\epsilon^6}{315}\right)\cos^2\varphi-\left(\frac{7\epsilon^4}{45}+\frac{40\epsilon^6}{189}\right)\cos^4\varphi+\frac{64\epsilon^6}{945}\cos^6\varphi,$$

$$\frac{a}{[1+(1-a^2)\tan^2\varphi]^{\frac{1}{2}}}=1-\left(\frac{2\epsilon^2}{3}+\frac{4\epsilon^4}{45}+\frac{38\epsilon^6}{945}\right)-\left(\frac{\epsilon^4}{15}+\frac{4\epsilon^6}{63}\right)\cos^2\varphi+\frac{34\epsilon^6}{945}\cos^4\varphi.$$

Therefore

$$b=\frac{2\epsilon^2}{3}+\frac{4\epsilon^4}{45}+\frac{38\epsilon^6}{945}+\left(\frac{\epsilon^4}{15}+\frac{4\epsilon^6}{63}\right)\cos^2\varphi-\frac{34\epsilon^6}{945}\cos^4\varphi.$$

But to the approximation required, we have

$$\varphi-\beta=\left(\frac{b}{2}+\frac{b^2}{4}+\frac{b^3}{8}\right)\sin 2\varphi-\left(\frac{b^2}{8}+\frac{b^3}{8}\right)\sin 4\varphi+\frac{b^3}{24}\sin 6\varphi.$$

$$b^2=\frac{4\epsilon^4}{9}+\frac{16\epsilon^6}{135}+\frac{4\epsilon^6}{45}\cos^2\varphi,$$

$$b^3=\frac{8\epsilon^6}{27}.$$

$$\varphi-\beta=\left[\frac{\epsilon^2}{3}+\frac{7\epsilon^4}{45}+\frac{82\epsilon^6}{945}+\left(\frac{\epsilon^4}{30}+\frac{17\epsilon^6}{315}\right)\cos^2\varphi-\frac{17\epsilon^6}{945}\cos^4\varphi\right]\sin 2\varphi-\left(\frac{\epsilon^4}{18}+\frac{7\epsilon^6}{135}+\frac{\epsilon^6}{90}\cos^2\varphi\right)\sin 4\varphi+\frac{\epsilon^6}{81}\sin 6\varphi,$$

or, finally, on reduction (for reductions, see table p. 88) we get the approximation

$$\varphi-\beta=\left(\frac{\epsilon^2}{3}+\frac{31\epsilon^4}{180}+\frac{59\epsilon^6}{560}\right)\sin 2\varphi-\left(\frac{17\epsilon^4}{360}+\frac{61\epsilon^6}{1260}\right)\sin 4\varphi$$
$$+\frac{383\epsilon^6}{45360}\sin 6\varphi.$$

DEVELOPMENT OF $\varphi-\beta$ IN TERMS OF φ—SECOND METHOD

The expansion of β in terms of φ can be carried through in a somewhat shorter way by the following method:

Set $\epsilon^2=h$ and the equation for $\sin\beta$ becomes (see p. 60)

$$\sin\beta=\sin\varphi\left[1-\frac{2}{3}h\cos^2\varphi-\left(\frac{7}{45}\cos^2\varphi+\frac{3}{5}\sin^2\varphi\cos^2\varphi\right)h^2\right.$$
$$-\left(\frac{64}{945}\cos^2\varphi+\frac{6}{35}\sin^2\varphi\cos^2\varphi+\frac{4}{7}\sin^4\varphi\cos^2\varphi\right)h^3$$
$$\left.-\ldots\right].$$

Differentiating this expression, considering β as a function of h, we get in succession

$$\cos\beta\frac{d\beta}{dh}=\sin\varphi\left[-\frac{2}{3}\cos^2\varphi-\left(\frac{14}{45}\cos^2\varphi+\frac{6}{5}\sin^2\varphi\cos^2\varphi\right)h\right.$$
$$\left.-\left(\frac{64}{315}\cos^2\varphi+\frac{18}{35}\sin^2\varphi\cos^2\varphi+\frac{12}{7}\sin^4\varphi\cos^2\varphi\right)h^2\right],$$

$$-\sin\beta\left(\frac{d\beta}{dh}\right)^2+\cos\beta\frac{d^2\beta}{dh^2}=\sin\varphi\left[-\left(\frac{14}{45}\cos^2\varphi+\frac{6}{5}\sin^2\varphi\cos^2\varphi\right)\right.$$
$$\left.-\left(\frac{128}{315}\cos^2\varphi+\frac{36}{35}\sin^2\varphi\cos^2\varphi+\frac{24}{7}\sin^4\varphi\cos^2\varphi\right)h\right],$$

$$-\cos\beta\left(\frac{d\beta}{dh}\right)^3-3\sin\beta\frac{d\beta}{dh}\frac{d^2\beta}{dh^2}+\cos\beta\frac{d^3\beta}{dh^3}$$
$$=-\frac{128}{315}\sin\varphi\cos^2\varphi-\frac{36}{35}\sin^3\varphi\cos^2\varphi-\frac{24}{7}\sin^5\varphi\cos^2\varphi.$$

These expressions must now be evaluated for $h=0$. (For the necessary reductions see the reduction table, p. 88.)

Denoting the value for $h=0$ by brackets we get

$$[\beta]=\varphi,$$

$$\left[\frac{d\beta}{dh}\right]=-\frac{2}{3}\sin\varphi\cos\varphi=-\frac{1}{3}\sin 2\varphi,$$

$$\left[\frac{d^2\beta}{dh^2}\right]=-\frac{14}{45}\sin\varphi\cos\varphi-\frac{34}{45}\sin^3\varphi\cos\varphi,$$

$$=-\frac{31}{90}\sin 2\varphi+\frac{17}{180}\sin 4\varphi,$$

$$\left[\frac{d^3\beta}{dh^3}\right]=-\frac{128}{315}\sin\varphi\cos\varphi-\frac{664}{945}\sin^3\varphi\cos\varphi-\frac{1532}{945}\sin^5\varphi\cos\varphi$$

$$=-\frac{177}{280}\sin 2\varphi+\frac{61}{210}\sin 4\varphi-\frac{383}{7560}\sin 6\varphi.$$

By Maclaurin's series we have, on replacing h by ϵ^2,

$$\beta=[\beta]+\frac{\epsilon^2}{1\,!}\left[\frac{d\beta}{dh}\right]+\frac{\epsilon^4}{2\,!}\left[\frac{d^2\beta}{dh^2}\right]+\frac{\epsilon^6}{3\,!}\left[\frac{d^3\beta}{dh^3}\right]+\cdots\cdots$$

By substituting the values in this expansion we obtain the desired approximation

$$\beta=\varphi-\frac{\epsilon^2}{3}\sin 2\varphi-\frac{31\epsilon^4}{180}\sin 2\varphi+\frac{17\epsilon^4}{360}\sin 4\varphi-\frac{59\epsilon^6}{560}\sin 2\varphi$$

$$+\frac{61\epsilon^6}{1260}\sin 4\varphi-\frac{383\epsilon^6}{45360}\sin 6\varphi,$$

or, as before, we obtain the approximation

$$\varphi-\beta=\left(\frac{\epsilon^2}{3}+\frac{31\epsilon^4}{180}+\frac{59\epsilon^6}{560}\right)\sin 2\varphi-\left(\frac{17\epsilon^4}{360}+\frac{61\epsilon^6}{1260}\right)\sin 4\varphi$$

$$+\frac{383\epsilon^6}{45360}\sin 6\varphi.$$

DEVELOPMENT OF $\varphi-\beta$ IN TERMS OF φ—THIRD METHOD.

We can develop the authalic latitude in terms of φ by a method similar to that used for the isometric latitude on page 18.

$$\sin(\varphi+h)=\sin\varphi\cos h+\cos\varphi\sin h.$$

If we assume

$$h = a\epsilon^2 + b\epsilon^4 + c\epsilon^6 + \cdots,$$

and if we wish to carry the approximation only far enough to include the sixth powers of ϵ, we get

$$\sin(\varphi + h) = \left(1 - \frac{h^2}{2}\right)\sin\varphi + \left(h - \frac{h^3}{6}\right)\cos\varphi.$$

After substitution of the value of h we obtain the approximation

$$\sin(\varphi + h) = \left(1 - \frac{a^2\epsilon^4}{2} - ab\epsilon^6\right)\sin\varphi + \left(a\epsilon^2 + b\epsilon^4 + c\epsilon^6 - \frac{a^3\epsilon^6}{6}\right)\cos\varphi.$$

Now, letting

$$\beta = \varphi + h = \varphi + a\epsilon^2 + b\epsilon^4 + c\epsilon^6 + \cdots,$$

we get

$$\sin\beta = \left(1 - \frac{a^2\epsilon^4}{2} - ab\epsilon^6\right)\sin\varphi + \left(a\epsilon^2 + b\epsilon^4 + c\epsilon^6 - \frac{a^3\epsilon^6}{6}\right)\cos\varphi.$$

If the equation of definition of authalic latitude is reduced to the form approximated to ϵ^6, inclusive,

$$\sin\beta = \sin\varphi\left[1 - \left(\frac{2\epsilon^2}{3} + \frac{34\epsilon^4}{45} + \frac{766\epsilon^6}{945}\right)\cos^2\varphi + \left(\frac{3\epsilon^4}{5} + \frac{46\epsilon^6}{35}\right)\cos^4\varphi - \frac{4\epsilon^6}{7}\cos^6\varphi\right],$$

we may equate the coefficients of like powers of ϵ in the two series, since the two series must be identically equal. In this way we may obtain equations for the determinations of the values of a, b, and c.

$$a\cos\varphi = -\frac{2}{3}\sin\varphi\cos^2\varphi,$$

$$-\frac{a^2}{2}\sin\varphi + b\cos\varphi = -\frac{34}{45}\sin\varphi\cos^2\varphi + \frac{3}{5}\sin\varphi\cos^4\varphi,$$

$$-ab\sin\varphi + c\cos\varphi - \frac{a^3}{6}\cos\varphi = -\frac{766}{945}\sin\varphi\cos^2\varphi + \frac{46}{35}\sin\varphi\cos^4\varphi - \frac{4}{7}\sin\varphi\cos^6\varphi.$$

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From these equations we obtain the values (for the necessary reductions see the reduction table, p. 88).

$$a=-\frac{2}{3}\sin\varphi\cos\varphi=-\frac{1}{3}\sin 2\varphi,$$

$$b=-\frac{8}{15}\sin\varphi\cos\varphi+\frac{17}{45}\sin\varphi\cos^3\varphi=-\frac{31}{180}\sin 2\varphi+\frac{17}{360}\sin 4\varphi,$$

$$c=-\frac{86}{189}\sin\varphi\cos\varphi+\frac{1864}{2835}\sin\varphi\cos^3\varphi-\frac{766}{2835}\sin\varphi\cos^5\varphi$$

$$=-\frac{59}{560}\sin 2\varphi+\frac{61}{1260}\sin 4\varphi-\frac{383}{45360}\sin 6\varphi,$$

When these values are substituted and rearranged we get, as before, the approximation

$$\varphi-\beta=\left(\frac{\epsilon^2}{3}+\frac{31\epsilon^4}{180}+\frac{59\epsilon^6}{560}\right)\sin 2\varphi-\left(\frac{17\epsilon^4}{360}+\frac{61\epsilon^6}{1260}\right)\sin 4\varphi$$
$$+\frac{383\epsilon^6}{45360}\sin 6\varphi.$$

DEVELOPMENT OF $\varphi-\beta$ IN TERMS OF φ—FOURTH METHOD.

The development of $\varphi-\beta$ in terms of β can first be made by the third, fourth, or fifth method (see pp. 72–79), and then this expression may be changed into terms of φ by Lagrange's theorem.

We are given the approximation

$$\beta=\varphi-\left(\frac{\epsilon^2}{3}+\frac{31\epsilon^4}{180}+\frac{517\epsilon^6}{5040}\right)\sin 2\beta-\left(\frac{23\epsilon^4}{360}+\frac{251\epsilon^6}{3780}\right)\sin 4\beta$$
$$-\frac{761\epsilon^6}{45360}\sin 6\beta.$$

By Lagrange's theorem we have

$$\beta=\varphi+\frac{1}{1!}g(\varphi)+\frac{1}{2!}\frac{d}{d\varphi}[g(\varphi)]^2+\frac{1}{3!}\frac{d^2}{d\varphi^2}[g(\varphi)]^3+\cdots,$$

in which

$$g(\varphi)=-\left(\frac{\epsilon^2}{3}+\frac{31\epsilon^4}{180}+\frac{517\epsilon^6}{5040}\right)\sin 2\varphi-\left(\frac{23\epsilon^4}{360}+\frac{251\epsilon^6}{3780}\right)\sin 4\varphi$$
$$-\frac{761\epsilon^6}{45360}\sin 6\varphi,$$

By squaring this expression, we get

$$[g(\varphi)]^2 = \left(\frac{\epsilon^4}{9} + \frac{31\epsilon^6}{270}\right)\sin^2 2\varphi + \frac{23\epsilon^6}{540}\sin 2\varphi \sin 4\varphi,$$

and by cubing it we obtain

$$[g(\varphi)]^3 = -\frac{\epsilon^6}{27}\sin^3 2\varphi.$$

Differentiating and reducing by the table on p. 88, we get

$$\frac{d}{d\varphi}[g(\varphi)]^2 = \left(\frac{4\epsilon^4}{9} + \frac{62\epsilon^6}{135}\right)\sin 2\varphi\cos 2\varphi + \frac{23\epsilon^6}{270}\cos 2\varphi\sin 4\varphi + \frac{23\epsilon^6}{135}\sin 2\varphi\cos 4\varphi$$

$$= -\frac{23\epsilon^6}{540}\sin 2\varphi + \left(\frac{2\epsilon^4}{9} + \frac{31\epsilon^6}{135}\right)\sin 4\varphi + \frac{23\epsilon^6}{180}\sin 6\varphi,$$

$$\frac{d^2}{d\varphi^2}[g(\varphi)]^3 = \frac{\epsilon^6}{9}\sin 2\varphi - \frac{\epsilon^6}{3}\sin 6\varphi.$$

Substituting these values in Lagrange's series, we get the approximation

$$\beta = \varphi - \left(\frac{\epsilon^2}{3} + \frac{31\epsilon^4}{180} + \frac{517\epsilon^6}{5040}\right)\sin 2\varphi - \left(\frac{23\epsilon^4}{360} + \frac{251\epsilon^6}{3780}\right)\sin 4\varphi - \frac{761\epsilon^6}{45360}\sin 6\varphi - \frac{23\epsilon^6}{1080}\sin 2\varphi + \left(\frac{\epsilon^4}{9} + \frac{31\epsilon^6}{270}\right)\sin 4\varphi + \frac{23\epsilon^6}{360}\sin 6\varphi + \frac{\epsilon^6}{54}\sin 2\varphi - \frac{\epsilon^6}{18}\sin 6\varphi.$$

By collecting and rearranging we get, as before, the approximation

$$\varphi - \beta = \left(\frac{\epsilon^2}{3} + \frac{31\epsilon^4}{180} + \frac{59\epsilon^6}{560}\right)\sin 2\varphi - \left(\frac{17\epsilon^4}{360} + \frac{61\epsilon^6}{1260}\right)\sin 4\varphi + \frac{383\epsilon^6}{45360}\sin 6\varphi.$$

DEVELOPMENT OF $\varphi-\beta$ IN TERMS OF φ—FIFTH METHOD.

$\varphi-\beta$ can be developed in terms of φ by Arbogast's rule (see p. 28). In this case the function to be developed is

$$\beta=\sin^{-1}\left[\sin\varphi-\frac{h}{1!}\frac{2}{3}\sin\varphi\cos^2\varphi-\frac{h^2}{2!}\left(\frac{14}{45}\sin\varphi\cos^2\varphi+\frac{6}{5}\sin^3\varphi\cos^2\varphi\right)-\frac{h^3}{3!}\left(\frac{128}{315}\sin\varphi\cos^2\varphi+\frac{36}{35}\sin^3\varphi\cos^2\varphi+\frac{24}{7}\sin^5\varphi\cos^2\varphi\right)+\ldots\right]$$

$$=A_0+A_1\frac{h}{1!}+A_2\frac{h^2}{2!}+A_3\frac{h^3}{3!}+\ldots\ldots,$$

with the A's defined as before.

$$A_0=f(a_0),$$
$$A_1=a_1f^1(a_0),$$
$$A_2=a_1^2f^2(a_0)+a_2f^1(a_0),$$
$$A_3=a_1^3f^3(a_0)+3a_1a_2f^2(a_0)+a_3f^1(a_0),$$
etc.

In this function we have

$$a_0=\sin\varphi,$$
$$\frac{da_0}{d\varphi}=\cos\varphi,$$

or

$$\frac{d\varphi}{da_0}=\sec\varphi.$$

$$f(a_0)=\varphi,$$
$$f^1(a_0)=\frac{d\varphi}{da_0}=\sec\varphi,$$
$$f^2(a_0)=\sec\varphi\tan\varphi\frac{d\varphi}{da_0}=\sec^2\varphi\tan\varphi,$$
$$f^3(a_0)=(2\sec^2\varphi\tan^2\varphi+\sec^4\varphi)\frac{d\varphi}{da_0}=2\sec^3\varphi\tan^2\varphi+\sec^5\varphi,$$

$$a_1=-\frac{2}{3}\sin\varphi\cos^2\varphi,$$
$$a_2=-\frac{14}{45}\sin\varphi\cos^2\varphi-\frac{6}{5}\sin^3\varphi\cos^2\varphi,$$
$$a_3=-\frac{128}{315}\sin\varphi\cos^2\varphi-\frac{36}{35}\sin^3\varphi\cos^2\varphi-\frac{24}{7}\sin^5\varphi\cos^2\varphi.$$

Substituting these values in the expressions for the A's and reducing by aid of the table on page 88, we get

$$A_0=\varphi,$$

$$A_1=-\frac{2}{3}\sin\varphi\cos\varphi=-\frac{1}{3}\sin 2\varphi,$$

$$\begin{aligned}A_2&=\frac{4}{9}\sin^3\varphi\cos\varphi-\frac{14}{45}\sin\varphi\cos\varphi-\frac{6}{5}\sin^3\varphi\cos\varphi\\&=-\frac{16}{15}\sin\varphi\cos\varphi+\frac{34}{45}\sin\varphi\cos^3\varphi\\&=-\frac{31}{90}\sin 2\varphi+\frac{17}{180}\sin 4\varphi,\end{aligned}$$

$$\begin{aligned}A_3=&-\frac{8}{27}\sin^3\varphi\cos^6\varphi\,(2\sec^3\varphi\tan^2\varphi+\sec^5\varphi)\\&+2\sin\varphi\cos^2\varphi\left(\frac{14}{45}\sin\varphi\cos^2\varphi\right.\\&\left.+\frac{6}{5}\sin^3\varphi\cos^2\varphi\right)\sec^2\varphi\tan\varphi\\&-\left(\frac{128}{315}\sin\varphi\cos^2\varphi+\frac{36}{35}\sin^3\varphi\cos^2\varphi+\frac{24}{7}\sin^5\varphi\cos^2\varphi\right)\sec\varphi\\=&-\frac{16}{27}\sin^5\varphi\cos\varphi-\frac{8}{27}\sin^3\varphi\cos\varphi+\frac{28}{45}\sin^3\varphi\cos\varphi\\&+\frac{12}{5}\sin^5\varphi\cos\varphi-\frac{128}{315}\sin\varphi\cos\varphi-\frac{36}{35}\sin^3\varphi\cos\varphi\\&-\frac{24}{7}\sin^5\varphi\cos\varphi\\=&-\frac{128}{315}\sin\varphi\cos\varphi-\frac{664}{945}\sin^3\varphi\cos\varphi-\frac{1532}{945}\sin^5\varphi\cos\varphi\\=&-\frac{177}{280}\sin 2\varphi+\frac{61}{210}\sin 4\varphi-\frac{383}{7560}\sin 6\varphi.\end{aligned}$$

Substituting these values for the A's in the development and replacing h by ϵ^2, we get, after rearrangement, the value obtained by the other methods as the approximation desired.

$$\varphi-\beta=\left(\frac{\epsilon^2}{3}+\frac{31\epsilon^4}{180}+\frac{59\epsilon^6}{560}\right)\sin 2\varphi-\left(\frac{17\epsilon^4}{360}+\frac{61\epsilon^6}{1260}\right)\sin 4\varphi+\frac{383\epsilon^6}{45360}\sin 6\varphi.$$

DEVELOPMENT OF $\varphi-\beta$ IN TERMS OF β—FIRST METHOD.

This quantity can now be expressed in terms of β by the application of Lagrange's series (see page 32). In this case

$$f(\varphi)=\left(\frac{\epsilon^2}{3}+\frac{31\epsilon^4}{180}+\frac{59\epsilon^6}{560}\right)\sin\varphi-\left(\frac{17\epsilon^4}{360}+\frac{61\epsilon^6}{1260}\right)\sin 4\varphi+\frac{383\epsilon^6}{45360}\sin 6\varphi.$$

Squaring this and reducing by aid of the table on page 88, we get

$$[f(\varphi)]^2=\frac{\epsilon^4}{18}+\frac{31\epsilon^6}{540}-\frac{17\epsilon^6}{1080}\cos 2\varphi-\left(\frac{\epsilon^4}{18}+\frac{31\epsilon^6}{540}\right)\cos 4\varphi+\frac{17\epsilon^6}{1080}\cos 6\varphi.$$

also by cubing and reducing

$$[f(\varphi)]^3=\frac{\epsilon^6}{36}\sin 2\varphi-\frac{\epsilon^6}{108}\sin 6\varphi.$$

$$\frac{d}{d\beta}[f(\beta)]^2=\frac{17\epsilon^6}{540}\sin\beta+\left(\frac{2\epsilon^4}{9}+\frac{31\epsilon^6}{135}\right)\sin 4\beta-\frac{17\epsilon^6}{180}\sin 6\beta,$$

$$\frac{d^2}{d\beta^2}[f(\beta)]^3=-\frac{\epsilon^6}{9}\sin 2\beta+\frac{\epsilon^6}{3}\sin 6\beta.$$

Substituting these values in Lagrange's series, we obtain the approximation

$$\varphi-\beta=\left(\frac{\epsilon^2}{3}+\frac{31\epsilon^4}{180}+\frac{59\epsilon^6}{560}\right)\sin 2\beta-\left(\frac{17\epsilon^4}{360}+\frac{61\epsilon^6}{1260}\right)\sin 4\beta$$
$$+\frac{383\epsilon^6}{45360}\sin 6\beta+\frac{17\epsilon^6}{1080}\sin 2\beta+\left(\frac{\epsilon^4}{9}+\frac{31\epsilon^6}{270}\right)\sin 4\beta$$
$$-\frac{17\epsilon^6}{360}\sin 6\beta-\frac{\epsilon^6}{54}\sin 2\beta+\frac{\epsilon^6}{18}\sin 6\beta,$$

or, after collecting similar terms, the approximation

$$\varphi-\beta=\left(\frac{\epsilon^2}{3}+\frac{31\epsilon^4}{180}+\frac{517\epsilon^6}{5040}\right)\sin 2\beta+\left(\frac{23\epsilon^4}{360}+\frac{251\epsilon^6}{3780}\right)\sin 4\beta+\frac{761\epsilon^6}{45360}\sin 6\beta.$$

DEVELOPMENT OF $\varphi-\beta$ IN TERMS OF β—SECOND METHOD.

In the series for $\varphi-\beta$ *in terms of* φ (see p. 69), *let* $\varphi=\beta+x$. Then

$$\varphi-\beta=\left(\frac{\epsilon^2}{3}+\frac{31\epsilon^4}{180}+\frac{59\epsilon^6}{560}\right)\sin\,(2\beta+2x)$$
$$-\left(\frac{17\epsilon^4}{360}+\frac{61\epsilon^6}{1260}\right)\sin\,(4\beta+4x)+\frac{383\epsilon^6}{45360}\sin\,(6\beta+6x),$$

or

$$\varphi-\beta=\left(\frac{\epsilon^2}{3}+\frac{31\epsilon^4}{180}+\frac{59\epsilon^6}{560}\right)(\sin 2\beta\cos 2x+\cos 2\beta\sin 2x)$$
$$-\left(\frac{17\epsilon^4}{360}+\frac{61\epsilon^6}{1260}\right)(\sin 4\beta\cos 4x+\cos 4\beta\sin 4x)$$
$$+\frac{383\epsilon^6}{45360}(\sin 6\beta\cos 6x+\cos 6\beta\sin 6x),$$

or, including all terms in ϵ^6,

$$\varphi-\beta=\left(\frac{\epsilon^2}{3}+\frac{31\epsilon^4}{180}+\frac{59\epsilon^6}{560}\right)(1-2x^2)\sin 2\beta+\left(\frac{\epsilon^2}{3}+\frac{31\epsilon^4}{180}\right)2x\cos 2\beta$$
$$-\left(\frac{17\epsilon^4}{360}+\frac{61\epsilon^6}{1260}\right)\sin 4\beta-\frac{17\epsilon^4}{360}4x\cos 4\beta+\frac{383\epsilon^6}{45360}\sin 6\beta.$$

But

$$x=\left(\frac{\epsilon^2}{3}+\frac{31\epsilon^4}{180}+\frac{59\epsilon^6}{560}\right)\sin 2\varphi-\left(\frac{17\epsilon^4}{360}+\frac{61\epsilon^6}{1260}\right)\sin 4\varphi$$
$$+\frac{383\epsilon^6}{45360}\sin 6\varphi.$$

Substituting this value and retaining all sixth powers of ϵ, we get

$$\varphi-\beta=\left(\frac{\epsilon^2}{3}+\frac{31\epsilon^4}{180}+\frac{59\epsilon^6}{560}\right)\sin 2\beta-\frac{2\epsilon^6}{27}\sin^3 2\beta$$
$$+\left(\frac{\epsilon^2}{3}+\frac{31\epsilon^4}{180}\right)\left(\frac{2\epsilon^2}{3}+\frac{31\epsilon^4}{90}\right)\sin 2\varphi\cos 2\beta-\frac{17\epsilon^6}{540}\sin 4\beta\cos 2\beta$$
$$-\left(\frac{17\epsilon^4}{360}+\frac{61\epsilon^6}{1260}\right)\sin 4\beta-\frac{17\epsilon^6}{270}\sin 2\beta\cos 4\beta+\frac{383\epsilon^6}{45360}\sin 6\beta.$$

To the required approximation we get

$$\sin 2\varphi=\sin 2\beta+\frac{2\epsilon^2}{3}\sin 2\beta\cos 2\beta.$$

Substituting this value, we get the approximation

$$\varphi-\beta=\left(\frac{\epsilon^2}{3}+\frac{31\epsilon^4}{180}+\frac{59\epsilon^6}{560}\right)\sin 2\beta-\frac{2\epsilon^6}{27}\sin^3 2\beta$$
$$+\left(\frac{2\epsilon^4}{9}+\frac{31\epsilon^6}{135}\right)\sin 2\beta\cos 2\beta+\frac{4\epsilon^6}{27}\sin 2\beta\cos^2 2\beta$$
$$-\frac{17\epsilon^6}{540}\sin 4\beta\cos 2\beta-\left(\frac{17\epsilon^4}{360}+\frac{61\epsilon^6}{1260}\right)\sin 4\beta$$
$$-\frac{17\epsilon^6}{270}\sin 2\beta\cos 4\beta+\frac{383\epsilon^6}{45360}\sin 6\beta.$$

Reduced to sines of multiple arcs (see reduction table, p. 88), this becomes the desired approximation

$$\varphi-\beta=\left(\frac{\epsilon^2}{3}+\frac{31\epsilon^4}{180}+\frac{517\epsilon^6}{5040}\right)\sin 2\beta+\left(\frac{23\epsilon^4}{360}+\frac{251\epsilon^6}{3780}\right)\sin 4\beta$$
$$+\frac{761\epsilon^6}{45360}\sin 6\beta.$$

This result agrees with that already obtained.

DEVELOPMENT OF $\varphi-\beta$ IN TERMS OF β—THIRD METHOD.

$\varphi-\beta$ can be developed in terms of β by a method similar to the third method of developing the same in terms of φ (see p. 64).

We have

$$\sin\varphi=\sin\beta+\left(\frac{2\epsilon^2}{3}+\frac{34\epsilon^4}{45}+\frac{766\epsilon^6}{945}\right)\sin\varphi\cos^2\varphi$$
$$-\left(\frac{3\epsilon^4}{5}+\frac{46\epsilon^6}{35}\right)\sin\varphi\cos^4\varphi+\frac{4\epsilon^6}{7}\sin\varphi\cos^6\varphi.$$

Assume

$$\varphi=\beta-h=\beta-a\epsilon^2-b\epsilon^4-c\epsilon^6-\cdots,$$

then

$$\sin\varphi=\sin(\beta-h)=\sin\beta\cos h-\cos\beta\sin h,$$

or approximately

$$\sin\varphi=\left(1-\frac{h^2}{2}\right)\sin\beta-\left(h-\frac{h^3}{6}\right)\cos\beta$$

$$=\left(1-\frac{a^2\epsilon^4}{2}-ab\epsilon^6\right)\sin\beta-\left(a\epsilon^2+b\epsilon^4+c\epsilon^6-\frac{a^3\epsilon^6}{6}\right)\cos\beta.$$

$$\cos\varphi=\cos(\beta-h)=\cos\beta\cos h+\sin\beta\sin h,$$

or approximately

$$\cos\varphi=\left(1-\frac{h^2}{2}\right)\cos\beta+\left(h-\frac{h^3}{6}\right)\sin\beta.$$

No powers of ϵ beyond the fourth are needed in this approximation.

$$\cos\varphi=\left(1-\frac{a^2\epsilon^4}{2}\right)\cos\beta+(a\epsilon^2+b\epsilon^4)\sin\beta,$$

$$\cos^2\varphi=\cos^2\beta+2\epsilon^2 a\sin\beta\cos\beta+\epsilon^4(a^2\sin^2\beta \\ +2b\sin\beta\cos\beta-a^2\cos^2\beta),$$

$$\sin\varphi\cos^2\varphi=\sin\beta\cos^2\beta+\epsilon^2(2a\sin^2\beta\cos\beta-a\cos^3\beta) \\ +\epsilon^4\left(a^2\sin^3\beta+2b\sin^2\beta\cos\beta-\frac{7a^2}{2}\sin\beta\cos^2\beta \right. \\ \left. -b\cos^3\beta\right),$$

$$\sin\varphi\cos^4\varphi=\sin\beta\cos^4\beta+\epsilon^2(4a\sin^2\beta\cos^3\beta-a\cos^5\beta),$$

$$\sin\varphi\cos^6\varphi=\sin\beta\cos^6\beta.$$

Substituting these approximations in the expression for $\sin\varphi$, we get

$$\sin\varphi=\sin\beta+\frac{2\epsilon^2}{3}\sin\beta\cos^2\beta+\epsilon^4\left(\frac{4a}{3}\sin^2\beta\cos\beta-\frac{2a}{3}\cos^3\beta \right. \\ \left. +\frac{34}{45}\sin\beta\cos^2\beta\right)+\epsilon^6\left(\frac{2a^2}{3}\sin^3\beta+\frac{4b}{3}\sin^2\beta\cos\beta \right. \\ -\frac{7a^2}{3}\sin\beta\cos^2\beta-\frac{2b}{3}\cos^3\beta+\frac{68a}{45}\sin^2\beta\cos\beta \\ \left. -\frac{34a}{45}\cos^3\beta\right)+\frac{766\epsilon^6}{945}\sin\beta\cos^2\beta-\frac{3\epsilon^4}{5}\sin\beta\cos^4\beta \\ -\epsilon^6\left(\frac{12a}{5}\sin^2\beta\cos^3\beta-\frac{3a}{5}\cos^5\beta+\frac{46}{35}\sin\beta\cos^4\beta\right) \\ +\frac{4\epsilon^6}{7}\sin\beta\cos^6\beta.$$

But this series in ϵ^2 must be identically equal to the series already determined in the form

$$\sin\varphi=\sin\beta-\left(\frac{a^2\epsilon^4}{2}+ab\epsilon^6\right)\sin\beta-\left(a\epsilon^2+b\epsilon^4+c\epsilon^6-\frac{a^3\epsilon^6}{6}\right)\cos\beta.$$

Equating the coefficients of like powers of ϵ, we get the equations

$$-a\cos\beta=\frac{2}{3}\sin\beta\cos^2\beta,$$

$$-\frac{a^2}{2}\sin\beta-b\cos\beta=\frac{4a}{3}\sin^2\beta\cos\beta-\frac{2a}{3}\cos^3\beta$$
$$+\frac{34}{45}\sin\beta\cos^2\beta-\frac{3}{5}\sin\beta\cos^4\beta,$$

$$-ab\sin\beta-c\cos\beta+\frac{a^3}{6}\cos\beta=\frac{2a^2}{3}\sin^3\beta+\frac{4b}{3}\sin^2\beta\cos\beta$$
$$-\frac{7a^2}{3}\sin\beta\cos^2\beta-\frac{2b}{3}\cos^3\beta+\frac{68a}{45}\sin^2\beta\cos\beta$$
$$-\frac{34a}{45}\cos^3\beta+\frac{766}{945}\sin\beta\cos^2\beta-\frac{12a}{5}\sin^2\beta\cos^3\beta$$
$$+\frac{3a}{5}\cos^5\beta-\frac{46}{35}\sin\beta\cos^4\beta+\frac{4}{7}\sin\beta\cos^6\beta.$$

From these equations by successive substitutions and reductions (see reduction table, p. 88) we derive the values of a, b, and c.

$$a=-\frac{2}{3}\sin\beta\cos\beta=-\frac{1}{3}\sin 2\beta,$$

$$b=-\frac{4}{45}\sin\beta\cos\beta-\frac{23}{45}\sin\beta\cos^3\beta=-\frac{31}{180}\sin 2\beta-\frac{23}{360}\sin 4\beta,$$

$$c=-\frac{38}{945}\sin\beta\cos\beta+\frac{16}{2835}\sin\beta\cos^3\beta-\frac{1522}{2835}\sin\beta\cos^5\beta$$
$$=-\frac{517}{5040}\sin 2\beta-\frac{251}{3780}\sin 4\beta-\frac{761}{45360}\sin 6\beta.$$

By substituting these values and rearranging we get, as before, the desired approximation

$$\varphi-\beta=\left(\frac{\epsilon^2}{3}+\frac{31\epsilon^4}{180}+\frac{517\epsilon^6}{5040}\right)\sin 2\beta+\left(\frac{23\epsilon^4}{360}+\frac{251\epsilon^6}{3780}\right)\sin 4\beta$$
$$+\frac{761\epsilon^6}{45360}\sin 6\beta.$$

DEVELOPMENT OF $\varphi-\beta$ IN TERMS OF β—FOURTH METHOD.

The difference between the geodetic latitude and the authalic latitude can be developed by direct differentiation. Let us write the approximate equation of definition in the form (for the reductions, see table p. 88)

$$\begin{aligned}\sin\beta=&\left(1-\frac{\epsilon^2}{6}-\frac{41\epsilon^4}{360}-\frac{251\epsilon^6}{3024}\right)\sin\varphi-\left(\frac{\epsilon^2}{6}+\frac{11\epsilon^4}{144}+\frac{79\epsilon^6}{2160}\right)\sin 3\varphi\\&+\left(\frac{3\epsilon^4}{80}+\frac{21\epsilon^6}{560}\right)\sin 5\varphi-\frac{\epsilon^6}{112}\sin 7\varphi.\end{aligned}$$

In this equation let $h=\epsilon^2$ and it becomes

$$\begin{aligned}\sin\beta=&\left(1-\frac{h}{6}-\frac{41h^2}{360}-\frac{251h^3}{3024}\right)\sin\varphi-\left(\frac{h}{6}+\frac{11h^2}{144}+\frac{79h^3}{2160}\right)\sin 3\varphi\\&+\left(\frac{3h^2}{80}+\frac{21h^3}{560}\right)\sin 5\varphi-\frac{h^3}{112}\sin 7\varphi.\end{aligned}$$

Differentiate this expression, considering φ as a function of h or ϵ^2, and we obtain in succession

$$\begin{aligned}&\left[\left(1-\frac{h}{6}-\frac{41h^2}{360}-\frac{251h^3}{3024}\right)\cos\varphi-\left(\frac{h}{2}+\frac{11h^2}{48}+\frac{79h^3}{720}\right)\cos 3\varphi\right.\\&\qquad\left.+\left(\frac{3h^2}{16}+\frac{21h^3}{112}\right)\cos 5\varphi-\frac{h^3}{16}\cos 7\varphi\right]\frac{d\varphi}{dh}-\left(\frac{1}{6}+\frac{41h}{180}\right.\\&\qquad\left.+\frac{251h^2}{1008}\right)\sin\varphi-\left(\frac{1}{6}+\frac{11h}{72}+\frac{79h^2}{720}\right)\sin 3\varphi\\&\qquad+\left(\frac{3h}{40}+\frac{63h^2}{560}\right)\sin 5\varphi-\frac{3h^2}{112}\sin 7\varphi=0,\end{aligned}$$

$$\begin{aligned}&\left[\left(1-\frac{h}{6}-\frac{41h^2}{360}-\frac{251h^3}{3024}\right)\cos\varphi-\left(\frac{h}{2}+\frac{11h^2}{48}+\frac{71h^3}{720}\right)\cos 3\varphi\right.\\&\qquad\left.+\left(\frac{3h^2}{16}+\frac{21h^3}{112}\right)\cos 5\varphi-\frac{h^3}{16}\cos 7\varphi\right]\frac{d^2\varphi}{dh^2}\\&\qquad-\left[\left(1-\frac{h}{6}-\frac{41h^2}{360}-\frac{251h^3}{3024}\right)\sin\varphi-\left(\frac{3h}{2}+\frac{11h^2}{16}\right.\right.\\&\qquad\left.+\frac{71h^3}{240}\right)\sin 3\varphi+\left(\frac{15h^2}{16}+\frac{105h^3}{112}\right)\sin 5\varphi\\&\qquad\left.-\frac{7h^3}{16}\sin 7\varphi\right]\left(\frac{d\varphi}{dh}\right)^2+2\left[\left(-\frac{1}{6}-\frac{41h}{180}-\frac{251h^2}{1008}\right)\cos\varphi\right.\\&\qquad-\left(\frac{1}{2}+\frac{11h}{24}+\frac{79h^2}{240}\right)\cos 3\varphi+\left(\frac{3h}{8}+\frac{63h^2}{112}\right)\cos 5\varphi\\&\qquad\left.-\frac{3h^2}{16}\cos 7\varphi\right]\frac{d\varphi}{dh}-\left(\frac{41}{180}+\frac{251h}{504}\right)\sin\varphi\end{aligned}$$

$$-\left(\frac{11}{72}+\frac{79h}{360}\right)\sin 3\varphi+\left(\frac{3}{40}+\frac{63h}{280}\right)\sin 5\varphi-\frac{3h}{56}\sin 7\varphi=0,$$

$$\left[\left(1-\frac{h}{6}-\frac{41h^2}{360}-\frac{251h^3}{3024}\right)\cos\varphi-\left(\frac{h}{2}+\frac{11h^2}{48}+\frac{71h^3}{720}\right)\cos 3\varphi\right.$$
$$\left.+\left(\frac{3h^2}{16}+\frac{21h^3}{112}\right)\cos 5\varphi-\frac{h^3}{16}\cos 7\varphi\right]\frac{d^3\varphi}{dh^3}-3\left[\left(1-\frac{h}{6}\right.\right.$$
$$\left.-\frac{41h^2}{360}-\frac{251h^3}{3024}\right)\sin\varphi-\left(\frac{3h}{2}+\frac{11h^2}{16}+\frac{71h^3}{240}\right)\sin 3\varphi$$
$$\left.+\left(\frac{15h^2}{16}+\frac{105h^3}{112}\right)\sin 5\varphi-\frac{7h^3}{16}\sin 7\varphi\right]\frac{d\varphi}{dh}\frac{d^2\varphi}{dh^2}$$
$$+3\left[-\left(\frac{1}{6}+\frac{41h}{180}+\frac{251h^2}{1008}\right)\cos\varphi-\left(\frac{1}{2}+\frac{11h}{24}\right.\right.$$
$$\left.\left.+\frac{71h^2}{240}\right)\cos 3\varphi+\left(\frac{3h}{8}+\frac{63h^2}{112}\right)\cos 5\varphi-\frac{3h^2}{16}\cos 7\varphi\right]\frac{d^2\varphi}{dh^2}$$
$$-\left[\left(1-\frac{h}{6}-\frac{41h^2}{360}-\frac{251h^3}{3024}\right)\cos\varphi-\left(\frac{9h}{2}+\frac{33h^2}{16}\right.\right.$$
$$\left.+\frac{71h^3}{80}\right)\cos 3\varphi+\left(\frac{75h^2}{16}+\frac{525h^3}{112}\right)\cos 5\varphi$$
$$\left.-\frac{49h^3}{16}\cos 7\varphi\right]\left(\frac{d\varphi}{dh}\right)^3-3\left[-\left(\frac{1}{6}+\frac{41h}{180}+\frac{251h^2}{1008}\right)\sin\varphi\right.$$
$$-\left(\frac{3}{2}+\frac{11h}{8}+\frac{71h^2}{80}\right)\sin 3\varphi+\left(\frac{15h}{8}+\frac{315h^2}{112}\right)\sin 5\varphi$$
$$\left.-\frac{21h^2}{16}\sin 7\varphi\right]\left(\frac{d\varphi}{dh}\right)^2+3\left[-\left(\frac{41}{180}+\frac{251h}{504}\right)\cos\varphi\right.$$
$$-\left(\frac{11}{24}+\frac{79h}{120}\right)\cos 3\varphi+\left(\frac{3}{8}+\frac{63h}{56}\right)\cos 5\varphi$$
$$\left.-\frac{3h}{8}\cos 7\varphi\right]\frac{d\varphi}{dh}-\frac{251}{504}\sin\varphi-\frac{79}{360}\sin 3\varphi$$
$$+\frac{63}{280}\sin 5\varphi-\frac{3}{56}\sin 7\varphi=0.$$

Denoting by brackets the value of the derivatives for $h=0$, and remembering that all functions of φ become functions of β for $h=0$, we obtain by successive substitutions and reductions (for the reductions see the reduction table, p. 88).

$$[\varphi]=\beta,$$

$$\left[\frac{d\varphi}{dh}\right]=\frac{2}{3}\sin\beta\cos\beta=\frac{1}{3}\sin 2\beta,$$

$$\left[\frac{d^2\varphi}{dh^2}\right]=\frac{8}{45}\sin\beta\cos\beta+\frac{46}{45}\sin\beta\cos^3\beta=\frac{31}{90}\sin 2\beta+\frac{23}{180}\sin 4\beta,$$

$$\left[\frac{d^3\varphi}{dh^3}\right]=\frac{76}{315}\sin\beta\cos\beta-\frac{32}{945}\sin\beta\cos^3\beta+\frac{3044}{945}\sin\beta\cos^5\beta$$
$$=\frac{517}{840}\sin 2\beta+\frac{251}{630}\sin 4\beta+\frac{761}{7560}\sin 6\beta.$$

By Maclaurin's theorem, we have

$$\varphi=[\varphi]+\frac{\epsilon^2}{1!}\left[\frac{d\varphi}{dh}\right]+\frac{\epsilon^4}{2!}\left[\frac{d^2\varphi}{dh^2}\right]+\frac{\epsilon^6}{3!}\left[\frac{d^3\varphi}{dh^3}\right]+\cdots\cdots.$$

By substituting the above values in this series and rearranging we obtain, as before, the approximation

$$\varphi-\beta=\left(\frac{\epsilon^2}{3}+\frac{31\epsilon^4}{180}+\frac{517\epsilon^6}{5040}\right)\sin 2\beta+\left(\frac{23\epsilon^4}{360}+\frac{251\epsilon^6}{3780}\right)\sin 4\beta$$
$$+\frac{761\epsilon^6}{45360}\sin 6\beta.$$

DEVELOPMENT OF $\varphi-\beta$ IN TERMS OF β—FIFTH METHOD.

By Lagrange's theorem $\varphi-\beta$ can be expressed in terms of β directly from the equation of definition. Let us take the equation in the form

$$x=y+\left(\frac{2\epsilon^2}{3}+\frac{34\epsilon^4}{45}+\frac{766\epsilon^6}{945}\right)\sin\varphi\cos^2\varphi-\left(\frac{3\epsilon^4}{5}+\frac{46\epsilon^6}{35}\right)\sin\varphi\cos^4\varphi$$
$$+\frac{4\epsilon^6}{7}\sin\varphi\cos^6\varphi,$$

in which $x=\sin\varphi$ and $y=\sin\beta$.

The series in ϵ^2 is a function of x through the functional relation between φ and x. We could substitute for the functions of φ their values in terms of x, but this is not necessary.

We wish to develop the function $\sin^{-1}x$ in terms of y or in terms of β through the functional relation between y and β. Since the series in ϵ^2 is a small quantity, Lagrange's series may be expressed in general terms in the form

$$f(x)=f(y)+\frac{1}{1!}g(y)f'(y)+\frac{1}{2!}\frac{d}{dy}\{[g(y)]^2f'(y)\}$$
$$+\frac{1}{3!}\frac{d^2}{dy^2}\{[g(y)]^3f'(y)\}+\frac{1}{4!}\frac{d^3}{dy^3}\{[g(y)]^4f'(y)\}+\cdots,$$

in which $f(x)$ represents the function of x to be developed, and $g(y)$ represents the series in ϵ^2 with φ replaced by β. The prime denotes differentiation with respect to y.
But

$$f(y)=\sin^{-1}y,$$

$$f'(y)=\frac{1}{(1-y^2)^{\frac{1}{2}}}=\frac{1}{\cos\beta}=\sec\beta.$$

Retaining all powers of ϵ up to the sixth, inclusive, we get

$$g(y)=\left(\frac{2\epsilon^2}{3}+\frac{34\epsilon^4}{45}+\frac{766\epsilon^6}{945}\right)\sin\beta\cos^2\beta-\left(\frac{3\epsilon^4}{5}+\frac{46\epsilon^6}{35}\right)\sin\beta\cos^4\beta+\frac{4\epsilon^6}{7}\sin\beta\cos^6\beta,$$

$$g(y)f'(y)=\left(\frac{2\epsilon^2}{3}+\frac{34\epsilon^4}{45}+\frac{766\epsilon^6}{945}\right)\sin\beta\cos\beta-\left(\frac{3\epsilon^4}{5}+\frac{46\epsilon^6}{35}\right)\sin\beta\cos^3\beta+\frac{4\epsilon^6}{7}\sin\beta\cos^5\beta,$$

$$[g(y)]^2=\left(\frac{4\epsilon^4}{9}+\frac{136\epsilon^6}{135}\right)\sin^2\beta\cos^4\beta-\frac{4\epsilon^6}{5}\sin^2\beta\cos^6\beta,$$

$$[g(y)]^2f'(y)=\left(\frac{4\epsilon^4}{9}+\frac{136\epsilon^6}{135}\right)\sin^2\beta\cos^3\beta-\frac{4\epsilon^6}{5}\sin^2\beta\cos^5\beta,$$

$$[g(y)]^3=\frac{8\epsilon^6}{27}\sin^3\beta\cos^6\beta,$$

$$[g(y)]^3f'(y)=\frac{8\epsilon^6}{27}\sin^3\beta\cos^5\beta.$$

To differentiate these expressions with respect to y, we merely differentiate with respect to β and multiply by $\frac{d\beta}{dy}$. But $\beta=f(y)$; so that $\frac{d\beta}{dy}=f'(y)=\sec\beta$.

For successive differentiations we differentiate with respect to β and multiply by $\sec\beta$, then differentiate again with respect to β and multiply again by $\sec\beta$, and so on for the remaining differentiations.

With this understanding, we get

$$\frac{d}{dy}\{[g(y)]^2f'(y)\}=\left(\frac{8\epsilon^4}{9}+\frac{272\epsilon^6}{135}\right)\sin\beta\cos^3\beta$$
$$-\left(\frac{4\epsilon^4}{3}+\frac{136\epsilon^6}{45}\right)\sin^3\beta\cos\beta-\frac{8\epsilon^6}{5}\sin\beta\cos^5\beta+4\epsilon^6\sin^3\beta\cos^3\beta$$
$$=-\left(\frac{4\epsilon^4}{3}+\frac{136\epsilon^6}{45}\right)\sin\beta\cos\beta+\left(\frac{20\epsilon^4}{9}+\frac{244\epsilon^6}{27}\right)\sin\beta\cos^3\beta$$
$$-\frac{28\epsilon^6}{5}\sin\beta\cos^5\beta,$$

$$\frac{d^2}{dy^2}\{[g(y)]^3f'(y)\}=\frac{16\epsilon^6}{9}\sin\beta\cos^5\beta-\frac{280\epsilon^6}{27}\sin^3\beta\cos^3\beta$$
$$+\frac{40\epsilon^6}{9}\sin^5\beta\cos\beta$$
$$=\frac{40\epsilon^6}{9}\sin\beta\cos\beta-\frac{520\epsilon^6}{27}\sin\beta\cos^3\beta$$
$$+\frac{448\epsilon^6}{27}\sin\beta\cos^5\beta.$$

Substituting these values in the Lagrange series above, we get the approximation

$$\varphi=\beta+\left(\frac{2\epsilon^2}{3}+\frac{34\epsilon^4}{45}+\frac{766\epsilon^6}{945}\right)\sin\beta\cos\beta-\left(\frac{3\epsilon^4}{5}+\frac{46\epsilon^6}{45}\right)\sin\beta\cos^3\beta$$
$$+\frac{4\epsilon^6}{7}\sin\beta\cos^5\beta-\left(\frac{2\epsilon^4}{3}+\frac{68\epsilon^6}{45}\right)\sin\beta\cos\beta+\left(\frac{10\epsilon^4}{9}\right.$$
$$\left.+\frac{122\epsilon^6}{27}\right)\sin\beta\cos^3\beta-\frac{14\epsilon^6}{5}\sin\beta\cos^5\beta+\frac{20\epsilon^6}{27}\sin\beta\cos\beta$$
$$-\frac{260\epsilon^6}{81}\sin\beta\cos^3\beta+\frac{224\epsilon^6}{81}\sin\beta\cos^5\beta.$$

By collecting similar terms, this becomes

$$\varphi=\beta+\left(\frac{2\epsilon^2}{3}+\frac{4\epsilon^4}{45}+\frac{38\epsilon^6}{945}\right)\sin\beta\cos\beta+\left(\frac{23\epsilon^4}{45}-\frac{16\epsilon^6}{2835}\right)\sin\beta\cos^3\beta$$
$$+\frac{1522\epsilon^6}{2835}\sin\beta\cos^5\beta.$$

On reduction by the table on page 88 and after rearrangement this becomes, as before, the desired approximation

$$\varphi-\beta=\left(\frac{\epsilon^2}{3}+\frac{31\epsilon^4}{180}+\frac{517\epsilon^6}{5040}\right)\sin 2\beta+\left(\frac{23\epsilon^4}{360}+\frac{251\epsilon^6}{3780}\right)\sin 4\beta$$
$$+\frac{761\epsilon^6}{45360}\sin 6\beta,$$

DEVELOPMENT OF $\varphi-\beta$ IN TERMS OF β—SIXTH METHOD.

The difference between the geodetic latitude and the authalic latitude can be developed in terms of the authalic latitude by Arbogast's rule. (See p. 28.) We may define the function to be developed in the form

$$\varphi=\sin^{-1}\Big[\sin\beta+\frac{h}{1!}\frac{2}{3}\sin\varphi\cos^2\varphi$$
$$+\frac{h^2}{2!}\Big(\frac{14}{45}\sin\varphi\cos^2\varphi+\frac{6}{5}\sin^3\varphi\cos^2\varphi\Big)$$
$$+\frac{h^3}{3!}\Big(\frac{128}{315}\sin\varphi\cos^2\varphi+\frac{36}{35}\sin^3\varphi\cos^2\varphi$$
$$+\frac{24}{7}\sin^5\varphi\cos^2\varphi\Big)+\quad\cdots\cdots\Big]=A_0+A_1\frac{h}{1!}$$
$$+A_2\frac{h^2}{2!}+A_3\frac{h^3}{3!}+\quad\cdots\cdots\cdots.$$

The A's are defined as before

$$A_0=f\,(a_0),$$
$$A_1=a_1f^1\,(a_0),$$
$$A_2=a_1^2f^2\,(a_0)+a_2f^1\,(a_0),$$
$$A_3=a_1^3f^3\,(a_0)+3a_1\,a_2f^2\,(a_0)+a_3f^1\,(a_0).$$

etc.

In this function we have

$$a_0=\sin\beta,$$
$$\frac{da_0}{d\beta}=\cos\beta,$$

or

$$\frac{d\beta}{da_0}=\sec\beta,$$

$$f\,(a_0)=\beta,$$

$$f^1\,(a_0)=\frac{d\beta}{da_0}=\sec\beta,$$

$$f^2\,(a_0)=\sec\beta\tan\beta\frac{d\beta}{da_0}=\sec^2\beta\tan\beta,$$

$$f^3\,(a_0)=(2\sec^2\beta\tan^2\beta+\sec^4\beta)\frac{d\beta}{da_0}$$
$$=2\sec^3\beta\tan^2\beta+\sec^5\beta,$$

$$a_1 = \frac{2}{3} \sin \varphi \cos^2 \varphi,$$

$$a_2 = \frac{14}{45} \sin \varphi \cos^2 \varphi + \frac{6}{5} \sin^3 \varphi \cos^2 \varphi,$$

$$a_3 = \frac{128}{315} \sin \varphi \cos^2 \varphi + \frac{36}{35} \sin^3 \varphi \cos^2 \varphi + \frac{24}{7} \sin^5 \varphi \cos^2 \varphi.$$

With these values of the a's and the above values of the derivatives of $f(a_0)$ we could compute the values of the A's, but if we should do so at this stage we should have a combination of functions of φ and β in the development. To obviate this difficulty we must obtain approximations for the a's in terms of functions of β and powers of h. The expression for a_1 must include all first and second powers of h and the expression for a_2 must include all first powers of h. In a_3 we need only to replace φ by β in the given expression.

From the definition of the function to be expanded we see that the approximation to the first power of h for $\sin \varphi$ is given by the expression

$$\sin \varphi = \sin \beta + \frac{2}{3} h \sin \beta \cos^2 \beta.$$

Substituting this value for $\sin \varphi$ in the expression for $\sin \varphi$ and retaining all second powers of h, as well as the first powers, we get

$$\begin{aligned}\sin \varphi = \sin \beta &+ \frac{2}{3} h \sin \beta \cos^2 \beta - \frac{4}{3} h^2 \sin^3 \beta \cos^2 \beta \\ &+ \frac{4}{9} h^2 \sin \beta \cos^2 \beta + \frac{7}{45} h^2 \sin \beta \cos^2 \beta \\ &+ \frac{3}{5} h^2 \sin^3 \beta \cos^2 \beta,\end{aligned}$$

or

$$\begin{aligned}\sin \varphi = \sin \beta &+ \frac{2}{3} h \sin \beta \cos^2 \beta + \frac{3}{5} h^2 \sin \beta \cos^2 \beta \\ &- \frac{11}{15} h^2 \sin^3 \beta \cos^2 \beta.\end{aligned}$$

Substituting this approximation in the expressions for the a's, we get, to the required degree of exactness,

$$a_1=\frac{2}{3}\Big(\sin\beta\ \cos^2\beta+\frac{2}{3}h\sin\beta\cos^2\beta-2h\sin^3\beta\cos^2\beta$$
$$+\frac{3}{5}h^2\sin\beta\cos^2\beta-\frac{58}{15}h^2\sin^3\beta\cos^2\beta$$
$$+\frac{53}{15}h^2\sin^5\beta\cos^2\beta\Big),$$

$$a_2=\frac{14}{45}\sin\beta\cos^2\beta+\frac{6}{5}\sin^3\beta\cos^2\beta+\frac{28}{135}h\sin\beta\cos^2\beta$$
$$+\frac{16}{9}h\sin^3\beta\cos^2\beta-4h\sin^5\beta\cos^2\beta,$$

$$a_3=\frac{128}{315}\sin\beta\cos^2\beta+\frac{36}{35}\sin^3\beta\cos^2\beta+\frac{24}{7}\sin^5\beta\cos^2\beta.$$

With these approximations, the approximations for the A's become (for the reductions see table p. 88)

$$A_0=\beta,$$

$$A_1=\frac{2}{3}\Big(\sin\beta\cos\beta+\frac{2}{3}h\sin\beta\cos\beta-2h\sin^3\beta\cos\beta$$
$$+\frac{3}{5}h^2\sin\beta\cos\beta-\frac{58}{15}h^2\sin^3\beta\cos\beta$$
$$+\frac{53}{15}h^2\sin^5\beta\cos\beta\Big),$$

$$A_2=\frac{4}{9}\Big(\sin^3\beta\cos\beta+\frac{4}{3}h\sin^3\beta\cos\beta-4h\sin^5\beta\cos\beta\Big)$$
$$+\frac{14}{45}\sin\beta\cos\beta+\frac{6}{5}\sin^3\beta\cos\beta+\frac{128}{315}h\sin\beta\cos\beta$$
$$+\frac{16}{9}h\sin^3\beta\cos\beta-4h\sin^5\beta\cos\beta$$
$$=\frac{14}{45}\sin\beta\cos\beta+\frac{74}{45}\sin^3\beta\cos\beta+\frac{28}{135}h\sin\beta\cos\beta$$
$$+\frac{64}{27}h\sin^3\beta\cos\beta-\frac{52}{9}h\sin^5\beta\cos\beta,$$

$$A_8=\frac{8}{27}\sin^8\beta\cos^6\beta\,(2\sec^3\beta\tan^2\beta+\sec^5\beta)$$
$$+\left(\frac{28}{45}\sin^2\beta\cos^4\beta+\frac{12}{5}\sin^4\beta\cos^4\beta\right)\sec^2\beta\tan\beta$$
$$+\frac{128}{315}\sin\beta\cos\beta+\frac{36}{35}\sin^3\beta\cos\beta+\frac{24}{7}\sin^5\beta\cos\beta$$
$$=\frac{16}{27}\sin^5\beta\cos\beta+\frac{8}{27}\sin^3\beta\cos\beta+\frac{28}{45}\sin^3\beta\cos\beta$$
$$+\frac{12}{5}\sin^5\beta\cos\beta+\frac{128}{315}\sin\beta\cos\beta+\frac{36}{35}\sin^3\beta\cos\beta$$
$$+\frac{24}{7}\sin^5\beta\cos\beta$$
$$=\frac{128}{315}\sin\beta\cos\beta+\frac{368}{189}\sin^3\beta\cos\beta$$
$$+\frac{6068}{945}\sin^5\beta\cos\beta.$$

Substituting these values in the development, we get the approximation

$$\varphi=\beta+\frac{2h}{3}\Big(\sin\beta\cos\beta+\frac{2}{3}h\sin\beta\cos\beta-2h\sin^3\beta\cos\beta$$
$$+\frac{3}{5}h^2\sin\beta\cos\beta-\frac{58}{15}h^2\sin^3\beta\cos\beta$$
$$+\frac{53}{15}h^2\sin^5\beta\cos\beta\Big)+\frac{h^2}{2}\Big(\frac{14}{45}\sin\beta\cos\beta$$
$$+\frac{74}{45}\sin^3\beta\cos\beta+\frac{28}{135}h\sin\beta\cos\beta$$
$$+\frac{64}{27}h\sin^3\beta\cos\beta-\frac{52}{9}h\sin^5\beta\cos\beta\Big)$$
$$+\frac{h^3}{6}\Big(\frac{128}{315}\sin\beta\cos\beta+\frac{368}{189}\sin^3\beta\cos\beta$$
$$+\frac{6068}{945}\sin^5\beta\cos\beta\Big).$$

Rearranging in powers of h and replacing h by ϵ^2 we get the approximation

$$\varphi=\beta+\frac{2\epsilon^2}{3}\sin\beta\cos\beta+\epsilon^4\left(\frac{3}{5}\sin\beta\cos\beta-\frac{23}{45}\sin^3\beta\cos\beta\right)$$
$$+\epsilon^6\Big(\frac{4}{7}\sin\beta\cos\beta-\frac{3028}{2835}\sin^3\beta\cos\beta$$
$$+\frac{1522}{2835}\sin^5\beta\cos\beta\Big).$$

Making the reductions by aid of the table on page 88 and rearranging, we obtain, as before, the desired approximation

$$\varphi-\beta=\left(\frac{\epsilon^2}{3}+\frac{31\epsilon^4}{180}+\frac{517\epsilon^6}{5040}\right)\sin 2\beta+\left(\frac{23\epsilon^4}{360}+\frac{251\epsilon^6}{3780}\right)\sin 4\beta+\frac{761\epsilon^6}{45360}\sin 6\beta.$$

TABULATION OF ALL THE DEVELOPMENTS.

For convenience of reference we shall now list all of the developments in a general table.

$$m=\frac{\epsilon^2}{2-\epsilon^2}$$

$$\varphi-\psi=m\sin 2\varphi-\frac{m^2}{2}\sin 4\varphi+\frac{m^3}{3}\sin 6\varphi-\cdots\cdot$$

$$\varphi-\psi=m\sin 2\psi+\frac{m^2}{2}\sin 4\psi+\frac{m^3}{3}\sin 6\psi+\cdots\cdot$$

$$n=\frac{1-(1-\epsilon^2)^{1/2}}{1+(1-\epsilon^2)^{1/2}}$$

$$\varphi-\theta=n\sin 2\varphi-\frac{n^2}{2}\sin 4\varphi+\frac{n^3}{3}\sin 6\varphi-\cdots\cdot$$

$$\varphi-\theta=n\sin 2\theta+\frac{n^2}{2}\sin 4\theta+\frac{n^3}{3}\sin 6\theta+\cdots\cdot$$

$$\theta-\psi=n\sin 2\theta-\frac{n^2}{2}\sin 4\theta+\frac{n^3}{3}\sin 6\theta-\cdots\cdot$$

$$\theta-\psi=n\sin 2\psi+\frac{n^2}{2}\sin 4\psi+\frac{n^3}{3}\sin 6\psi+\cdots\cdot$$

$$\varphi-\chi=\left(\frac{\epsilon^2}{2}+\frac{5\epsilon^4}{24}+\frac{3\epsilon^6}{32}+\frac{281\epsilon^8}{5760}+\cdots\cdot\right)\sin 2\varphi-\left(\frac{5\epsilon^4}{48}+\frac{7\epsilon^6}{80}+\frac{697\epsilon^8}{11520}+\cdots\cdot\right)\sin 4\varphi+\left(\frac{13\epsilon^6}{480}+\frac{461\epsilon^8}{13440}+\cdots\cdot\right)\sin 6\varphi-\left(\frac{1237\epsilon^8}{161280}+\cdots\cdot\right)\sin 8\varphi+\cdots\cdot$$

$$\varphi-\chi=\left(\frac{\epsilon^2}{2}+\frac{5\epsilon^4}{24}+\frac{\epsilon^6}{12}+\frac{13\epsilon^8}{360}+\dots\right)\sin 2\chi$$
$$+\left(\frac{7\epsilon^4}{48}+\frac{29\epsilon^6}{240}+\frac{811\epsilon^8}{11520}+\dots\right)\sin 4\chi+\left(\frac{7\epsilon^6}{120}\right.$$
$$\left.+\frac{81\epsilon^8}{1120}+\dots\right)\sin 6\chi$$
$$+\left(\frac{4279\epsilon^8}{161280}+\dots\right)\sin 8\chi+\dots$$

$$\varphi-\beta=\left(\frac{\epsilon^2}{3}+\frac{31\epsilon^4}{180}+\frac{59\epsilon^6}{560}+\dots\right)\sin 2\varphi-\left(\frac{17\epsilon^4}{360}\right.$$
$$\left.+\frac{61\epsilon^6}{1260}+\dots\right)\sin 4\varphi+\left(\frac{383\epsilon^6}{45360}+\dots\right)\sin 6\varphi-\dots$$

$$\varphi-\beta=\left(\frac{\epsilon^2}{3}+\frac{31\epsilon^4}{180}+\frac{517\epsilon^6}{5040}+\dots\right)\sin 2\beta+\left(\frac{23\epsilon^4}{360}\right.$$
$$\left.+\frac{251\epsilon^6}{3780}+\dots\right)\sin 4\beta+\left(\frac{761\epsilon^6}{45360}+\dots\right)\sin 6\beta+\dots$$

If the various differences of latitude were computed as they are here given, the results would be expressed in radians. It is most convenient to have them expressed in seconds of arc; the results would therefore have to be divided by the arc of one second or by the sine of one second, since the arc and sine of one second are much more nearly equal than the approximation requires. In practice it is better to divide each of the coefficients in the above developments by the sine of one second, since in this manner we may transform from radians to seconds of arc by one operation. The various coefficients will then be expressed in seconds and the result of any computation will be in seconds of arc.

DETERMINATION OF THE NUMERICAL VALUE OF THE COEFFICIENTS IN THE DEVELOPMENTS FOR THE CLARKE SPHEROID OF 1866.

For computation purposes it is necessary to have the coefficients in these developments expressed as numbers or as logarithms.

The division of each of the coefficients by sin 1″ is indicated in the first development given below, and the same process must be applied in the case of each of the other developments.

We assume ϵ^2 as defined by

$$\log \epsilon^2 = 7.83050257 - 10;$$

hence

$$\epsilon^2 = 0.006768658$$

$$m = \frac{\epsilon^2}{2-\epsilon^2}$$

$$\log m = 7.53094486 - 10.$$

If $\varphi - \psi$ is to be expressed in seconds, we get

$$\varphi - \psi = \frac{m}{\sin 1''} \sin 2\varphi - \frac{m^2}{2 \sin 1''} \sin 4\varphi + \frac{m^3}{3 \sin 1''} \sin 6\varphi - \dots$$

or,

$$\varphi - \psi = 700''.4385 \sin 2\varphi - 1''.1893 \sin 4\varphi + 0''.0027 \sin 6\varphi,$$

or in terms of logarithms

$$\varphi - \psi = [2.8453700] \sin 2\varphi - [0.075285] \sin 4\varphi + [7.430 - 10] \sin 6\varphi.$$

Also

$$\varphi - \psi = 700''.4385 \sin 2\psi + 1''.1893 \sin 4\psi + 0''.0027 \sin 6\psi$$

and

$$\varphi - \psi = [2.8453700] \sin 2\psi + [0.075285] \sin 4\psi + [7.430 - 10] \sin 6\psi.$$

Furthermore

$$n = \frac{1-(1-\epsilon^2)^{1/2}}{1+(1-\epsilon^2)^{1/2}} = \frac{\epsilon^2}{4} + \frac{\epsilon^4}{8} + \frac{5\epsilon^6}{64} + \frac{7\epsilon^8}{128} + \dots$$

$$\log n = 7.22991610 - 10,$$

$$\varphi - \theta = 350''.2202 \sin 2\varphi - 0''.2973 \sin 4\varphi + 0''.0003 \sin 6\varphi,$$

$$\varphi - \theta = [2.5443412] \sin 2\varphi - [9.47323 - 10] \sin 4\varphi + [6.527 - 10] \sin 6\varphi.$$

Also

$$\varphi-\theta=350''.2202 \sin 2\theta+0''.2973 \sin 4\theta+0''.0003 \sin 6\theta,$$

$$\varphi-\theta=[2.5443412] \sin 2\theta+[9.47323-10] \sin 4\theta$$
$$+[6.527-10] \sin 6\theta.$$

In a similar manner we have

$$\theta-\psi=350''.2202 \sin 2\theta-0''.2973 \sin 4\theta+0''.0003 \sin 6\theta,$$

$$\theta-\psi=[2.5443412] \sin 2\theta-[9.47323-10] \sin 4\theta$$
$$+[6.527-10] \sin 6\theta,$$

and

$$\theta-\psi=350''.2202 \sin 2\psi+0''.2973 \sin 4\psi+0''.0003 \sin 6\psi,$$

$$\theta-\psi=[2.5443412] \sin 2\psi+[9.47323-10] \sin 4\psi$$
$$+[6.527-10] \sin 6\psi,$$

$$\varphi-\chi=700''.0427 \sin 2\varphi-0''.9900 \sin 4\varphi+0''.0017 \sin 6\varphi,$$

$$\varphi-\chi=[2.84512455] \sin 2\varphi-[9.99563-10] \sin 4\varphi$$
$$+[7.238-10] \sin 6\varphi,$$

$$\varphi-\chi=700''.0420 \sin 2\chi+1''.3859 \sin 4\chi+0''.0037 \sin 6\chi,$$

$$\varphi-\chi=[2.84512413] \sin 2\chi+[0.141726] \sin 4\chi$$
$$+[7.572-10] \sin 6\chi,$$

$$\varphi-\beta=467''.0129 \sin 2\varphi-0''.4494 \sin 4\varphi+0''.0005 \sin 6\varphi,$$

$$\varphi-\beta=[2.6693289] \sin 2\varphi-[9.65258-10] \sin 4\varphi$$
$$+[6.732-10] \sin 6\varphi,$$

and

$$\varphi-\beta=467''.0127 \sin 2\beta+0''.6080 \sin 4\beta+0''.0011 \sin 6\beta,$$

$$\varphi-\beta=[2.6693287] \sin 2\beta+[9.78390-10] \sin 4\beta$$
$$+[7.031-10] \sin 6\beta.$$

The radius of the sphere equivalent in area to the ellipsoid of revolution is defined by the formula

$$R=a\left(1-\frac{\epsilon^2}{6}-\frac{17\epsilon^4}{360}-\frac{67\epsilon^6}{3024}- \quad . \quad . \quad . \quad .\right),$$

in which a is the equatorial radius of the ellipsoid.

For the Clarke ellipsoid of 1866

$$\log R=6.80420742$$

$$R=6370997.2 \text{ meters.}$$

By using the authalic latitudes and this value of R, the spheroid can be treated as a sphere in all questions of equivalent or equal-area mapping.

The appended tables of the various latitude differences are computed for the Clarke spheroid of 1866. Interpolation could be made with the table by the consideration of second differences to a degree of accuracy sufficient for ordinary purposes. The differences are given in thousandths of a second in order that the differences between adjacent values may be better preserved. The table of transformation to isometric latitude is given in a form somewhat different from the others. The isometric colatitude is given and in another column this value divided by two. This is done because in the most important applications of this latitude it is most convenient to use the semicolatitude. The table thus gives at once the value that is needed for use.

REDUCTION TABLE.

$$\sin^2 a = \frac{1}{2}(1 - \cos 2a)$$

$$\sin^3 a = \frac{1}{4}(3 \sin a - \sin 3a)$$

$$\sin^4 a = \frac{1}{8}(3 - 4 \cos 2a + \cos 4a)$$

$$\sin^5 a = \frac{1}{16}(10 \sin a - 5 \sin 3a + \sin 5a)$$

$$\sin^6 a = \frac{1}{32}(10 - 15 \cos 2a + 6 \cos 4a - \cos 6a)$$

$$\sin^7 a = \frac{1}{64}(35 \sin a - 21 \sin 3a + 7 \sin 5a - \sin 7a)$$

$$\cos^2 a = \frac{1}{2}(1 + \cos 2a)$$

$$\cos^3 \quad = \frac{1}{4}(3 \cos a + \cos 3a)$$

$$\cos^4 a = \frac{1}{8}(3 + 4 \cos 2a + \cos 4a)$$

$$\cos^5 a = \frac{1}{16}(10 \cos a + 5 \cos 3a + \cos 5a)$$

$$\cos^6 a = \frac{1}{32}(10 + 15 \cos 2a + 6 \cos 4a + \cos 6a)$$

$$\cos^7 a = \frac{1}{64}(35 \cos a + 21 \cos 3a + 7 \cos 5a + \cos 7a)$$

$$\sin 2a = 2 \sin a \cos a,$$

$$\sin 3a = 4 \sin a \cos^2 a - \sin a,$$

$$\sin 4a = 8 \sin a \cos^3 a - 4 \sin a \cos a,$$

$$\sin 5a = 16 \sin a \cos^4 a - 12 \sin a \cos^2 a + \sin a,$$

$$\sin 6a = 32 \sin a \cos^5 a - 32 \sin a \cos^3 a + 6 \sin a \cos a,$$

$$\sin 7a = 64 \sin a \cos^6 a - 80 \sin a \cos^4 a + 24 \sin a \cos^2 a - \sin a,$$

$$\sin 8a = 128 \sin a \cos^7 a - 192 \sin a \cos^5 a + 80 \sin a \cos^3 a + 8 \sin a \cos a,$$

$$\cos 2a = 2 \cos^2 a - 1,$$

$$\cos 3a = 4 \cos^3 a - 3 \cos a,$$

$$\cos 4a = 8 \cos^4 a - 8 \cos^2 a + 1,$$

$$\cos 5a = 16 \cos^5 a - 20 \cos^3 a + 5 \cos a,$$

$$\cos 6a = 32 \cos^6 a - 48 \cos^4 a + 18 \cos^2 a - 1,$$

$$\cos 7a = 64 \cos^7 a - 112 \cos^5 a + 56 \cos^3 a - 7 \cos a,$$

$$\cos 8a = 128 \cos^8 a - 256 \cos^6 a + 160 \cos^4 a - 32 \cos^2 a + 1,$$

$$\sin a \cos b = \frac{1}{2}[\sin (a+b) + \sin (a-b)],\ a > b,$$

$$\sin a \cos b = \frac{1}{2}[\sin (b+a) - \sin (b-a)],\ b > a,$$

$$\cos a \cos b = \frac{1}{2}[\cos (a-b) + \cos (a+b)],$$

$$\sin a \sin b = \frac{1}{2} [\cos (a-b) - \cos (a+b)],$$

$$\tan \frac{a}{2} = \frac{\sin a}{1+\cos a} = \frac{1-\cos a}{\sin a},$$

$$\tan \frac{a}{2} \sin a = 1 - \cos a,$$

$$\sec^2 \frac{a}{2} \sin^2 a = 2 (1 - \cos a),$$

$$\sin a + \sin b = 2 \sin \frac{1}{2} (a+b) \cos \frac{1}{2} (a-b),$$

$$\sin a - \sin b = 2 \sin \frac{1}{2} (a-b) \cos \frac{1}{2} (a+b),$$

$$\cos a + \cos b = 2 \cos \frac{1}{2} (a+b) \cos \frac{1}{2} (a-b),$$

$$\cos a - \cos b = -2 \sin \frac{1}{2} (a+b) \sin \frac{1}{2} (a-b),$$

$$\sin a \cos a = \frac{1}{2} \sin 2a,$$

$$\sin a \cos^2 a = \frac{1}{4} \sin a + \frac{1}{4} \sin 3a,$$

$$\sin a \cos^3 a = \frac{1}{4} \sin 2a + \frac{1}{8} \sin 4a,$$

$$\sin a \cos^4 a = \frac{1}{8} \sin a + \frac{3}{16} \sin 3a + \frac{1}{16} \sin 5a,$$

$$\sin a \cos^5 a = \frac{5}{32} \sin 2a + \frac{1}{8} \sin 4a + \frac{1}{32} \sin 6a,$$

$$\sin a \cos^6 a = \frac{5}{64} \sin a + \frac{9}{64} \sin 3a + \frac{5}{64} \sin 5a + \frac{1}{64} \sin 7a,$$

$$\sin a \cos^7 a = \frac{7}{64} \sin 2a + \frac{7}{64} \sin 4a + \frac{3}{64} \sin 6a + \frac{1}{128} \sin 8a.$$

LATITUDE TRANSFORMATION.

Geodetic to geocentric.

Geodetic latitude. φ	Geodetic minus geocentric. $\varphi-\psi$	Geocentric latitude. ψ	Geodetic latitude. φ	Geodetic minus geocentric. $\varphi-\psi$	Geocentric latitude. ψ
° ′	′ ″	° ′ ″	° ′	′ ″	° ′ ″
0 00	0 00.000	0 00 00.00	22 30	8 14.097	22 21 45.90
0 30	0 12.183	0 29 47.82	23 00	8 22.666	22 51 37.33
1 00	0 24.362	0 59 35.64	23 30	8 31.084	23 21 28.92
1 30	0 36.534	1 29 23.47	24 00	8 39.346	23 51 20.65
2 00	0 48.695	1 59 11.30	24 30	8 47.451	24 21 12.55
2 30	1 00.841	2 28 59.16	25 00	8 55.397	24 51 04.60
3 00	1 12.969	2 58 47.03	25 30	9 03.181	25 20 56.82
3 30	1 25.075	3 28 34.92	26 00	9 10.800	25 50 49.20
4 00	1 37.156	3 58 22.84	26 30	9 18.253	26 20 41.75
4 30	1 49.206	4 28 10.79	27 00	9 25.536	26 50 34.46
5 00	2 01.224	4 57 58.78	27 30	9 32.649	27 20 27.35
5 30	2 13.206	5 27 46.79	28 00	9 39.588	27 50 20.41
6 00	2 25.147	5 57 34.85	28 30	9 46.351	28 20 13.65
6 30	2 37.045	6 27 22.96	29 00	9 52.937	28 50 07.06
7 00	2 48.895	6 57 11.11	29 30	9 59.343	29 20 00.66
7 30	3 00.694	7 26 59.31	30 00	10 05.568	29 49 54.43
8 00	3 12.439	7 56 47.56	30 30	10 11.609	30 19 48.39
8 30	3 24.126	8 26 35.87	31 00	10 17.464	30 49 42.54
9 00	3 35.750	8 56 24.25	31 30	10 23.133	31 19 36.87
9 30	3 47.311	9 26 12.69	32 00	10 28.612	31 49 31.39
10 00	3 58.802	9 56 01.20	32 30	10 33.901	32 19 26.10
10 30	4 10.221	10 25 49.78	33 00	10 38.998	32 49 21.00
11 00	4 21.565	10 55 38.43	33 30	10 43.900	33 19 16.10
11 30	4 32.830	11 25 27.17	34 00	10 48.608	33 49 11.39
12 00	4 44.013	11 55 15.99	34 30	10 53.119	34 19 06.88
12 30	4 55.110	12 25 04.89	35 00	10 57.431	34 49 02.57
13 00	5 06.117	12 54 53.88	35 30	11 01.544	35 18 58.46
13 30	5 17.033	13 24 42.97	36 00	11 05.456	35 48 54.54
14 00	5 27.853	13 54 32.15	36 30	11 09.166	36 18 50.83
14 30	5 38.573	14 24 21.43	37 00	11 12.673	36 48 47.33
15 00	5 49.192	14 54 10.81	37 30	11 15.975	37 18 44.02
15 30	5 59.705	15 24 00.29	38 00	11 19.072	37 48 40.93
16 00	6 10.110	15 53 49.89	38 30	11 21.963	38 18 38.04
16 30	6 20.402	16 23 39.60	39 00	11 24.646	38 48 35.35
17 00	6 30.580	16 53 29.42	39 30	11 27.122	39 18 32.88
17 30	6 40.640	17 23 19.36	40 00	11 29.388	39 48 30.61
18 00	6 50.579	17 53 09.42	40 30	11 31.445	40 18 28.56
18 30	7 00.394	18 22 59.61	41 00	11 33.292	40 48 26.71
19 00	7 10.082	18 52 49.92	41 30	11 34.927	41 18 25.07
19 30	7 19.639	19 22 40.36	42 00	11 36.352	41 48 23.65
20 00	7 29.064	19 52 30.94	42 30	11 37.564	42 18 22.44
20 30	7 38.354	20 22 21.65	43 00	11 38.564	42 48 21.44
21 00	7 47.504	20 52 12.50	43 30	11 39.352	43 18 20.65
21 30	7 56.513	21 22 03.49	44 00	11 39.926	43 48 20.07
22 00	8 05.379	21 51 54.62	44 30	11 40.288	44 18 19.71
22 30	8 14.097	22 21 45.90	45 00	11 40.436	44 48 19.56

$\varphi-\psi = +700''.4385 \sin 2\varphi - 1''.1893 \sin 4\varphi + 0''.0027 \sin 6\varphi.$

$\varphi-\psi = [2.8453700] \sin 2\varphi - [0.075285] \sin 4\varphi + [7.430-10] \sin 6\varphi.$

LATITUDE TRANSFORMATION—Continued.

Geodetic to geocentric—Continued.

Geodetic latitude. φ	Geodetic minus geocentric. $\varphi-\psi$	Geocentric latitude. ψ	Geodetic latitude. φ	Geodetic minus geocentric. $\varphi-\psi$	Geocentric latitude. ψ
° ′	′ ″	° ′ ″	° ′	′ ″	° ′ ″
45 00	11 40.436	44 48 19.56	67 30	8 16.476	67 21 43.52
45 30	11 40.371	45 18 19.63	68 00	8 07.756	67 51 52.24
46 00	11 40.092	45 48 19.91	68 30	7 58.886	68 22 01.11
46 30	11 39.600	46 18 20.40	69 00	7 49.870	68 52 10.13
47 00	11 38.895	46 48 21.10	69 30	7 40.709	69 22 19.29
47 30	11 37.977	47 18 22.02	70 00	7 31.407	69 52 28.59
48 00	11 36.846	47 48 23.15	70 30	7 21.966	70 22 38.03
48 30	11 35.503	48 18 24.50	71 00	7 12.390	70 52 47.61
49 00	11 33.947	48 48 26.05	71 30	7 02.680	71 22 57.32
49 30	11 32.180	49 18 27.82	72 00	6 52.841	71 53 07.16
50 00	11 30.202	49 48 29.80	72 30	6 42.875	72 23 17.12
50 30	11 28.013	50 18 31.99	73 00	6 32.786	72 53 27.21
51 00	11 25.614	50 48 34.39	73 30	6 22.575	73 23 37.42
51 30	11 23.006	51 18 36.99	74 00	6 12.248	73 53 47.75
52 00	11 20.189	51 48 39.81	74 30	6 01.805	74 23 58.19
52 30	11 17.164	52 18 42.84	75 00	5 51.252	74 54 08.75
53 00	11 13.933	52 48 46.07	75 30	5 40.591	75 24 19.41
53 30	11 10.496	53 18 49.50	76 00	5 29.825	75 54 30.17
54 00	11 06.854	53 48 53.15	76 30	5 18.957	76 24 41.04
54 30	11 03.008	54 18 56.99	77 00	5 07.992	76 54 52.01
55 00	10 58.960	54 49 01.04	77 30	4 56.932	77 25 03.07
55 30	10 54.710	55 19 05.29	78 00	4 45.780	77 55 14.22
56 00	10 50.260	55 49 09.74	78 30	4 34.541	78 25 25.46
56 30	10 45.612	56 19 14.39	79 00	4 23.218	78 55 36.78
57 00	10 40.765	56 49 19.23	79 30	4 11.813	79 25 48.19
57 30	10 35.723	57 19 24.28	80 00	4 00.331	79 55 59.67
58 00	10 30.487	57 49 29.51	80 30	3 48.775	80 26 11.22
58 30	10 25.057	58 19 34.94	81 00	3 37.149	80 56 22.85
59 00	10 19.436	58 49 40.56	81 30	3 25.456	81 26 34.54
59 30	10 13.626	59 19 46.37	82 00	3 13.699	81 56 46.30
60 00	10 07.628	59 49 52.37	82 30	3 01.883	82 26 58.12
60 30	10 01.443	60 19 58.56	83 00	2 50.012	82 57 09.99
61 00	9 55.075	60 50 04.93	83 30	2 38.088	83 27 21.91
61 30	9 48.524	61 20 11.48	84 00	2 26.115	83 57 33.88
62 00	9 41.793	61 50 18.21	84 30	2 14.097	84 27 45.90
62 30	9 34.884	62 20 25.12	85 00	2 02.038	84 57 57.96
63 00	9 27.799	62 50 32.20	85 30	1 49.941	85 28 10.06
63 30	9 20.539	63 20 39.46	86 00	1 37.811	85 58 22.19
64 00	9 13.108	63 50 46.89	86 30	1 25.651	86 28 34.35
64 30	9 05.508	64 20 54.49	87 00	1 13.464	86 58 46.54
65 00	8 57.740	64 51 02.26	87 30	1 01.254	87 28 58.75
65 30	8 49.807	65 21 10.19	88 00	0 49.026	87 59 10.97
66 00	8 41.712	65 51 18.29	88 30	0 36.783	88 29 23.22
66 30	8 33.456	66 21 26.54	89 00	0 24.528	88 59 35.47
67 00	8 25.044	66 51 34.96	89 30	0 12.266	89 29 47.73
67 30	8 16.476	67 21 43.52	90 00	0 00.000	90 00 00.00

$\varphi-\psi=+700''.4385 \sin 2\varphi-1''.1893 \sin 4\varphi+0''.0027 \sin 6\varphi.$

$\varphi-\psi=[2.8453700] \sin 2\varphi-[0.075285] \sin 4\varphi+[7.430-10] \sin 6\varphi.$

LATITUDE TRANSFORMATION—Continued.

Geocentric to geodetic.

Geocentric latitude. ψ	Geodetic minus geocentric. $\varphi-\psi$	Geodetic latitude. φ	Geocentric latitude. ψ	Geodetic minus geocentric. $\varphi-\psi$	Geodetic latitude. φ
° '	' "	° ' "	° '	' "	° ' "
0 00	0 00.000	0 00 00.00	22 30	8 16.476	22 38 16.48
0 30	0 12.266	0 30 12.27	23 00	8 25.044	23 08 25.04
1 00	0 24.528	1 00 24.53	23 30	8 33.456	23 38 33.46
1 30	0 36.783	1 30 36.78	24 00	8 41.712	24 08 41.71
2 00	0 49.026	2 00 49.03	24 30	8 49.807	24 38 49.81
2 30	1 01.254	2 31 01.25	25 00	8 57.740	25 08 57.74
3 00	1 13.464	3 01 13.46	25 30	9 05.508	25 39 05.51
3 30	1 25.651	3 31 25.65	26 00	9 13.108	26 09 13.11
4 00	1 37.811	4 01 37.81	26 30	9 20.539	26 39 20.54
4 30	1 49.941	4 31 49.94	27 00	9 27.799	27 09 27.80
5 00	2 02.038	5 02 02.04	27 30	9 34.884	27 39 34.88
5 30	2 14.097	5 32 14.10	28 00	9 41.793	28 09 41.79
6 00	2 26.115	6 02 26.12	28 30	9 48.524	28 39 48.52
6 30	2 38.088	6 32 38.09	29 00	9 55.075	29 09 55.07
7 00	2 50.012	7 02 50.01	29 30	10 01.443	29 40 01.44
7 30	3 01.883	7 33 01.88	30 00	10 07.628	30 10 07.63
8 00	3 13.699	8 03 13.70	30 30	10 13.626	30 40 13.63
8 30	3 25.456	8 33 25.46	31 00	10 19.436	31 10 19.44
9 00	3 37.149	9 03 37.15	31 30	10 25.057	31 40 25.06
9 30	3 48.775	9 33 48.78	32 00	10 30.487	32 10 30.49
10 00	4 00 331	10 04 00.33	32 30	10 35.723	32 40 35.72
10 30	4 11.813	10 34 11.81	33 00	10 40.765	33 10 40.77
11 00	4 23.218	11 04 23.22	33 30	10 45.612	33 40 45.61
11 30	4 34.541	11 34 34.54	34 00	10 50.260	34 10 50.26
12 00	4 45.780	12 04 45.78	34 30	10 54.710	34 40 54.71
12 30	4 56.932	12 34 56.93	35 00	10 58.960	35 10 58.96
13 00	5 07.992	13 05 07.99	35 30	11 03.008	35 41 03.01
13 30	5 18.957	13 35 18.96	36 00	11 06.854	36 11 06.85
14 00	5 29.825	14 05 29.83	36 30	11 10.496	36 41 10.50
14 30	5 40.591	14 35 40.59	37 00	11 13.933	37 11 13.93
15 00	5 51.252	15 05 51.25	37 30	11 17.164	37 41 17.16
15 30	6 01.805	15 36 01.81	38 00	11 20.189	38 11 20.19
16 00	6 12.248	16 06 12.25	38 30	11 23.006	38 41 23.01
16 30	6 22.575	16 36 22.58	39 00	11 25.614	39 11 25.61
17 00	6 32.786	17 06 32.79	39 30	11 28.013	39 41 28.01
17 30	6 42.875	17 36 42.88	40 00	11 30.202	40 11 30.20
18 00	6 52.841	18 06 52.84	40 30	11 32.180	40 41 32.18
18 30	7 02.680	18 37 02.68	41 00	11 33.947	41 11 33.95
19 00	7 12.390	19 07 12.39	41 30	11 35.503	41 41 35.50
19 30	7 21.966	19 37 21.97	42 00	11 36.846	42 11 36.85
20 00	7 31.407	20 07 31.41	42 30	11 37.977	42 41 37.98
20 30	7 40.709	20 37 40.71	43 00	11 38.895	43 11 38.90
21 00	7 49.870	21 07 49.87	43 30	11 39.600	43 41 39.60
21 30	7 58.886	21 37 58.89	44 00	11 40.092	44 11 40.09
22 00	8 07.756	22 08 07.76	44 30	11 40.371	44 41 40.37
22 30	8 16.476	22 38 16.48	45 00	11 40.436	45 11 40.44

$\varphi-\psi=+700''.4385 \sin 2\psi+1''.1893 \sin 4\psi+0''.0027 \sin 6\psi.$

$\varphi-\psi=[2.8453700] \sin 2\psi+[0.075285] \sin 4\psi+[7.430-10] \sin 6\psi.$

LATITUDE TRANSFORMATION—Continued.

Geocentric to geodetic—Continued.

Geocentric latitude. ψ	Geodetic minus geocentric. $\varphi-\psi$	Geodetic latitude. φ	Geocentric latitude. ψ	Geodetic minus geocentric. $\varphi-\psi$	Geodetic latitude. φ
° ′	′ ″	° ′ ″	° ′	′ ″	° ′ ″
45 00	11 40.436	45 11 40.44	67 30	8 14.097	67 38 14.10
45 30	11 40.288	45 41 40.29	68 00	8 05.379	68 08 05.38
46 00	11 39.926	46 11 39.93	68 30	7 56.513	68 37 56.51
46 30	11 39.352	46 41 39.35	69 00	7 47.504	69 07 47.50
47 00	11 38.564	47 11 38.56	69 30	7 38.354	69 37 38.35
47 30	11 37.564	47 41 37.56	70 00	7 29.064	70 07 29.06
48 00	11 36.352	48 11 36.35	70 30	7 19.639	70 37 19.64
48 30	11 34.927	48 41 34.93	71 00	7 10.082	71 07 10.08
49 00	11 33.292	49 11 33.29	71 30	7 00.394	71 37 00.39
49 30	11 31.445	49 41 31.44	72 00	6 50.579	72 06 50.58
50 00	11 29.388	50 11 29.39	72 30	6 40.640	72 36 40.64
50 30	11 27.122	50 41 27.12	73 00	6 30.580	73 06 30.58
51 00	11 24.646	51 11 24.65	73 30	6 20.402	73 36 20.40
51 30	11 21.963	51 41 21.96	74 00	6 10.110	74 06 10.11
52 00	11 19.072	52 11 19.07	74 30	5 59.705	74 35 59.71
52 30	11 15.975	52 41 15.98	75 00	5 49.192	75 05 49.19
53 00	11 12.673	53 11 12.67	75 30	5 38.573	75 35 38.57
53 30	11 09.166	53 41 09.17	76 00	5 27.853	76 05 27.85
54 00	11 05.456	54 11 05.46	76 30	5 17.033	76 35 17.03
54 30	11 01.544	54 41 01.54	77 00	5 06.117	77 05 06.12
55 00	10 57.431	55 10 57.43	77 30	4 55.110	77 34 55.11
55 30	10 53.119	55 40 53.12	78 00	4 44.013	78 04 44.01
56 00	10 48.608	56 10 48.61	78 30	4 32.830	78 34 32.83
56 30	10 43.900	56 40 43.90	79 00	4 21.565	79 04 21.57
57 00	10 38.998	57 10 39.00	79 30	4 10.221	79 34 10.22
57 30	10 33.901	57 40 33.90	80 00	3 58.802	80 03 58.80
58 00	10 28.612	58 10 28.61	80 30	3 47.311	80 33 47.31
58 30	10 23.133	58 40 23.13	81 00	3 35.750	81 03 35.75
59 00	10 17.464	59 10 17.46	81 30	3 24.126	81 33 24.13
59 30	10 11.609	59 40 11.61	82 00	3 12.439	82 03 12.44
60 00	10 05.568	60 10 05.57	82 30	3 00.694	82 33 00.69
60 30	9 59.343	60 39 59.34	83 00	2 48.895	83 02 48.89
61 00	9 52.937	61 09 52.94	83 30	2 37.045	83 32 37.04
61 30	9 46.351	61 39 46.35	84 00	2 25.147	84 02 25.15
62 00	9 39.588	62 09 39.59	84 30	2 13.206	84 32 13.21
62 30	9 32.649	62 39 32.65	85 00	2 01.224	85 02 01.22
63 00	9 25.536	63 09 25.54	85 30	1 49.206	85 31 49.21
63 30	9 18.253	63 39 18.25	86 00	1 37.156	86 01 37.16
64 00	9 10.800	64 09 10.80	86 30	1 25.075	86 31 25.08
64 30	9 03.181	64 39 03.18	87 00	1 12.969	87 01 12.97
65 00	8 55.397	65 08 55.40	87 30	1 00.841	87 31 00.84
65 30	8 47.451	65 38 47.45	88 00	0 48.695	88 00 48.70
66 00	8 39.346	66 08 39.35	88 30	0 36.534	88 30 36.53
66 30	8 31.084	66 38 31.08	89 00	0 24.362	89 00 24.36
67 00	8 22.666	67 08 22.67	89 30	0 12.183	89 30 12.18
67 30	8 14.097	67 38 14.10	90 00	0 00.000	90 00 00.00

$\varphi-\psi=+700''.4385 \sin 2\psi+1''.1893 \sin 4\psi+0''.0027 \sin 6\psi.$

$\varphi-\psi=[2.8453700] \sin 2\psi+[0.075285] \sin 4\psi+[7.430-10] \sin 6\psi.$

LATITUDE TRANSFORMATION—Continued.

Geodetic to parametric.

Geodetic latitude. φ	Geodetic minus parametric. $\varphi-\theta$	Parametric latitude. θ	Geodetic latitude. φ	Geodetic minus parametric. $\varphi-\theta$	Parametric latitude. θ
° ′	′ ″	° ′ ″	° ′	′ ″	° ′ ″
0 00	0 00.000	0 00 00.00	22 30	4 07.346	22 25 52.65
0 30	0 06.102	0 29 53.90	23 00	4 11.630	22 55 48.37
1 00	0 12.202	0 59 47.80	23 30	4 15.838	23 25 44.16
1 30	0 18.298	1 29 41.70	24 00	4 19.969	23 55 40.03
2 00	0 24.389	1 59 35.61	24 30	4 24.020	24 25 35.98
2 30	0 30.472	2 29 29.53	25 00	4 27.992	24 55 32.01
3 00	0 36.546	2 59 23.45	25 30	4 31.882	25 25 28.12
3 30	0 42.609	3 29 17.39	26 00	4 35.689	25 55 24.31
4 00	0 48.659	3 59 11.34	26 30	4 39.413	26 25 20.59
4 30	0 54.695	4 29 05.31	27 00	4 43.052	26 55 16.95
5 00	1 00.714	4 58 59.29	27 30	4 46.604	27 25 13.40
5 30	1 06.714	5 28 53.29	28 00	4 50.070	27 55 09.93
6 00	1 12.694	5 58 47.31	28 30	4 53.448	28 25 06.55
6 30	1 18.652	6 28 41.35	29 00	4 56.736	28 55 03.26
7 00	1 24.587	6 58 35.41	29 30	4 59.935	29 25 00.06
7 30	1 30.495	7 28 29.50	30 00	5 03 042	29 54 56.96
8 00	1 36.377	7 58 23.62	30 30	5 06.057	30 24 53.94
8 30	1 42.228	8 28 17.77	31 00	5 08.980	30 54 51.02
9 00	1 48.050	8 58 11.95	31 30	5 11.808	31 24 48.19
9 30	1 53.838	9 28 06.16	32 00	5 14.541	31 54 45.46
10 00	1 59.592	9 58 00.41	32 30	5 17 180	32 24 42.82
10 30	2 05.309	10 27 54.69	33 00	5 19.721	32 54 40.28
11 00	2 10.989	10 57 49.01	33 30	5 22.165	33 24 37.84
11 30	2 16.628	11 27 43.37	34 00	5 24.512	33 54 35.49
12 00	2 22.227	11 57 37.77	34 30	5 26.760	34 24 33.24
12 30	2 27.782	12 27 32.22	35 00	5 28.908	34 54 31.09
13 00	2 33.292	12 57 26.71	35 30	5 30.957	35 24 29.04
13 30	2 38.756	13 27 21.24	36 00	5 32.904	35 54 27.10
14 00	2 44.172	13 57 15.83	36 30	5 34.751	36 24 25.25
14 30	2 49.538	14 27 10.46	37 00	5 36.496	36 54 23.50
15 00	2 54.853	14 57 05.15	37 30	5 38.138	37 24 21.86
15 30	3 00.114	15 26 59.89	38 00	5 39.677	37 54 20.32
16 00	3 05.321	15 56 54.68	38 30	5 41.114	38 24 18.89
16 30	3 10.472	16 26 49.53	39 00	5 42.446	38 54 17.55
17 00	3 15.565	16 56 44.43	39 30	5 43.674	39 24 16.33
17 30	3 20.599	17 26 39.40	40 00	5 44.798	39 54 15.20
18 00	3 25.572	17 56 34.43	40 30	5 45.816	40 24 14.18
18 30	3 30.482	18 26 29.52	41 00	5 46.730	40 54 13.27
19 00	3 35.329	18 56 24.67	41 30	5 47.538	41 24 12.46
19 30	3 40.110	19 26 19.89	42 00	5 48.240	41 54 11.76
20 00	3 44.825	19 56 15.18	42 30	5 48.836	42 24 11.16
20 30	3 49.471	20 26 10.53	43 00	5 49.325	42 54 10.67
21 00	3 54.048	20 56 05.95	43 30	5 49.709	43 24 10.29
21 30	3 58.553	21 26 01.45	44 00	5 49.986	43 54 10.01
22 00	4 02.987	21 55 57.01	44 30	5 50.156	44 24 09.84
22 30	4 07.346	22 25 52.65	45 00	5 50.220	44 54 09.78

$\varphi-\theta=+350''.2202 \sin 2\varphi-0''.2973 \sin 4\varphi+0''.0003 \sin 6\varphi.$

$\varphi-\theta=[2.5443412] \sin 2\varphi-[9.47323-10] \sin 4\varphi+[6.527-10] \sin 6\varphi.$

LATITUDE TRANSFORMATION—Continued.

Geodetic to parametric—Continued.

Geodetic latitude. φ	Geodetic minus parametric. φ−θ	Parametric latitude. θ	Geodetic latitude. φ	Geodetic minus parametric. φ−θ	Parametric latitude. θ
° ′	′ ″	° ′ ″	° ′	′ ″	° ′ ″
45 00	5 50.220	44 54 09.78	67 30	4 07.941	67 25 52.06
45 30	5 50.177	45 24 09.82	68 00	4 03.581	67 55 56.42
46 00	5 50.027	45 54 09.97	68 30	3 59.146	68 26 00.85
46 30	5 49.771	46 24 10.23	69 00	3 54.639	68 56 05.36
47 00	5 49.408	46 54 10.59	69 30	3 50.060	69 26 09.94
47 30	5 48.939	47 24 11.06	70 00	3 45.410	69 56 14.59
48 00	5 48.363	47 54 11.64	70 30	3 40.692	70 26 19.31
48 30	5 47.681	48 24 12.32	71 00	3 35.906	70 56 24.09
49 00	5 46.894	48 54 13.11	71 30	3 31.054	71 26 28.95
49 30	5 46.000	49 24 14.00	72 00	3 26.137	71 56 33.86
50 00	5 45.001	49 54 15.00	72 30	3 21.158	72 26 38.84
50 30	5 43.897	50 24 16.10	73 00	3 16.117	72 56 43.88
51 00	5 42.688	50 54 17.31	73 30	3 11.016	73 26 48.98
51 30	5 41.374	51 24 18.63	74 00	3 05.856	73 56 54.14
52 00	5 39.957	51 54 20.04	74 30	3 00.640	74 26 59.36
52 30	5 38.435	52 24 21.56	75 00	2 55.368	74 57 04.63
53 00	5 36.811	52 54 23.19	75 30	2 50.042	75 27 09.96
53 30	5 35.083	53 24 24.92	76 00	2 44.665	75 57 15.34
54 00	5 33.254	53 54 26.75	76 30	2 39.237	76 27 20.76
54 30	5 31.322	54 24 28.68	77 00	2 33.761	76 57 26.24
55 00	5 29.290	54 54 30.71	77 30	2 28.238	77 27 31.76
55 30	5 27.158	55 24 32.84	78 00	2 22.669	77 57 37.33
56 00	5 24.925	55 54 35.08	78 30	2 17.056	78 27 42.94
56 30	5 22.593	56 24 37.41	79 00	2 11.402	78 57 48.60
57 00	5 20.163	56 54 39.84	79 30	2 05.707	79 27 54.29
57 30	5 17.635	57 24 42.36	80 00	1 59.974	79 58 00.03
58 00	5 15.010	57 54 44.99	80 30	1 54.204	80 28 05.80
58 30	5 12.289	58 24 47.71	81 00	1 48.399	80 58 11.60
59 00	5 09 473	58 54 50.53	81 30	1 42.561	81 28 17.44
59 30	5 06.562	59 24 53.44	82 00	1 36.692	81 58 23.31
60 00	5 03.557	59 54 56.44	82 30	1 30.792	82 28 29.21
60 30	5 00.460	60 24 59.54	83 00	1 24.866	82 58 35.13
61 00	4 57.271	60 55 02.73	83 30	1 18.913	83 28 41.09
61 30	4 53.991	61 25 06.01	84 00	1 12.936	83 58 47.06
62 00	4 50.622	61 55 09.38	84 30	1 06.937	84 28 53.06
62 30	4 47.163	62 25 12.84	85 00	1 00.917	84 58 59.08
63 00	4 43.617	62 55 16.38	85 30	0 54.878	85 29 05.12
63 30	4 39.984	63 25 20.02	86 00	0 48.823	85 59 11.18
64 00	4 36.266	63 55 23.73	86 30	0 42.753	86 29 17.25
64 30	4 32.463	64 25 27.54	87 00	0 36.670	86 59 23.33
65 00	4 28.577	64 55 31.42	87 30	0 30.575	87 29 29.42
65 30	4 24.609	65 25 35.39	88 00	0 24.472	87 59 35.53
66 00	4 20.560	65 55 39.44	88 30	0 18.360	88 29 41.64
66 30	4 16.432	66 25 43.57	89 00	0 12.243	88 59 47.76
67 00	4 12.225	66 55 47.78	89 30	0 06.123	89 29 53.88
67 30	4 07.941	67 25 52.06	90 00	0 00.000	90 00 00.00

$\varphi-\theta = +350''.2202 \sin 2\varphi - 0''.2973 \sin 4\varphi + 0''.0003 \sin 6\varphi.$

$\varphi-\theta = [2.5443412] \sin 2\varphi - [9.47323-10] \sin 4\varphi + [6.527-10] \sin 6\varphi.$

LATITUDE TRANSFORMATION—Continued.

Parametric to geodetic.

Parametric latitude. θ	Geodetic minus parametric. $\varphi-\theta$	Geodetic latitude. φ	Parametric latitude. θ	Geodetic minus parametric. $\varphi-\theta$	Geodetic latitude. φ
° ′	′ ″	° ′ ″	° ′	′ ″	° ′ ″
0 00	0 00.000	0 00 00.00	22 30	4 07.941	22 34 07.94
0 30	0 06.123	0 30 06.12	23 00	4 12.225	23 04 12.22
1 00	0 12.243	1 00 12.24	23 30	4 16.432	23 34 16.43
1 30	0 18.360	1 30 18.36	24 00	4 20.560	24 04 20.56
2 00	0 24.472	2 00 24.47	24 30	4 24.609	24 34 24.61
2 30	0 30.575	2 30 30.58	25 00	4 28.577	25 04 28.58
3 00	0 36.670	3 00 36.67	25 30	4 32.463	25 34 32.46
3 30	0 42.753	3 30 42.75	26 00	4 36.266	26 04 36.27
4 00	0 48.823	4 00 48.82	26 30	4 39.984	26 34 39.98
4 30	0 54.878	4 30 54.88	27 00	4 43.617	27 04 43.62
5 00	1 00.917	5 01 00.92	27 30	4 47.163	27 34 47.16
5 30	1 06.937	5 31 06.94	28 00	4 50.622	28 04 50.62
6 00	1 12.936	6 01 12.94	28 30	4 53.991	28 34 53.99
6 30	1 18.913	6 31 18.91	29 00	4 57.271	29 04 57.27
7 00	1 24.866	7 01 24.87	29 30	5 00.460	29 35 00.46
7 30	1 30.792	7 31 30.79	30 00	5 03.557	30 05 03.56
8 00	1 36.692	8 01 36.69	30 30	5 06.562	30 35 06.56
8 30	1 42.561	8 31 42.56	31 00	5 09.473	31 05 09.47
9 00	1 48.399	9 01 48.40	31 30	5 12.289	31 35 12.29
9 30	1 54.204	9 31 54.20	32 00	5 15.010	32 05 15.01
10 00	1 59.974	10 01 59.97	32 30	5 17.635	32 35 17.64
10 30	2 05.707	10 32 05.71	33 00	5 20.163	33 05 20.16
11 00	2 11.402	11 02 11.40	33 30	5 22.593	33 35 22.59
11 30	2 17.056	11 32 17.06	34 00	5 24.925	34 05 24.92
12 00	2 22.669	12 02 22.67	34 30	5 27.158	34 35 27.16
12 30	2 28.238	12 32 28.24	35 00	5 29.290	35 05 29.29
13 00	2 33.761	13 02 33.76	35 30	5 31.322	35 35 31.32
13 30	2 39.237	13 32 39.24	36 00	5 33.254	36 05 33.25
14 00	2 44.665	14 02 44.66	36 30	5 35.083	36 35 35.08
14 30	2 50.042	14 32 50.04	37 00	5 36.811	37 05 36.81
15 00	2 55.368	15 02 55.37	37 30	5 38.435	37 35 38.44
15 30	3 00.640	15 33 00.64	38 00	5 39.957	38 05 39.96
16 00	3 05.856	16 03 05.86	38 30	5 41.374	38 35 41.37
16 30	3 11.016	16 33 11.02	39 00	5 42.688	39 05 42.69
17 00	3 16.117	17 03 16.12	39 30	5 43.897	39 35 43.90
17 30	3 21.158	17 33 21.16	40 00	5 45.001	40 05 45.00
18 00	3 26.137	18 03 26.14	40 30	5 46.000	40 35 46.00
18 30	3 31.054	18 33 31.05	41 00	5 46.894	41 05 46.89
19 00	3 35.906	19 03 35.91	41 30	5 47.681	41 35 47.68
19 30	3 40.692	19 33 40.69	42 00	5 48.363	42 05 48.36
20 00	3 45.410	20 03 45.41	42 30	5 48.939	42 35 48.94
20 30	3 50.060	20 33 50.06	43 00	5 49.408	43 05 49.41
21 00	3 54.639	21 03 54.64	43 30	5 49.771	43 35 49.77
21 30	3 59.146	21 33 59.15	44 00	5 50.027	44 05 50.03
22 00	4 03.581	22 04 03.58	44 30	5 50.177	44 35 50.18
22 30	4 07.941	22 34 07.94	45 00	5 50.220	45 05 50.22

$\varphi-\theta = +350.''2202 \sin 2\theta + 0.''2973 \sin 4\theta + 0.''0003 \sin 6\theta.$

$\varphi-\theta = [2.5443412] \sin 2\theta + [9.47323-10] \sin 4\theta + [6.527-10] \sin 6\theta.$

LATITUDE TRANSFORMATION—Continued.

Parametric to geodetic—Continued.

Parametric latitude. θ	Geodetic minus parametric. $\varphi-\theta$	Geodetic latitude. φ	Parametric latitude. θ	Geodetic minus parametric. $\varphi-\theta$	Geodetic latitude. φ
° ′	′ ″	° ′ ″	° ′	′ ″	° ′ ″
45 00	5 50.220	45 05 50.22	67 30	4 07.346	67 34 07.35
45 30	5 50.156	45 35 50.16	68 00	4 02.987	68 04 02.99
46 00	5 49.986	46 05 49.99	68 30	3 58.553	68 33 58.55
46 30	5 49.709	46 35 49.71	69 00	3 54.048	69 03 54.05
47 00	5 49.325	47 05 49.33	69 30	3 49.471	69 33 49.47
47 30	5 48.836	47 35 48.84	70 00	3 44.825	70 03 44.82
48 00	5 48.240	48 05 48.24	70 30	3 40.110	70 33 40.11
48 30	5 47.538	48 35 47.54	71 00	3 35.329	71 03 35.33
49 00	5 46.730	49 05 46.73	71 30	3 30 482	71 33 30.48
49 30	5 45.816	49 35 45.82	72 00	3 25.572	72 03 25.57
50 00	5 44.798	50 05 44.80	72 30	3 20.599	72 33 20.60
50 30	5 43.674	50 35 43.67	73 00	3 15.565	73 03 15.57
51 00	5 42.446	51 05 42.45	73 30	3 10.472	73 33 10.47
51 30	5 41.114	51 35 41.11	74 00	3 05.321	74 03 05.32
52 00	5 39.677	52 05 39.68	74 30	3 00.114	74 33 00.11
52 30	5 38.138	52 35 38.14	75 00	2 54.853	75 02 54.85
53 00	5 36.496	53 05 36.50	75 30	2 49.538	75 32 49.54
53 30	5 34.751	53 35 34.75	76 00	2 44.172	76 02 44.17
54 00	5 32.904	54 05 32.90	76 30	2 38.756	76 32 38.76
54 30	5 30.957	54 35 30.96	77 00	2 33.292	77 02 33.29
55 00	5 28.908	55 05 28.91	77 30	2 27.782	77 32 27.78
55 30	5 26.760	55 35 26.76	78 00	2 22.227	78 02 22.23
56 00	5 24.512	56 05 24.51	78 30	2 16.628	78 32 16.63
56 30	5 22.165	56 35 22.16	79 00	2 10.989	79 02 10.99
57 00	5 19.721	57 05 19.72	79 30	2 05.309	79 32 05.31
57 30	5 17.180	57 35 17.18	80 00	1 59.592	80 01 59.59
58 00	5 14.541	58 05 14.54	80 30	1 53.838	80 31 53.84
58 30	5 11.808	58 35 11.81	81 00	1 48.050	81 01 48.05
59 00	5 08.980	59 05 08.98	81 30	1 42.228	81 31 42.23
59 30	5 06.057	59 35 06.06	82 00	1 36.377	82 01 36.38
60 00	5 03.042	60 05 03.04	82 30	1 30.495	82 31 30.50
60 30	4 59.935	60 34 59.94	83 00	1 24.587	83 01 24.59
61 00	4 56.736	61 04 56.74	83 30	1 18.652	83 31 18.65
61 30	4 53.448	61 34 53.45	84 00	1 12.694	84 01 12.69
62 00	4 50.070	62 04 50.07	84 30	1 06.714	84 31 06.71
62 30	4 46.604	62 34 46.60	85 00	1 00.714	85 01 00.71
63 00	4 43.052	63 04 43.05	85 30	0 54.695	85 30 54.69
63 30	4 39.413	63 34 39.41	86 00	0 48.659	86 00 48.66
64 00	4 35.689	64 04 35.69	86 30	0 42.609	86 30 42.61
64 30	4 31.882	64 34 31.88	87 00	0 36.546	87 00 36.55
65 00	4 27.992	65 04 27.99	87 30	0 30.472	87 30 30.47
65 30	4 24.020	65 34 24.02	88 00	0 24.389	88 00 24.39
66 00	4 19.969	66 04 19.97	88 30	0 18.298	88 30 18.30
66 30	4 15.838	66 34 15.84	89 00	0 12.202	89 00 12.20
67 00	4 11.630	67 04 11.63	89 30	0 06.102	89 30 06.10
67 30	4 07.346	67 34 07.35	90 00	0 00.000	90 00 00.00

$\varphi-\theta = +350''.2202 \sin 2\theta + 0''.2973 \sin 4\theta + 0''.0003 \sin 6\theta.$

$\varphi-\theta = [2.5443412] \sin 2\theta + [9.47323-10] \sin 4\theta + [6.527-10] \sin 6\theta,$

LATITUDE TRANSFORMATION—Continued.

Parametric to geocentric.

Parametric latitude. θ	Parametric minus geocentric. $\theta-\psi$	Geocentric latitude. ψ	Parametric latitude. θ	Parametric minus geocentric. $\theta-\psi$	Geocentric latitude. ψ
° ′	′ ″	° ′ ″	° ′	′ ″	° ′ ″
0 00	0 00.000	0 00 00.00	22 30	4 07.346	22 25 52.65
0 30	0 06.102	0 29 53.90	23 00	4 11.630	22 55 48.37
1 00	0 12.202	0 59 47.80	23 30	4 15.838	23 25 44.16
1 30	0 18.298	1 29 41.70	24 00	4 19.969	23 55 40.03
2 00	0 24.389	1 59 35.61	24 30	4 24.020	24 25 35.98
2 30	0 30.472	2 29 29.53	25 00	4 27.992	24 55 32.01
3 00	0 36.546	2 59 23.45	25 30	4 31.882	25 25 28.12
3 30	0 42.609	3 29 17.39	26 00	4 35.689	25 55 24.31
4 00	0 48.659	3 59 11.34	26 30	4 39.413	26 25 20.59
4 30	0 54.695	4 29 05.31	27 00	4 43.052	26 55 16.95
5 00	1 00.714	4 58 59.29	27 30	4 46.604	27 25 13.40
5 30	1 06.714	5 28 53.29	28 00	4 50.070	27 55 09.93
6 00	1 12.694	5 58 47.31	28 30	4 53.448	28 25 06.55
6 30	1 18.652	6 28 41.35	29 00	4 56.736	28 55 03.26
7 00	1 24.587	6 58 35.41	29 30	4 59.935	29 25 00.06
7 30	1 30.495	7 28 29.50	30 00	5 03.042	29 54 56.96
8 00	1 36.377	7 58 23.62	30 30	5 06.057	30 24 53.94
8 30	1 42.228	8 28 17.77	31 00	5 08.980	30 54 51.02
9 00	1 48.050	8 58 11.95	31 30	5 11.808	31 24 48.19
9 30	1 53.838	9 28 06.16	32 00	5 14.541	31 54 45.46
10 00	1 59.592	9 58 00.41	32 30	5 17.180	32 24 42.82
10 30	2 05.309	10 27 54.69	33 00	5 19.721	32 54 40.28
11 00	2 10.989	10 57 49.01	33 30	5 22.165	33 24 37.84
11 30	2 16.628	11 27 43.37	34 00	5 24.512	33 54 35.49
12 00	2 22.227	11 57 37.77	34 30	5 26.760	34 24 33.24
12 30	2 27.782	12 27 32.22	35 00	5 28.908	34 54 31.09
13 00	2 33.292	12 57 26.71	35 30	5 30.957	35 24 29.04
13 30	2 38.756	13 27 21.24	36 00	5 32.904	35 54 27.10
14 00	2 44.172	13 57 15.83	36 30	5 34.751	36 24 25.25
14 30	2 49.538	14 27 10.46	37 00	5 36.496	36 54 23.50
15 00	2 54.853	14 57 05.15	37 30	5 38.138	37 24 21.86
15 30	3 00.114	15 26 59.89	38 00	5 39.677	37 54 20.32
16 00	3 05.321	15 56 54.68	38 30	5 41.114	38 24 18.89
16 30	3 10.472	16 26 49.53	39 00	5 42.446	38 54 17.55
17 00	3 15.565	16 56 44.43	39 30	5 43.674	39 24 16.33
17 30	3 20.599	17 26 39.40	40 00	5 44.798	39 54 15.20
18 00	3 25.572	17 56 34.43	40 30	5 45.816	40 24 14.18
18 30	3 30.482	18 26 29.52	41 00	5 46.730	40 54 13.27
19 00	3 35.329	18 56 24.67	41 30	5 47.538	41 24 12.46
19 30	3 40.110	19 26 19.89	42 00	5 48.240	41 54 11.76
20 00	3 44.825	19 56 15.18	42 30	5 48.836	42 24 11.16
20 30	3 49.471	20 26 10.53	43 00	5 49.325	42 54 10.67
21 00	3 54.048	20 56 05.95	43 30	5 49.709	43 24 10.29
21 30	3 58.553	21 26 01.45	44 00	5 49.986	43 54 10.01
22 00	4 02.987	21 55 57.01	44 30	5 50.156	44 24 09.84
22 30	4 07.346	22 25 52.65	45 00	5 50.220	44 54 09.78

$\theta-\psi=+350''.2202 \sin 2\theta-0''.2973 \sin 4\theta+0''.0003 \sin 6\theta.$

$\theta-\psi=[2.5443412] \sin 2\theta-[9.47323-10] \sin 4\theta+[6.527-10] \sin 6\theta.$

LATITUDE TRANSFORMATION—Continued.

Parametric to geocentric—Continued.

Parametric latitude. θ	Parametric minus geocentric. $\theta-\psi$	Geocentric latitude. ψ	Parametric latitude. θ	Parametric minus geocentric. $\theta-\psi$	Geocentric latitude. ψ
° ′	′ ″	° ′ ″	° ′	′ ″	° ′ ″
45 00	5 50.220	44 54 09.78	67 30	4 07.941	67 25 52.06
45 30	5 50.177	45 24 09.82	68 00	4 03.581	67 55 56.42
46 00	5 50.027	45 54 09.97	68 30	3 59.146	68 26 00.85
46 30	5 49.771	46 24 10.23	69 00	3 54.639	68 56 05.36
47 00	5 49.408	46 54 10.59	69 30	3 50.060	69 26 09.94
47 30	5 48.939	47 24 11.06	70 00	3 45.410	69 56 14.59
48 00	5 48.363	47 54 11.64	70 30	3 40.692	70 26 19.31
48 30	5 47.681	48 24 12.32	71 00	3 35.906	70 56 24.09
49 00	5 46.894	48 54 13.11	71 30	3 31.054	71 26 28.95
49 30	5 46.000	49 24 14.00	72 00	3 26.137	71 56 33.86
50 00	5 45.001	49 54 15.00	72 30	3 21.158	72 26 38.84
50 30	5 43.897	50 24 16.10	73 00	3 16.117	72 56 43.88
51 00	5 42.688	50 54 17.31	73 30	3 11.016	73 26 48.98
51 30	5 41.374	51 24 18.63	74 00	3 05.856	73 56 54.14
52 00	5 39.957	51 54 20.04	74 30	3 00.640	74 26 59.36
52 30	5 38.435	52 24 21.56	75 00	2 55.368	74 57 04.63
53 00	5 36.811	52 54 23.19	75 30	2 50.042	75 27 09.96
53 30	5 35.083	53 24 24.92	76 00	2 44.665	75 57 15.34
54 00	5 33.254	53 54 26.75	76 30	2 39.237	76 27 20.76
54 30	5 31.322	54 24 28.68	77 00	2 33.761	76 57 26.24
55 00	5 29.290	54 54 30.71	77 30	2 28.238	77 27 31.76
55 30	5 27.158	55 24 32.84	78 00	2 22.669	77 57 37.33
56 00	5 24.925	55 54 35.08	78 30	2 17.056	78 27 42.94
56 30	5 22.593	56 24 37.41	79 00	2 11.402	78 57 48.60
57 00	5 20.163	56 54 39.84	79 30	2 05.707	79 27 54.29
57 30	5 17.635	57 24 42.36	80 00	1 59.974	79 58 00.03
58 00	5 15.010	57 54 44.99	80 30	1 54.204	80 28 05.80
58 30	5 12.289	58 24 47.71	81 00	1 48.399	80 58 11.60
59 00	5 09.473	58 54 50.53	81 30	1 42.561	81 28 17.44
59 30	5 06.562	59 24 53.44	82 00	1 36.692	81 58 23.31
60 00	5 03.557	59 54 56.44	82 30	1 30.792	82 28 29.21
60 30	5 00.460	60 24 59.54	83 00	1 24.866	82 58 35.13
61 00	4 57.271	60 55 02.73	83 30	1 18.913	83 28 41.09
61 30	4 53.991	61 25 06.01	84 00	1 12.936	83 58 47.06
62 00	4 50.622	61 55 09.38	84 30	1 06.937	84 28 53.06
62 30	4 47.163	62 25 12.84	85 00	1 00.917	84 58 59.08
63 00	4 43.617	62 55 16.38	85 30	0 54.878	85 29 05.12
63 30	4 39.984	63 25 20.02	86 00	0 48.823	85 59 11.18
64 00	4 36.266	63 55 23.73	86 30	0 42.753	86 29 17.25
64 30	4 32.463	64 25 27.54	87 00	0 36.670	86 59 23.33
65 00	4 28.577	64 55 31.42	87 30	0 30.575	87 29 29.42
65 30	4 24.609	65 25 35.39	88 00	0 24.472	87 59 35.53
66 00	4 20.560	65 55 39.44	88 30	0 18.360	88 29 41.64
66 30	4 16.432	66 25 43.57	89 00	0 12.243	88 59 47.76
67 00	4 12.225	66 55 47.78	89 30	0 06.123	89 29 53.88
67 30	4 07.941	67 25 52.06	90 00	0 00.000	90 00 00.00

$\theta-\psi=+350''.2202 \sin 2\theta-0''.2973 \sin 4\theta+0''.0003 \sin 6\theta.$

$\theta-\psi=[2.5443412] \sin 2\theta-[9.47323-10] \sin 4\theta+[6.527-10] \sin 6\theta,$

LATITUDE TRANSFORMATION—Continued.

Geocentric to parametric.

Geocentric latitude. ψ	Parametric minus geocentric. $\theta-\psi$	Parametric latitude. θ	Geocentric latitude. ψ	Parametric minus geocentric. $\theta-\psi$	Parametric latitude. θ
° ′	′ ″	° ′ ″	° ′	′ ″	° ′ ″
0 00	0 00.000	0 00 00.00	22 30	4 07.941	22 34 07.94
0 30	0 06.123	0 30 06.12	23 00	4 12.225	23 04 12.22
1 00	0 12.243	1 00 12.24	23 30	4 16.432	23 34 16.43
1 30	0 18.360	1 30 18.36	24 00	4 20.560	24 04 20.56
2 00	0 24.472	2 00 24.47	24 30	4 24.609	24 34 24.61
2 30	0 30.575	2 30 30.58	25 00	4 28.577	25 04 28.58
3 00	0 36.670	3 00 36.67	25 30	4 32.463	25 34 32.46
3 30	0 42.753	3 30 42.75	26 00	4 36.266	26 04 36.27
4 00	0 48.823	4 00 48.82	26 30	4 39.984	26 34 39.98
4 30	0 54.878	4 30 54.88	27 00	4 43.617	27 04 43.62
5 00	1 00.917	5 01 00.92	27 30	4 47.163	27 34 47.16
5 30	1 06.937	5 31 06.94	28 00	4 50.622	28 04 50.62
6 00	1 12.936	6 01 12.94	28 30	4 53.991	28 34 53.99
6 30	1 18.913	6 31 18.91	29 00	4 57.271	29 04 57.27
7 00	1 24.866	7 01 24.87	29 30	5 00.460	29 35 00.46
7 30	1 30.792	7 31 30.79	30 00	5 03.557	30 05 03.56
8 00	1 36.692	8 01 36.69	30 30	5 06.562	30 35 06.56
8 30	1 42.561	8 31 42.56	31 00	5 09.473	31 05 09.47
9 00	1 48.399	9 01 48.40	31 30	5 12.289	31 35 12.29
9 30	1 54.204	9 31 54.20	32 00	5 15.010	32 05 15.01
10 00	1 59.974	10 01 59.97	32 30	5 17.635	32 35 17.64
10 30	2 05.707	10 32 05.71	33 00	5 20.163	33 05 20.16
11 00	2 11.402	11 02 11.40	33 30	5 22.593	33 35 22.59
11 30	2 17.056	11 32 17.06	34 00	5 24.925	34 05 24.92
12 00	2 22.669	12 02 22.67	34 30	5 27.158	34 35 27.16
12 30	2 28.238	12 32 28.24	35 00	5 29.290	35 05 29.29
13 00	2 33.761	13 02 33.76	35 30	5 31.322	35 35 31.32
13 30	2 39.237	13 32 39.24	36 00	5 33.254	36 05 33.25
14 00	2 44.665	14 02 44.66	36 30	5 35.083	36 35 35.08
14 30	2 50.042	14 32 50.04	37 00	5 36.811	37 05 36.81
15 00	2 55.368	15 02 55.37	37 30	5 38.435	37 35 38.44
15 30	3 00.640	15 33 00.64	38 00	5 39.957	38 05 39.96
16 00	3 05.856	16 03 05.86	38 30	5 41.374	38 35 41.37
16 30	3 11.016	16 33 11.02	39 00	5 42.688	39 05 42.69
17 00	3 16.117	17 03 16.12	39 30	5 43.897	39 35 43.90
17 30	3 21.158	17 33 21.16	40 00	5 45.001	40 05 45.00
18 00	3 26.137	18 03 26.14	40 30	5 46.000	40 35 46.00
18 30	3 31.054	18 33 31.05	41 00	5 46.894	41 05 46.89
19 00	3 35.906	19 03 35.91	41 30	5 47.681	41 35 47.68
19 30	3 40.692	19 33 40.69	42 00	5 48.363	42 05 48.36
20 00	3 45.410	20 03 45.41	42 30	5 48.939	42 35 48.94
20 30	3 50.060	20 33 50.06	43 00	5 49.408	43 05 49.41
21 00	3 54.639	21 03 54.64	43 30	5 49.771	43 35 49.77
21 30	3 59.146	21 33 59.15	44 00	5 50.027	44 05 50.03
22 00	4 03.581	22 04 03.58	44 30	5 50.177	44 35 50.18
22 30	4 07.941	22 34 07.94	45 00	5 50.220	45 05 50.22

$\theta-\psi=+350''.2022 \sin 2\psi+0''.2973 \sin 4\psi+0''.0003 \sin 6\psi.$

$\theta-\psi=[2.5443412] \sin 2\psi+[9.47323-10] \sin 4\psi+[6.527-10] \sin 6\psi.$

LATITUDE TRANSFORMATION—Continued.

Geocentric to parametric—Continued.

Geocentric latitude. ψ	Parametric minus geocentric. $\theta-\psi$	Parametric latitude. θ	Geocentric latitude. ψ	Parametric minus geocentric. $\theta-\psi$	Parametric latitude. θ
° '	' "	° ' "	° '	' "	° ' "
45 00	5 50.220	45 05 50.22	67 30	4 07.346	67 34 07.35
45 30	5 50.156	45 35 50.16	68 00	4 02.987	68 04 02.99
46 00	5 49.986	46 05 49.99	68 30	3 58.553	68 33 58.55
46 30	5 49.709	46 35 49.71	69 00	3 54.048	69 03 54.05
47 00	5 49.325	47 05 49.33	69 30	3 49.471	69 33 49.47
47 30	5 48.836	47 35 48.84	70 00	3 44.825	70 03 44.82
48 00	5 48.240	48 05 48.24	70 30	3 40.110	70 33 40.11
48 30	5 47.538	48 35 47.54	71 00	3 35.329	71 03 35.33
49 00	5 46.730	49 05 46.73	71 30	3 30.482	71 33 30.48
49 30	5 45.816	49 35 45.82	72 00	3 25.572	72 03 25.57
50 00	5 44.798	50 05 44.80	72 30	3 20.599	72 33 20.60
50 30	5 43.674	50 35 43.67	73 00	3 15.565	73 03 15.57
51 00	5 42.446	51 05 42.45	73 30	3 10.472	73 33 10.47
51 30	5 41.114	51 35 41.11	74 00	3 05.321	74 03 05.32
52 00	5 39.677	52 05 39.68	74 30	3 00.114	74 33 00.11
52 30	5 38.138	52 35 38.14	75 00	2 54.853	75 02 54.85
53 00	5 36.496	53 05 36.50	75 30	2 49.538	75 32 49.54
53 30	5 34.751	53 35 34.75	76 00	2 44.172	76 02 44.17
54 00	5 32.904	54 05 32.90	76 30	2 38.756	76 32 38.76
54 30	5 30.957	54 35 30.96	77 00	2 33.292	77 02 33.29
55 00	5 28.908	55 05 28.91	77 30	2 27.782	77 32 27.78
55 30	5 26.760	55 35 26.76	78 00	2 22.227	78 02 22.23
56 00	5 24.512	56 05 24.51	78 30	2 16.628	78 32 16.63
56 30	5 22.165	56 35 22.16	79 00	2 10.989	79 02 10.99
57 00	5 19.721	57 05 19.72	79 30	2 05.309	79 32 05.31
57 30	5 17.180	57 35 17.18	80 00	1 59.592	80 01 59.59
58 00	5 14.541	58 05 14.54	80 30	1 53.838	80 31 53.84
58 30	5 11.808	58 35 11.81	81 00	1 48.050	81 01 48.05
59 00	5 08.980	59 05 08.98	81 30	1 42.228	81 31 42.23
59 30	5 06.057	59 35 06.06	82 00	1 36.377	82 01 36.38
60 00	5 03.042	60 05 03.04	82 30	1 30.495	82 31 30.50
60 30	4 59.935	60 34 59.94	83 00	1 24.587	83 01 24.59
61 00	4 56.736	61 04 56.74	83 30	1 18.652	83 31 18.65
61 30	4 53.448	61 34 53.45	84 00	1 12.694	84 01 12.69
62 00	4 50.070	62 04 50.07	84 30	1 06.714	84 31 06.71
62 30	4 46.604	62 34 46.60	85 00	1 00.714	85 01 00.71
63 00	4 43.052	63 04 43.05	85 30	0 54.695	85 30 54.69
63 30	4 39.413	63 34 39.41	86 00	0 48.659	86 00 48.66
64 00	4 35.689	64 04 35.69	86 30	0 42.609	86 30 42.61
64 30	4 31.882	64 34 31.88	87 00	0 36.546	87 00 36.55
65 00	4 27.992	65 04 27.99	87 30	0 30.472	87 30 30.47
65 30	4 24.020	65 34 24.02	88 00	0 24.389	88 00 24.39
66 00	4 19.969	66 04 19.97	88 30	0 18.298	88 30 18.30
66 30	4 15.838	66 34 15.84	89 00	0 12.202	89 00 12.20
67 00	4 11.630	67 04 11.63	89 30	0 06.102	89 30 06.10
67 30	4 07.346	67 34 07.35	90 00	0 00.000	90 00 00.00

$\theta-\psi=+350''.2022 \sin 2\psi+0''.2973 \sin 4\psi+0''.0003 \sin 6\psi.$

$\theta-\psi=[2.5443412] \sin 2\psi+[9.47323-10] \sin 4\psi+[6.527-10] \sin 6\psi.$

LATITUDE TRANSFORMATION—Continued.

Geodetic to isometric.

Geodetic latitude. φ		Geodetic colatitude. p		Geodetic minus isometric. $\varphi-\chi$		Isometric colatitude. z			$\frac{z}{2}$		
°	′	°	′	′	″	°	′	″	°	′	″
0	00	90	00	0	00.000	90	00	00.00	45	00	00.00
0	30	89	30	0	12.183	89	30	12.18	44	45	06.09
1	00	89	00	0	24.362	89	00	24.36	44	30	12.18
1	30	88	30	0	36.534	88	30	36.53	44	15	18.27
2	00	88	00	0	48.695	88	00	48.70	44	00	24.35
2	30	87	30	1	00.841	87	31	00.84	43	45	30.42
3	00	87	00	1	12.969	87	01	12.97	43	30	36.48
3	30	86	30	1	25.075	86	31	25.08	43	15	42.54
4	00	86	00	1	37.155	86	01	37.16	43	00	48.58
4	30	85	30	1	49.206	85	31	49.21	42	45	54.60
5	00	85	00	2	01.223	85	02	01.22	42	31	00.61
5	30	84	30	2	13.204	84	32	13.20	42	16	06.60
6	00	84	00	2	25.145	84	02	25.14	42	01	12.57
6	30	83	30	2	37.042	83	32	37.04	41	46	18.52
7	00	83	00	2	48.892	83	02	48.89	41	31	24.45
7	30	82	30	3	00.691	82	33	00.69	41	16	30.35
8	00	82	00	3	12.435	82	03	12.44	41	01	36.22
8	30	81	30	3	24.120	81	33	24.12	40	46	42.06
9	00	81	00	3	35.746	81	03	35.75	40	31	47.87
9	30	80	30	3	47.304	80	33	47.30	40	16	53.65
10	00	80	00	3	58.794	80	03	58.79	40	01	59.40
10	30	79	30	4	10.212	79	34	10.21	39	47	05.11
11	00	79	00	4	21.554	79	04	21.55	39	32	10.78
11	30	78	30	4	32.818	78	34	32.82	39	17	16.41
12	00	78	00	4	43.999	78	04	44.00	39	02	22.00
12	30	77	30	4	55.094	77	34	55.09	38	47	27.55
13	00	77	00	5	06.100	77	05	06.10	38	32	33.05
13	30	76	30	5	17.014	76	35	17.01	38	17	38.51
14	00	76	00	5	27.831	76	05	27.83	38	02	43.92
14	30	75	30	5	38.550	75	35	38.55	37	47	49.28
15	00	75	00	5	49.166	75	05	49.17	37	32	54.58
15	30	74	30	5	59.676	74	35	59.68	37	17	59.84
16	00	74	00	6	10.078	74	06	10.08	37	03	05.04
16	30	73	30	6	20.368	73	36	20.37	36	48	10.18
17	00	73	00	6	30.543	73	06	30.54	36	33	15.27
17	30	72	30	6	40.599	72	36	40.60	36	18	20.30
18	00	72	00	6	50.535	72	06	50.54	36	03	25.27
18	30	71	30	7	00.346	71	37	00.35	35	48	30.17
19	00	71	00	7	10.030	71	07	10.03	35	33	35.02
19	30	70	30	7	19.584	70	37	19.58	35	18	39.79
20	00	70	00	7	29.005	70	07	29.00	35	03	44.50
20	30	69	30	7	38.290	69	37	38.29	34	48	49.15
21	00	69	00	7	47.437	69	07	47.44	34	33	53.72
21	30	68	30	7	56.442	68	37	56.44	34	18	58.22
22	00	68	00	8	05.302	68	08	05.30	34	04	02.65
22	30	67	30	8	14.016	67	38	14.02	33	49	07.01

$\varphi-\chi=+700''.0427 \sin 2\varphi-0''.9900 \sin 4\varphi+0''.0017 \sin 6\varphi.$

$\varphi-\chi=[2.84512455] \sin 2\varphi-[9.99563-10] \sin 4\varphi+[7.238-10] \sin 6\varphi.$

LATITUDE TRANSFORMATION—Continued.

Geodetic to isometric—Continued.

Geodetic latitude. φ	Geodetic colatitude p	Geodetic minus isometric. $\varphi-\chi$	Isometric colatitude. z	$\frac{z}{2}$
° ′	° ′	′ ″	° ′ ″	° ′ ″
22 30	67 30	8 14.016	67 38 14.02	33 49 07.01
23 00	67 00	8 22.580	67 08 22.58	33 34 11.29
23 30	66 30	8 30.992	66 38 30.99	33 19 15.50
24 00	66 00	8 39.250	66 08 39.25	33 04 19.62
24 30	65 30	8 47.349	65 38 47.35	32 49 23.67
25 00	65 00	8 55.290	65 08 55.29	32 34 27.64
25 30	64 30	9 03.068	64 39 03.07	32 19 31.53
26 00	64 00	9 10.681	64 09 10.68	32 04 35.34
26 30	63 30	9 18.128	63 39 18.13	31 49 39.06
27 00	63 00	9 25.405	63 09 25.41	31 34 42.70
27 30	62 30	9 32.512	62 39 32.51	31 19 46.26
28 00	62 00	9 39.444	62 09 39.44	31 04 49.72
28 30	61 30	9 46.201	61 39 46.20	30 49 53.10
29 00	61 00	9 52.780	61 09 52.78	30 34 56.39
29 30	60 30	9 59.179	60 39 59.18	30 19 59.59
30 00	60 00	10 05.397	60 10 05.40	30 05 02.70
30 30	59 30	10 11.431	59 40 11.43	29 50 05.72
31 00	59 00	10 17.280	59 10 17.28	29 35 08.64
31 30	58 30	10 22.941	58 40 22.94	29 20 11.47
32 00	58 00	10 28.414	58 10 28.41	29 05 14.21
32 30	57 30	10 33.695	57 40 33.70	28 50 16.85
33 00	57 00	10 38.785	57 10 38.78	28 35 19.39
33 30	56 30	10 43.680	56 40 43.68	28 20 21.84
34 00	56 00	10 48.380	56 10 48.38	28 05 24.19
34 30	55 30	10 52.883	55 40 52.88	27 50 26.44
35 00	55 00	10 57.188	55 10 57.19	27 35 28.59
35 30	54 30	11 01.293	54 41 01.29	27 20 30.65
36 00	54 00	11 05.198	54 11 05.20	27 05 32.60
36 30	53 30	11 08.900	53 41 08.90	26 50 34.45
37 00	53 00	11 12.398	53 11 12.40	26 35 36.20
37 30	52 30	11 15.693	52 41 15.69	26 20 37.85
38 00	52 00	11 18.782	52 11 18.78	26 05 39.39
38 30	51 30	11 21.665	51 41 21.66	25 50 40.83
39 00	51 00	11 24.341	51 11 24.34	25 35 42.17
39 30	50 30	11 26.809	50 41 26.81	25 20 43.40
40 00	50 00	11 29.067	50 11 29.07	25 05 44.53
40 30	49 30	11 31.117	49 41 31.12	24 50 45.56
41 00	49 00	11 32.955	49 11 32.96	24 35 46.48
41 30	48 30	11 34.584	48 41 34.58	24 20 47.29
42 00	48 00	11 36.000	48 11 36.00	24 05 48.00
42 30	47 30	11 37.205	47 41 37.20	23 50 48.60
43 00	47 00	11 38.198	47 11 38.20	23 35 49.10
43 30	46 30	11 38.978	46 41 38.98	23 20 49.49
44 00	46 00	11 39.546	46 11 39.55	23 05 49.77
44 30	45 30	11 39.900	45 41 39.90	22 50 49.95
45 00	45 00	11 40.041	45 11 40.04	22 35 50.02

$\varphi-\chi=+700''.0427 \sin 2\varphi-0''.9900 \sin 4\varphi+0''.0017 \sin 6\varphi.$

$\varphi-\chi=[2.84512455] \sin 2\varphi-[9.99563-10] \sin 4\varphi+[7.238-10] \sin 6\varphi.$

LATITUDE TRANSFORMATION—Continued.

Geodetic to isometric—Continued.

Geodetic latitude. φ	Geodetic colatitude p	Geodetic minus isometric. $\varphi-\chi$	Isometric colatitude. z	$\frac{z}{2}$
° ′	° ′	′ ″	° ′ ″	° ′ ″
45 00	45 00	11 40.041	45 11 40.04	22 35 50.02
45 30	44 30	11 39.969	44 41 39.97	22 20 49.98
46 00	44 00	11 39.684	44 11 39.68	22 05 49.84
46 30	43 30	11 39.185	43 41 39.18	21 50 49.59
47 00	43 00	11 38.474	43 11 38.47	21 35 49.24
47 30	42 30	11 37.549	42 41 37.55	21 20 48.77
48 00	42 00	11 36.412	42 11 36.41	21 05 48.21
48 30	41 30	11 35.063	41 41 35.06	20 50 47.53
49 00	41 00	11 33.501	41 11 33.50	20 35 46.75
49 30	40 30	11 31.728	40 41 31.73	20 20 45.86
50 00	40 00	11 29.745	40 11 29.74	20 05 44.87
50 30	39 30	11 27.550	39 41 27.55	19 50 43.78
51 00	39 00	11 25.146	39 11 25.15	19 35 42.57
51 30	38 30	11 22.533	38 41 22.53	19 20 41.27
52 00	38 00	11 19.712	38 11 19.71	19 05 39.86
52 30	37 30	11 16.683	37 41 16.68	18 50 38.34
53 00	37 00	11 13.448	37 11 13.45	18 35 36.72
53 30	36 30	11 10.007	36 41 10.01	18 20 35.00
54 00	36 00	11 06.360	36 11 06.36	18 05 33.18
54 30	35 30	11 02.512	35 41 02.51	17 50 31.26
55 00	35 00	10 58.461	35 10 58.46	17 35 29.23
55 30	34 30	10 54.208	34 40 54.21	17 20 27.10
56 00	34 00	10 49.755	34 10 49.76	17 05 24.88
56 30	33 30	10 45.104	33 40 45.10	16 50 22.55
57 00	33 00	10 40.256	33 10 40.26	16 35 20.13
57 30	32 30	10 35.212	32 40 35.21	16 20 17.61
58 00	32 00	10 29.974	32 10 29.97	16 05 14.99
58 30	31 30	10 24.543	31 40 24.54	15 50 12.27
59 00	31 00	10 18.922	31 10 18.92	15 35 09.46
59 30	30 30	10 13.111	30 40 13.11	15 20 06.56
60 00	30 00	10 07.112	30 10 07.11	15 05 03.56
60 30	29 30	10 00.928	29 40 00.93	14 50 00.46
61 00	29 00	9 54.560	29 09 54.56	14 34 57.28
61 30	28 30	9 48.010	28 39 48.01	14 19 54.00
62 00	28 00	9 41.280	28 09 41.28	14 04 50.64
62 30	27 30	9 34.372	27 39 34.37	13 49 47.19
63 00	27 00	9 27.288	27 09 27.29	13 34 43.64
63 30	26 30	9 20.031	26 39 20.03	13 19 40.02
64 00	26 00	9 12.602	26 09 12.60	13 04 36.30
64 30	25 30	9 05.005	25 39 05.00	12 49 32.50
65 00	25 00	8 57.240	25 08 57.24	12 34 28.62
65 30	24 30	8 49.310	24 38 49.31	12 19 24.66
66 00	24 00	8 41.219	24 08 41.22	12 04 20.61
66 30	23 30	8 32.968	23 38 32.97	11 49 16.48
67 00	23 00	8 24.559	23 08 24.56	11 34 12.28
67 30	22 30	8 15.996	22 38 16.00	11 19 08.00

$\varphi-\chi=+700''.0427 \sin 2\varphi-0''.9900 \sin 4\varphi+0''.0017 \sin 6\varphi.$
$\varphi-\chi=[2.84512455] \sin 2\varphi-[9.99563-10] \sin 4\varphi+[7.238-10] \sin 6\varphi.$

LATITUDE TRANSFORMATION—Continued.

Geodetic to isometric—Continued.

Geodetic latitude. φ	Geodetic colatitude p	Geodetic minus isometric. $\varphi-\chi$	Isometric colatitude. z	$\frac{z}{2}$
° ′	° ′	′ ″	° ′ ″	° ′ ″
67 30	22 30	8 15.996	22 38 16.00	11 19 08.00
68 00	22 00	8 07.281	22 08 07.28	11 04 03.64
68 30	21 30	7 58.417	21 37 58.42	10 48 59.21
69 00	21 00	7 49.406	21 07 49.41	10 33 54.70
69 30	20 30	7 40.251	20 37 40.25	10 18 50.13
70 00	20 00	7 30.955	20 07 30.96	10 03 45.48
70 30	19 30	7 21.521	19 37 21.52	9 48 40.76
71 00	19 00	7 11.952	19 07 11.95	9 33 35.98
71 30	18 30	7 02.249	18 37 02.25	9 18 31.12
72 00	18 00	6 52.418	18 06 52.42	9 03 26.21
72 30	17 30	6 42.460	17 36 42.46	8 48 21.23
73 00	17 00	6 32.378	17 06 32.38	8 33 16.19
73 30	16 30	6 22.177	16 36 22.18	8 18 11.09
74 00	16 00	6 11.858	16 06 11.86	8 03 05.93
74 30	15 30	6 01.424	15 36 01 42	7 48 00.71
75 00	15 00	5 50.880	15 05 50.88	7 32 55.44
75 30	14 30	5 40.229	14 35 40.23	7 17 50.11
76 00	14 00	5 29.472	14 05 29.47	7 02 44.74
76 30	13 30	5 18.615	13 35 18.62	6 47 29.31
77 00	13 00	5 07.660	13 05 07.66	6 32 33.83
77 30	12 30	4 56.611	12 34 56.61	6 17 28.31
78 00	12 00	4 45.470	12 04 45.47	6 02 22.74
78 30	11 30	4 34.242	11 34 34.24	5 47 17.12
79 00	11 00	4 22.930	11 04 22.93	5 32 11.46
79 30	10 30	4 11.537	10 34 11.54	5 17 05.77
80 00	10 00	4 00.067	10 04 00.07	5 02 00.03
80 30	9 30	3 48.522	9 33 48.52	4 46 54.26
81 00	9 00	3 36.907	9 03 36.91	4 31 48.45
81 30	8 30	3 25.228	8 33 25.23	4 16 42.61
82 00	8 00	3 13.484	8 03 13.48	4 01 36.74
82 30	7 30	3 01.681	7 33 01.68	3 46 30.84
83 00	7 00	2 49.822	7 02 49.82	3 31 24.91
83 30	6 30	2 37.910	6 32 37.91	3 16 18.96
84 00	6 00	2 25.951	6 02 25.95	3 01 12.98
84 30	5 30	2 13.945	5 32 13.94	2 46 06.97
85 00	5 00	2 01.900	5 02 01.90	2 31 00.95
85 30	4 30	1 49.818	4 31 49.82	2 15 54.91
86 00	4 00	1 37.701	4 01 37.70	2 00 48.85
86 30	3 30	1 25.554	3 31 25.55	1 45 42.78
87 00	3 00	1 13.381	3 01 13.38	1 30 36.69
87 30	2 30	1 01.184	2 31 01.18	1 15 30.59
88 00	2 00	0 48.971	2 00 48.97	1 00 24.49
88 30	1 30	0 36.741	1 30 36.74	0 45 18.37
89 00	1 00	0 24.500	1 00 24.50	0 30 12.25
89 30	0 30	0 12.252	0 30 12.25	0 15 06.13
90 00	0 00	0 00.000	0 00 00.00	0 00 00.00

$\varphi-\chi=+700''.0427 \sin 2\varphi-0''.9900 \sin 4\varphi+0''.0017 \sin 6\varphi.$

$\varphi-\chi=[2.84512455] \sin 2\varphi-[9.99563-10] \sin 4\varphi+[7.238-10] \sin 6\varphi.$

LATITUDE TRANSFORMATION—Continued.

Isometric to geodetic.

Isometric latitude. χ	Geodetic minus isometric. $\varphi-\chi$	Geodetic latitude. φ	Isometric latitude. χ	Geodetic minus isometric. $\varphi-\chi$	Geodetic latitude. φ
° ′	′ ″	° ′ ″	° ′	′ ″	° ′ ″
0 00	0 00.000	0 00 00.00	22 30	8 16.393	22 38 16.39
0 30	0 12.266	0 30 12.27	23 00	8 24.956	23 08 24.96
1 00	0 24.528	1 00 24.53	23 30	8 33.363	23 38 33.36
1 30	0 36.783	1 30 36.78	24 00	8 41.613	24 08 41.61
2 00	0 49.026	2 00 49.03	24 30	8 49.703	24 38 49.70
2 30	1 01.254	2 31 01.25	25 00	8 57.630	25 08 57.63
3 00	1 13.463	3 01 13.46	25 30	9 05.392	25 39 05.39
3 30	1 25.650	3 31 25.65	26 00	9 12.987	26 09 12.99
4 00	1 37.810	4 01 37.81	26 30	9 20.412	26 39 20.41
4 30	1 49.941	4 31 49.94	27 00	9 27.665	27 09 27.66
5 00	2 02.037	5 02 02.04	27 30	9 34.744	27 39 34.74
5 30	2 14.097	5 32 14.10	28 00	9 41.647	28 09 41.65
6 00	2 26.113	6 02 26.11	28 30	9 48.371	28 39 48.37
6 30	2 38.085	6 32 38.08	29 00	9 54.915	29 09 54.92
7 00	2 50.009	7 02 50.01	29 30	10 01.277	29 40 01.28
7 30	3 01.880	7 33 01.88	30 00	10 07.454	30 10 07.45
8 00	3 13.695	8 03 13.70	30 30	10 13.446	30 40 13.45
8 30	3 25.450	8 33 25.45	31 00	10 19.249	31 10 19.25
9 00	3 37.142	9 03 37.14	31 30	10 24.863	31 40 24.86
9 30	3 48.768	9 33 48.77	32 00	10 30.285	32 10 30.28
10 00	4 00.322	10 04 00.32	32 30	10 35.514	32 40 35.51
10 30	4 11.803	10 34 11.80	33 00	10 40.549	33 10 40.55
11 00	4 23.206	11 04 23.21	33 30	10 45.388	33 40 45.39
11 30	4 34.529	11 34 34.53	34 00	10 50.029	34 10 50.03
12 00	4 45.766	12 04 45.77	34 30	10 54.471	34 40 54.47
12 30	4 56.916	12 34 56.92	35 00	10 58.713	35 10 58.71
13 00	5 07.974	13 05 07.97	35 30	11 02.754	35 41 02.75
13 30	5 18.937	13 35 18.94	36 00	11 06.592	36 11 06.59
14 00	5 29.802	14 05 29.80	36 30	11 10.226	36 41 10.23
14 30	5 40.566	14 35 40.57	37 00	11 13.656	37 11 13.66
15 00	5 51.225	15 05 51.22	37 30	11 16.879	37 41 16.88
15 30	6 01.776	15 36 01.78	38 00	11 19.896	38 11 19.90
16 00	6 12.215	16 06 12.22	38 30	11 22.705	38 41 22.70
16 30	6 22.540	16 36 22.54	39 00	11 25.305	39 11 25.30
17 00	6 32.747	17 06 32.75	39 30	11 27.696	39 41 27.70
17 30	6 42.834	17 36 42.83	40 00	11 29.878	40 11 29.88
18 00	6 52.796	18 06 52.80	40 30	11 31.848	40 41 31.85
18 30	7 02.631	18 37 02.63	41 00	11 33.607	41 11 33.61
19 00	7 12.337	19 07 12.34	41 30	11 35.150	41 41 35.16
19 30	7 21.910	19 37 21.91	42 00	11 36.492	42 11 36.49
20 00	7 31.346	20 07 31.35	42 30	11 37.615	42 41 37.62
20 30	7 40.644	20 37 40.64	43 00	11 38.526	43 11 38.53
21 00	7 49.801	21 07 49.80	43 30	11 39.224	43 41 39.22
21 30	7 58.813	21 37 58.81	44 00	11 39.709	44 11 39.71
22 00	8 07.678	22 08 07.68	44 30	11 39.980	44 41 39.98
22 30	8 16.393	22 38 16.39	45 00	11 40.038	45 11 40.04

$\varphi-\chi=+700''.0420 \sin 2\chi+1''.3859 \sin 4\chi+0''.0037 \sin 6\chi.$

$\varphi-\chi=[2.84512413] \sin 2\chi+[0.141726] \sin 4\chi+[7.572-10] \sin 6\chi.$

LATITUDE TRANSFORMATION—Continued.

Isometric to geodetic—Continued.

Isometric latitude. χ	Geodetic minus isometric. $\varphi-\chi$	Geodetic latitude. φ	Isometric latitude. χ	Geodetic minus isometric. $\varphi-\chi$	Geodetic latitude. φ
° ′	′ ″	° ′ ″	° ′	′ ″	° ′ ″
45 00	11 40.038	45 11 40.04	67 30	8 13.621	67 38 13.62
45 30	11 39.883	45 41 39.88	68 00	8 04.908	68 08 04.91
46 00	11 39.515	46 11 39.52	68 30	7 56.048	68 37 56.05
46 30	11 38.934	46 41 38.93	69 00	7 47.044	69 07 47.04
47 00	11 38.140	47 11 38.14	69 30	7 37.900	69 37 37.90
47 30	11 37.134	47 41 37.13	70 00	7 28.617	70 07 28.62
48 00	11 35.916	48 11 35.92	70 30	7 19.198	70 37 19.20
48 30	11 34.485	48 41 34.48	71 00	7 09.648	71 07 09.65
49 00	11 32.844	49 11 32.84	71 30	6 59.967	71 36 59.97
49 30	11 30.992	49 41 30.99	72 00	6 50.160	72 06 50.16
50 00	11 28.930	50 11 28.93	72 30	6 40.229	72 36 40.23
50 30	11 26.658	50 41 26.66	73 00	6 30.177	73 06 30.18
51 00	11 24.178	51 11 24.18	73 30	6 20.008	73 36 20.01
51 30	11 21.490	51 41 21.49	74 00	6 09.724	74 06 09.72
52 00	11 18.595	52 11 18.60	74 30	5 59.328	74 35 59.33
52 30	11 15.493	52 41 15.49	75 00	5 48.824	75 05 48.82
53 00	11 12.187	53 11 12.19	75 30	5 38.216	75 35 38.22
53 30	11 08.676	53 41 08.68	76 00	5 27.504	76 05 27.50
54 00	11 04.963	54 11 04.96	76 30	5 16.695	76 35 16.70
54 30	11 01.048	54 41 01.05	77 00	5 05.790	77 05 05.79
55 00	10 56.932	55 10 56.93	77 30	4 54.792	77 34 54.79
55 30	10 52.617	55 40 52.62	78 00	4 43.706	78 04 43.71
56 00	10 48.103	56 10 48.10	78 30	4 32.535	78 34 32.54
56 30	10 43.394	56 40 43.39	79 00	4 21.281	79 04 21.28
57 00	10 38.489	57 10 38.49	79 30	4 09.949	79 34 09.95
57 30	10 33.391	57 40 33.39	80 00	3 58.541	80 03 58.54
58 00	10 28.101	58 10 28.10	80 30	3 47.061	80 33 47.06
58 30	10 22.620	58 40 22.62	81 00	3 35.513	81 03 35.51
59 00	10 16.951	59 10 16.95	81 30	3 23.900	81 33 23.90
59 30	10 11.095	59 40 11.10	82 00	3 12.226	82 03 12.23
60 00	10 05.054	60 10 05.05	82 30	3 00.494	82 33 00.49
60 30	9 58.830	60 39 58.83	83 00	2 48.707	83 02 48.71
61 00	9 52.424	61 09 52.42	83 30	2 36.870	83 32 36.87
61 30	9 45.839	61 39 45.84	84 00	2 24.985	84 02 24.99
62 00	9 39.077	62 09 39.08	84 30	2 13.057	84 32 13.06
62 30	9 32.140	62 39 32.14	85 00	2 01.089	85 02 01.09
63 00	9 25.029	63 09 25.03	85 30	1 49.084	85 31 49.08
63 30	9 17.748	63 39 17.75	86 00	1 37.046	86 01 37.05
64 00	9 10.297	64 09 10.30	86 30	1 24.980	86 31 24.98
64 30	9 02.681	64 39 02.68	87 00	1 12.887	87 01 12.89
65 00	8 54.900	65 08 54.90	87 30	1 00.773	87 31 00.77
65 30	8 46.958	65 38 46.96	88 00	0 48.640	88 00 48.64
66 00	8 38.857	66 08 38.86	88 30	0 36.493	88 30 36.49
66 30	8 30.598	66 38 30.60	89 00	0 24.335	89 00 24.34
67 00	8 22.186	67 08 22.19	89 30	0 12.169	89 30 12.17
67 30	8 13.621	67 38 13.62	90 00	0 00.000	90 00 00.00

$\varphi-\chi=+700.''0420 \sin 2\chi+1.''3859 \sin 4\chi+0.''0037 \sin 6\chi.$

$\varphi-\chi=[2.84512413] \sin 2\chi+[0.141726] \sin 4\chi+[7.572-10] \sin 6\chi.$

LATITUDE TRANSFORMATION—Continued.

Geodetic to authalic.

Geodetic latitude. φ	Geodetic minus authalic. $\varphi-\beta$	Authalic latitude. β		Geodetic latitude. φ	Geodetic minus authalic. $\varphi-\beta$	Authalic latitude. β
° ′	′ ″	° ′ ″		° ′	′ ″	° ′ ″
0 00	0 00.000	0 00 00.00		22 30	5 29.779	22 24 30.22
0 30	0 08.135	0 29 51.87		23 00	5 35.492	22 54 24.51
1 00	0 16.267	0 59 43.73		23 30	5 41.104	23 24 18.90
1 30	0 24.395	1 29 35.61		24 00	5 46.612	23 54 13.39
2 00	0 32.515	1 59 27.49		24 30	5 52.014	24 24 07.99
2 30	0 40.625	2 29 19.38		25 00	5 57.310	24 54 02.69
3 00	0 48.723	2 59 11.28		25 30	6 02.498	25 23 57.50
3 30	0 56.806	3 29 03.19		26 00	6 07.575	25 53 52.42
4 00	1 04.872	3 58 55.13		26 30	6 12.541	26 23 47.46
4 30	1 12.918	4 28 47.08		27 00	6 17.394	26 53 42.61
5 00	1 20.942	4 58 39.06		27 30	6 22.132	27 23 37.87
5 30	1 28.942	5 28 31.06		28 00	6 26.755	27 53 33.24
6 00	1 36.915	5 58 23.08		28 30	6 31.260	28 23 28.74
6 30	1 44.858	6 28 15.14		29 00	6 35.646	28 53 24.35
7 00	1 52.770	6 58 07.23		29 30	6 39.911	29 23 20.09
7 30	2 00.648	7 27 59.35		30 00	6 44.056	29 53 15.94
8 00	2 08.488	7 57 51.51		30 30	6 48.078	30 23 11.92
8 30	2 16.290	8 27 43.71		31 00	6 51.975	30 53 08.02
9 00	2 24.051	8 57 35.95		31 30	6 55.748	31 23 04.25
9 30	2 31.768	9 27 28.23		32 00	6 59.394	31 53 00.61
10 00	2 39.439	9 57 20.56		32 30	7 02.913	32 22 57.09
10 30	2 47.062	10 27 12.94		33 00	7 06.303	32 52 53.70
11 00	2 54.634	10 57 05.37		33 30	7 09.564	33 22 50.44
11 30	3 02.154	11 26 57.85		34 00	7 12.694	33 52 47.31
12 00	3 09.618	11 56 50.38		34 30	7 15.693	34 22 44.31
12 30	3 17.024	12 26 42.98		35 00	7 18.560	34 52 41.44
13 00	3 24.371	12 56 35.63		35 30	7 21.292	35 22 38.71
13 30	3 31.656	13 26 28.34		36 00	7 23.891	35 52 36.11
14 00	3 38.877	13 56 21.12		36 30	7 26.355	36 22 33.64
14 30	3 46.032	14 26 13.97		37 00	7 28.683	36 52 31.32
15 00	3 53.118	14 56 06.88		37 30	7 30.875	37 22 29.12
15 30	4 00.133	15 25 59.87		38 00	7 32.929	37 52 27.07
16 00	4 07.076	15 55 52.92		38 30	7 34.846	38 22 25.15
16 30	4 13.944	16 25 46.06		39 00	7 36.624	38 52 23.38
17 00	4 20.734	16 55 39.27		39 30	7 38.264	39 22 21.74
17 30	4 27.446	17 25 32.55		40 00	7 39.764	39 52 20.24
18 00	4 34.076	17 55 25.92		40 30	7 41.124	40 22 18.88
18 30	4 40.624	18 25 19.38		41 00	7 42.344	40 52 17.66
19 00	4 47.086	18 55 12.91		41 30	7 43.423	41 22 16.58
19 30	4 53.462	19 25 06.54		42 00	7 44.361	41 52 15.64
20 00	4 59.748	19 55 00.25		42 30	7 45.157	42 22 14.84
20 30	5 05.944	20 24 54.06		43 00	7 45.812	42 52 14.19
21 00	5 12.046	20 54 47.95		43 30	7 46.325	43 22 13.68
21 30	5 18.054	21 24 41.95		44 00	7 46.696	43 52 13.30
22 00	5 23.966	21 54 36.03		44 30	7 46.926	44 22 13.07
22 30	5 29.779	22 24 30.22		45 00	7 47.012	44 52 12.99

$$\varphi-\beta=+467''.0129 \sin 2\varphi-0''.4494 \sin 4\varphi+0''.0005 \sin 6\varphi.$$

$$\varphi-\beta=[2.6693289] \sin 2\varphi-[9.65258-10] \sin 4\varphi+[6.732-10] \sin 6\varphi.$$

LATITUDE TRANSFORMATION—Continued.

Geodetic to authalic—Continued.

Geodetic latitude. φ	Geodetic minus authalic. $\varphi-\beta$	Authalic latitude. β	Geodetic latitude. φ	Geodetic minus authalic. $\varphi-\beta$	Authalic latitude. β
° ′	′ ″	° ′ ″	° ′	′ ″	° ′ ″
45 00	7 47.012	44 52 12.99	67 30	5 30.678	67 24 29.32
45 30	7 46.957	45 22 13.04	68 00	5 24.864	67 54 35.14
46 00	7 46.759	45 52 13.24	68 30	5 18.951	68 24 41.05
46 30	7 46.419	46 22 13.58	69 00	5 12.940	68 54 47.06
47 00	7 45.937	46 52 14.06	69 30	5 06.833	69 24 53.17
47 30	7 45.313	47 22 14.69	70 00	5 00.633	69 54 59.37
48 00	7 44.547	47 52 15.45	70 30	4 54.341	70 25 05.66
48 30	7 43.640	48 22 16.36	71 00	4 47.958	70 55 12.04
49 00	7 42.591	48 52 17.41	71 30	4 41.488	71 25 18.51
49 30	7 41.402	49 22 18.60	72 00	4 34.931	71 55 25.07
50 00	7 40.071	49 52 19.93	72 30	4 28.290	72 25 31.71
50 30	7 38.600	50 22 21.40	73 00	4 21.567	72 55 38.43
51 00	7 36.990	50 52 23.01	73 30	4 14.764	73 25 45.24
51 30	7 35.240	51 22 24.76	74 00	4 07.883	73 55 52.12
52 00	7 33.351	51 52 26.65	74 30	4 00.927	74 25 59.07
52 30	7 31.324	52 22 28.68	75 00	3 53.896	74 56 06.10
53 00	7 29.159	52 52 30.84	75 30	3 46.794	75 26 13.21
53 30	7 26.858	53 22 33.14	76 00	3 39.622	75 56 20.38
54 00	7 24.419	53 52 35.58	76 30	3 32.383	76 26 27.62
54 30	7 21.846	54 22 38.15	77 00	3 25.080	76 56 34.92
55 00	7 19.137	54 52 40.86	77 30	3 17.713	77 26 42.29
55 30	7 16.294	55 22 43.71	78 00	3 10.286	77 56 49.71
56 00	7 13.319	55 52 46.68	78 30	3 02.800	78 26 57.20
56 30	7 10.210	56 22 49.79	79 00	2 55.259	78 57 04.74
57 00	7 06.971	56 52 53.03	79 30	2 47.663	79 27 12.34
57 30	7 03.501	57 22 56.50	80 00	2 40.017	79 57 19.98
58 00	7 00.102	57 52 59.90	80 30	2 32.322	80 27 27.68
58 30	6 56.475	58 23 03.52	81 00	2 24.579	80 57 35.42
59 00	6 52.720	58 53 07.28	81 30	2 16.793	81 27 43.21
59 30	6 48.840	59 23 11.16	82 00	2 08.965	81 57 51.04
60 00	6 44.834	59 53 15.17	82 30	2 01.097	82 27 58.90
60 30	6 40.705	60 23 19.30	83 00	1 53.192	82 58 06.81
61 00	6 36.453	60 53 23.55	83 30	1 45.252	83 28 14.75
61 30	6 32.080	61 23 27.92	84 00	1 37.280	83 58 22.72
62 00	6 27.588	61 53 32.41	84 30	1 29.279	84 28 30.72
62 30	6 22.977	62 23 37.02	85 00	1 21.250	84 58 38.75
63 00	6 18.249	62 53 41.75	85 30	1 13.196	85 28 46.80
63 30	6 13.405	63 23 46.60	86 00	1 05.120	85 58 54.88
64 00	6 08.447	63 53 51.55	86 30	0 57.023	86 29 02.98
64 30	6 03.377	64 23 56.62	87 00	0 48.910	86 59 11.09
65 00	5 58.195	64 54 01.80	87 30	0 40.781	87 29 19.22
65 30	5 52.904	65 24 07.10	88 00	0 32.640	87 59 27.36
66 00	5 47.505	65 54 12.50	88 30	0 24.489	88 29 35.51
66 30	5 42.000	66 24 18.00	89 00	0 16.330	88 59 43.67
67 00	5 36.390	66 54 23.61	89 30	0 08.166	89 29 51.87
67 30	5 30.678	67 24 29.32	90 00	0 00.000	90 00 00.00

$\varphi-\beta=+467.''0129 \sin 2\varphi-0.''4494 \sin 4\varphi+0.''0005 \sin 6\varphi.$
$\varphi-\beta=[2.6693289] \sin 2\varphi-[9.65258-10] \sin 4\varphi+[6.732-10] \sin 6\varphi.$

LATITUDE TRANSFORMATION—Continued.

Authalic to geodetic.

Authalic latitude. β	Geodetic minus authalic. $\varphi-\beta$	Geodetic latitude. φ	Authalic latitude. β	Geodetic minus authalic. $\varphi-\beta$	Geodetic latitude. φ
° ′	′ ″	° ′ ″	° ′	′ ″	° ′ ″
0 00	0 00.000	0 00 00.00	22 30	5 30.837	22 35 30.84
0 30	0 08.172	0 30 08.17	23 00	5 36.549	23 05 36.55
1 00	0 16.341	1 00 16.34	23 30	5 42.159	23 35 42.16
1 30	0 24.505	1 30 24.50	24 00	5 47.663	24 05 47.66
2 00	0 32.662	2 00 32.66	24 30	5 53.062	24 35 53.06
2 30	0 40.809	2 30 40.81	25 00	5 58.352	25 05 58.35
3 00	0 48.943	3 00 48.94	25 30	6 03.532	25 36 03.53
3 30	0 57.062	3 30 57.06	26 00	6 08.601	26 06 08.60
4 00	1 05.164	4 01 05.16	26 30	6 13.558	26 36 13.56
4 30	1 13.245	4 31 13.24	27 00	6 18.400	27 06 18.40
5 00	1 21.304	5 01 21.30	27 30	6 23 126	27 36 23.13
5 30	1 29.339	5 31 29.34	28 00	6 27.735	28 06 27.74
6 00	1 37.345	6 01 37.34	28 30	6 32.225	28 36 32.22
6 30	1 45.322	6 31 45.32	29 00	6 36.596	29 06 36.60
7 00	1 53.267	7 01 53.27	29 30	6 40.845	29 36 40.84
7 30	2 01.177	7 32 01.18	30 00	6 44.971	30 06 44.97
8 00	2 09.049	8 02 09.05	30 30	6 48.974	30 36 48.97
8 30	2 16.882	8 32 16.88	31 00	6 52.852	31 06 52.85
9 00	2 24.673	9 02 24.67	31 30	6 56.603	31 36 56.60
9 30	2 32.420	9 32 32.42	32 00	7 00.227	32 07 00.23
10 00	2 40.120	10 02 40.12	32 30	7 03.723	32 37 03.73
10 30	2 47.770	10 32 47.77	33 00	7 07.089	33 07 07.09
11 00	2 55.369	11 02 55.37	33 30	7 10.324	33 37 10.32
11 30	3 02.915	11 33 02.92	34 00	7 13.429	34 07 13.43
12 00	3 10.404	12 03 10.40	34 30	7 16.400	34 37 16.40
12 30	3 17.835	12 33 17.84	35 00	7 19.239	35 07 19.24
13 00	3 25.205	13 03 25.20	35 30	7 21.943	35 37 21.94
13 30	3 32.512	13 33 32.51	36 00	7 24.512	36 07 24.51
14 00	3 39.754	14 03 30.75	36 30	7 26.946	36 37 26.95
14 30	3 46.929	14 33 46.93	37 00	7 29.243	37 07 29.24
15 00	3 54.034	15 03 54.03	37 30	7 31.403	37 37 31.40
15 30	4 01.067	15 34 01.07	38 00	7 33.425	38 07 33.42
16 00	4 08.027	16 04 08.03	38 30	7 35.309	38 37 35.31
16 30	4 14.910	16 34 14.91	39 00	7 37.054	39 07 37.05
17 00	4 21.715	17 04 21.72	39 30	7 38.659	39 37 38.66
17 30	4 28.440	17 34 28.44	40 00	7 40.125	40 07 40.12
18 00	4 35.082	18 04 35.08	40 30	7 41.450	40 37 41.45
18 30	4 41.641	18 34 41.64	41 00	7 42.634	41 07 42.63
19 00	4 48.113	19 04 48.11	41 30	7 43.678	41 37 43.68
19 30	4 54.496	19 34 54.50	42 00	7 44.580	42 07 44.58
20 00	5 00.790	20 05 00.79	42 30	7 45.340	42 37 45.34
20 30	5 06.991	20 35 06.99	43 00	7 45.959	43 07 45.96
21 00	5 13.098	21 05 13.10	43 30	7 46.435	43 37 46.44
21 30	5 19.109	21 35 19.11	44 00	7 46.770	44 07 46.77
22 00	5 25 023	22 05 25.02	44 30	7 46.962	44 37 46.96
22 30	5 30.837	22 35 30.84	45 00	7 47.012	45 07 47.01

$\varphi-\beta=+467.''0127 \sin 2\beta+0.''6080 \sin 4\beta+0.''0011 \sin 6\beta.$

$\varphi-\beta=[2.6693287] \sin 2\beta+[9.78390-10] \sin 4\beta+[7.031-10] \sin 6\beta.$

LATITUDE TRANSFORMATION—Continued.

Authalic to geodetic—Continued.

Authalic latitude. β	Geodetic minus authalic. φ−β	Geodetic latitude. φ	Authalic latitude. β	Geodetic minus authalic. φ−β	Geodetic latitude. φ
° ′	′ ″	° ′ ″	° ′	′ ″	° ′ ″
45 00	7 47.012	45 07 47.01	67 30	5 29.621	67 35 29.62
45 30	7 46.919	45 37 46.92	68 00	5 23.808	68 05 23.81
46 00	7 46.685	46 07 46.68	68 30	5 17.896	68 35 17.90
46 30	7 46.308	46 37 46.31	69 00	5 11.889	69 05 11.89
47 00	7 45.790	47 07 45.79	69 30	5 05.787	69 35 05.79
47 30	7 45.129	47 37 45.13	70 00	4 59.592	70 04 59.59
48 00	7 44.327	48 07 44.33	70 30	4 53.307	70 34 53.31
48 30	7 43.384	48 37 43.38	71 00	4 46.933	71 04 46.93
49 00	7 42.299	49 07 42.30	71 30	4 40.472	71 34 40.47
49 30	7 41.074	49 37 41.07	72 00	4 33.926	72 04 33.93
50 00	7 39.709	50 07 39.71	72 30	4 27.297	72 34 27.30
50 30	7 38.205	50 37 38.20	73 00	4 20.588	73 04 20.59
51 00	7 36.559	51 07 36.56	73 30	4 13.799	73 34 13.80
51 30	7 34.776	51 37 34 78	74 00	4 06.934	74 04 06.93
52 00	7 32.854	52 07 32.85	74 30	3 59.994	74 33 59.99
52 30	7 30.795	52 37 30.80	75 00	3 52.981	75 03 52.98
53 00	7 28.599	53 07 28.60	75 30	3 45.898	75 33 45.90
53 30	7 26.266	53 37 26.27	76 00	3 38.746	76 03 38.75
54 00	7 23.798	54 07 23.80	76 30	3 31.529	76 33 31.53
54 30	7 21.194	54 37 21.19	77 00	3 24.247	77 03 24.25
55 00	7 18.457	55 07 18.46	77 30	3 16.903	77 33 16.90
55 30	7 15.587	55 37 15.59	78 00	3 09.500	78 03 09.50
56 00	7 12.584	56 07 12.58	78 30	3 02.040	78 33 02.04
56 30	7 09.450	56 37 09.45	79 00	2 54.525	79 02 54.52
57 00	7 06.185	57 07 06.18	79 30	2 46.957	79 32 46.96
57 30	7 02.791	57 37 02.79	80 00	2 39.338	80 02 39.34
58 00	6 59.269	58 06 59.27	80 30	2 31.671	80 32 31.67
58 30	6 55.619	58 36 55.62	81 00	2 23.958	81 02 23.96
59 00	6 51.844	59 06 51.84	81 30	2 16.202	81 32 16.20
59 30	6 47.943	59 36 47.94	82 00	2 08.405	82 02 08.40
60 00	6 43.918	60 06 43.92	82 30	2 00.569	82 32 00.57
60 30	6 39.771	60 36 39.77	83 00	1 52.696	83 01 52.70
61 00	6 35.503	61 06 35.50	83 30	1 44.789	83 31 44.79
61 30	6 31.115	61 36 31.12	84 00	1 36.851	84 01 36.85
62 00	6 26.608	62 06 26.61	84 30	1 28.883	84 31 28.88
62 30	6 21.983	62 36 21.98	85 00	1 20.889	85 01 20.89
63 00	6 17.243	63 06 17.24	85 30	1 12.870	85 31 12.87
63 30	6 12 389	63 36 12.39	86 00	1 04.828	86 01 04.83
64 00	6 07.422	64 06 07.42	86 30	0 56.768	86 30 56.77
64 30	6 02.343	64 36 02.34	87 00	0 48.690	87 00 48.69
65 00	5 57.154	65 05 57.15	87 30	0 40.598	87 30 40.60
65 30	5 51.858	65 35 51.86	88 00	0 32.493	88 00 32.49
66 00	5 46.454	66 05 46.45	88 30	0 24.378	88 30 24.38
66 30	5 40.946	66 35 40.95	89 00	0 16.256	89 00 16.26
67 00	5 35.334	67 05 35.33	89 30	0 08.129	89 30 08.13
67 30	5 29.621	67 35 29.62	90 00	0 00.000	90 00 00.00

$\varphi-\beta=+467''.0127 \sin 2\beta+0''.6080 \sin 4\beta+0''.0011 \sin 6\beta.$

$\varphi-\beta=[2.6693287] \sin 2\beta+[9.78390-10] \sin 4\beta+[7.031-10] \sin 6\beta,$

TRANSFORMATION FROM GEOGRAPHICAL TO AZIMUTHAL COORDINATES—CENTER ON THE EQUATOR.

Values of the great circle central distance, ζ. $\cos \zeta = \cos \lambda \cos \varphi$.

Long.	Lat. 0°.	Lat. 5°.	Lat. 10°.	Lat. 15°.	Lat. 20°.
°	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″
0...	0 00 00.0	5 00 00.0	10 00 00.0	15 00 00.0	20 00 00.0
5...	5 00 00.0	7 04 00.0	11 10 08.2	15 47 35.7	20 35 26.5
10...	10 00 00.0	11 10 08.2	14 06 21.6	17 57 49.8	22 16 07.4
15...	15 00 00.0	15 47 35.7	17 57 49.8	21 05 26.0	24 48 51.2
20...	20 00 00.0	20 35 26.5	22 16 07.4	24 48 51.2	27 59 27.3
25...	25 00 00.0	25 27 48.8	26 48 21.4	28 54 16.4	31 36 30.0
30...	30 00 00.0	30 22 31.8	31 28 29.8	33 13 33.4	35 31 52.9
35...	35 00 00.0	35 18 36.7	36 13 28.3	37 41 54.4	39 40 06.4
40...	40 00 00.0	40 15 32.9	41 01 35.2	42 16 24.6	43 57 29.6
45...	45 00 00.0	45 13 03.4	45 51 50.3	46 55 13.7	48 21 31.9
50...	50 00 00.0	50 10 57.7	50 43 35.6	51 37 09.1	52 50 29.2
55...	55 00 00.0	55 09 09.1	55 36 26.1	56 21 21.3	57 23 07.4
60...	60 00 00.0	60 07 32.9	60 30 04.6	61 07 15.3	61 58 32.4
65...	65 00 00.0	65 06 05.8	65 24 18.8	65 54 25.4	66 36 03.7
70...	70 00 00.0	70 04 45.6	70 18 59.4	70 42 32.4	71 15 10.0
75...	75 00 00.0	75 03 30.3	75 13 59.2	75 31 21.0	75 55 26.1
80...	80 00 00.0	80 02 18.4	80 09 12.4	80 20 38.6	80 36 31.4
85...	85 00 00.0	85 01 08.7	85 05 15.0	85 10 14.8	85 18 08.1
90...	90 00 00.0	90 00 00.0	90 00 00.0	90 00 00.0	90 00 00.0

Long.	Lat. 25°.	Lat. 30°.	Lat. 35°.	Lat. 40°.	Lat. 45°.
0...	25 00 00.0	30 00 00.0	35 00 00.0	40 00 00.0	45 00 00.0
5...	25 27 48.8	30 22 31.8	35 18 36.7	40 15 32.9	45 13 03.4
10...	26 48 21.4	31 28 29.8	36 13 28.3	41 01 35.2	45 51 50.3
15...	28 54 16.4	33 13 33.4	37 41 54.4	42 16 24.6	46 55 13.7
20...	31 36 30.0	35 31 52.9	39 40 06.4	43 57 29.6	48 21 31.9
25...	34 46 31.6	38 17 23.7	42 03 48.3	46 01 50.7	50 02 02.2
30...	38 17 23.7	41 24 34.7	44 48 48.1	48 26 21.2	52 14 19.5
35...	42 03 48.3	44 48 48.1	47 51 17.7	51 08 00.9	54 36 13.5
40...	46 01 50.7	48 26 21.2	51 08 00.9	54 04 04.9	57 12 08.1
45...	50 02 02.2	52 14 19.5	54 36 13.5	57 12 08.1	60 00 00.0
50...	54 22 08.2	56 10 27.0	58 13 40.7	60 30 04.6	62 57 57.5
55...	58 40 43.3	60 12 57.6	61 58 32.4	63 56 07.3	66 04 21.1
60...	63 03 13.6	64 20 28.0	65 45 44.9	67 28 44.4	69 17 42.7
65...	67 28 44.4	68 31 51.5	69 44 44.3	71 06 37.7	72 36 44.2
70...	71 56 32.1	72 46 14.2	73 43 47.5	74 48 39.9	76 00 16.4
75...	76 26 01.5	77 02 50.9	77 45 34.9	78 33 51.7	79 27 16.9
80...	80 56 42.8	81 21 03.0	81 49 20.2	82 21 20.6	82 56 49.1
85...	85 28 10.2	85 40 16.5	85 54 21.4	86 10 18.4	86 28 00.2
90...	90 00 00.0	90 00 00.0	90 00 00.0	90 00 00.0	90 00 00.0

TRANSFORMATION FROM GEOGRAPHICAL TO AZIMUTHAL COORDINATES—CENTER ON THE EQUATOR—Continued.

Values of the great circle central distance, ζ. $\cos \zeta = \cos \lambda \cos \varphi$—Continued.

Long.	Lat. 45°.	Lat. 50°.	Lat. 55°.	Lat. 60°.	Lat. 65°.
°	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″
0...	45 00 00.0	50 00 00.0	55 00 00.0	60 00 00.0	65 00 00.0
5...	45 13 03.4	50 10 57.7	55 09 09.1	60 07 32.9	65 06 05.8
10...	45 51 50.3	50 43 35.6	55 36 26.1	60 30 04.6	65 24 18.8
15...	46 55 13.7	51 37 09.1	56 21 21.3	61 07 15.3	65 54 25.4
20...	48 21 31.9	52 50 29.2	57 23 07.4	61 58 32.4	66 36 03.7
25...	50 02 02.2	54 22 08.2	58 40 43.3	63 03 13.6	67 28 44.4
30...	52 14 19.5	56 10 27.0	60 12 57.6	64 20 28.0	68 31 51.5
35...	54 36 13.5	58 13 40.7	61 58 32.4	65 45 44.9	69 44 44.3
40...	57 12 08.1	60 30 04.6	63 56 07.3	67 28 44.4	71 06 37.7
45...	60 00 00.0	62 57 57.5	66 04 21.1	69 [illegible]	[illegible] 28 [illegible].2
50...	62 57 57.5	65 35 43.8	68 21 55.0	71 15 10.0	74 14 14.3
55...	66 04 21.1	68 21 55.0	70 47 33.1	73 20 03.2	75 58 17.5
60...	69 17 42.7	71 15 10.0	73 20 03.2	75 31 21.0	77 48 03.3
65...	72 36 44.2	74 14 14.3	75 58 17.5	77 48 03.3	79 42 41.1
70...	76 00 16.4	77 18 00.0	78 41 11.9	80 09 12.4	81 41 20.9
75...	79 27 16.9	80 25 24.3	81 27 45.9	82 33 52.3	83 43 13.2
80...	82 56 49.1	83 35 28.9	84 17 01.8	85 01 08.7	85 47 29.2
85...	86 28 00.2	86 47 18.5	87 08 04.4	87 30 08.6	87 53 20.8
90...	90 00 00.0	90 00 00.0	90 00 00.0	90 00 00.0	90 00 00.0

	Lat. 70°.	Lat. 75°.	Lat. 80°.	Lat. 85°.	Lat. 90°.
0...	70 00 00.0	75 00 00.0	80 00 00.0	85 00 00.0	90 00 00.0
5...	70 04 45.6	75 03 30.3	80 02 18.4	85 01 08.7	90 00 00.0
10...	70 18 59.4	75 13 59.2	80 09 12.4	85 05 15.0	90 00 00.0
15...	70 42 32.4	75 31 21.0	80 20 38.6	85 10 14.8	90 00 00.0
20...	71 15 10.0	75 55 26.1	80 36 31.4	85 18 08.1	90 00 00.0
25...	71 56 32.1	76 26 01.5	80 56 42.8	85 28 10.2	90 00 00.0
30...	72 46 14.2	77 02 50.9	81 21 03.0	85 40 16.5	90 00 00.0
35...	73 43 47.5	77 45 34.9	81 49 20.2	85 54 21.4	90 00 00.0
40...	74 48 39.9	78 33 51.7	82 21 20.6	86 10 18.4	90 00 00.0
45...	76 00 16.4	79 27 16.9	82 56 49.1	86 28 00.2	90 00 00.0
50...	77 18 00.0	80 25 24.3	83 35 28.9	86 47 18.5	90 00 00.0
55...	78 41 11.9	81 27 45.9	84 17 01.8	87 08 04.4	90 00 00.0
60...	80 09 12.4	82 33 52.3	85 01 08.7	87 30 08.6	90 00 00.0
65...	81 41 20.9	83 43 13.2	85 47 29.2	87 53 20.8	90 00 00.0
70...	83 16 56.2	84 55 17.2	86 35 42.5	88 17 30.5	90 00 00.0
75...	84 55 17.2	86 09 32.5	87 25 26.6	88 42 26.8	90 00 00.0
80...	86 35 42.5	87 25 26.6	88 16 19.4	89 07 58.2	90 00 00.0
85...	88 17 30.5	88 42 26.8	89 07 58.2	89 33 53.2	90 00 00.0
90...	90 00 00.0	90 00 00.0	90 00 00.0	90 00 00.0	90 00 00.0

TRANSFORMATION FROM GEOGRAPHICAL TO AZIMUTHAL COORDINATES—CENTER ON THE EQUATOR—Continued.

Values of the azimuth reckoned from the north, α. $\tan \alpha = \sin \lambda \cot \varphi$.

Long.	Lat. 0°.	Lat. 5°.	Lat. 10°.	Lat. 15°.	Lat. 20°.
°	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″
0...	....	0 00 00.0	0 00 00.0	0 00 00.0	0 00 00.0
5...	90 00 00.0	44 53 26.8	26 18 08.9	18 01 05.3	13 27 59.0
10...	90 00 00.0	63 15 35.2	44 33 41.2	32 56 44.9	25 30 20.0
15...	90 00 00.0	71 19 23.5	55 44 03.7	44 00 25.3	35 24 59.8
20...	90 00 00.0	75 30 05.2	62 43 36.6	51 55 25.5	43 13 09.0
25...	90 00 00.0	78 18 14.7	67 21 10.4	57 37 27.9	49 15 50.7
30...	90 00 00.0	80 04 30.0	70 34 28.6	61 48 47.6	53 56 51.4
35...	90 00 00.0	81 19 38.7	72 54 42.1	64 57 36.5	57 36 08.3
40...	90 00 00.0	82 14 57.1	74 39 36.7	67 22 15.4	60 28 47.4
45...	90 00 00.0	82 56 48.4	75 59 53.0	69 14 47.1	62 45 49.3
50...	90 00 00.0	83 29 04.5	77 02 15.1	70 43 15.7	64 35 10.4
55...	90 00 00.0	83 54 13.3	77 51 07.7	71 53 12.4	66 02 35.5
60...	90 00 00.0	84 13 52.9	78 29 29.8	72 48 28.4	67 12 14.8
65...	90 00 00.0	84 29 10.1	78 59 25.2	73 31 47.0	68 07 11.2
70...	90 00 00.0	84 40 51.2	79 22 20.7	74 05 05.0	68 49 37.8
75...	90 00 00.0	84 49 28.4	79 30 17.0	74 29 45.3	69 21 11.2
80...	90 00 00.0	84 55 23.8	79 50 56.1	74 46 45.3	69 42 59.2
85...	90 00 00.0	84 58 51.6	79 57 45.3	74 56 43.1	69 55 46.9
90...	90 00 00.0	85 00 00.0	80 00 00.0	75 00 00.0	70 00 00.0

	Lat. 25°.	Lat. 30°.	Lat. 35°.	Lat. 40°.	Lat. 45°.
0...	0 00 00.0	0 00 00.0	0 00 00.0	0 00 00.0	0 00 00.0
5...	10 35 12.4	8 35 04.0	7 05 42.7	5 55 47.8	4 58 51.8
10...	20 25 29.3	16 44 22.5	13 55 41.1	11 41 31.5	9 51 03.9
15...	29 01 55.2	24 08 46.0	20 17 09.3	17 08 32.3	14 30 38.9
20...	36 15 31.4	30 38 32.4	26 02 00.4	22 10 33.6	18 52 54.2
25...	42 11 10.6	36 12 14.4	31 06 48.8	26 43 56.8	22 54 35.3
30...	46 59 49.0	40 53 36.2	35 31 46.7	30 47 23.0	26 33 54.2
35...	50 53 22.2	44 48 43.7	39 19 21.7	34 21 18.1	29 50 15.2
40...	54 02 28.1	48 04 11.6	42 33 06.5	37 27 13.4	32 43 56.7
45...	56 35 48.5	50 46 06.5	45 16 51.2	40 07 14.7	35 15 51.8
50...	58 40 12.9	52 59 43.8	47 34 15.4	42 23 38.7	37 27 13.4
55...	60 20 56.5	54 49 23.7	49 28 34.8	44 18 38.9	39 19 21.7
60...	61 41 59.8	56 18 35.8	51 02 36.3	45 54 16.9	40 53 36.2
65...	62 46 24.8	57 30 05.1	52 18 38.0	47 12 18.5	42 11 10.5
70...	63 36 28.2	58 25 59.8	53 18 30.7	48 14 12.1	43 13 09.0
75...	64 13 50.7	59 07 57.1	54 03 40.8	49 01 09.0	44 00 25.3
80...	64 39 44.6	59 37 07.5	54 35 12.5	49 34 03.2	44 33 41.2
85...	64 54 58.4	59 54 19.1	54 53 50.3	49 53 32.7	44 53 26.8
90...	65 00 00.0	60 00 00.0	55 00 00.0	50 00 00.0	45 00 00.0

TRANSFORMATION FROM GEOGRAPHICAL TO AZIMUTHAL COORDINATES—CENTER ON THE EQUATOR—Continued.

Values of the azimuth reckoned from the north, α. tan α=sin λ cot φ— Continued.

Long.	Lat. 45°.	Lat. 50°.	Lat. 55°.	Lat. 60°.	Lat. 65°.
°	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″
0...	0 00 00.0	0 00 00.0	0 00 00.0	0 00 00.0	0 00 00.0
5...	4 58 51.8	4 10 57.8	3 29 32.2	2 52 50.4	2 19 38.3
10...	9 51 03.9	8 17 24.4	6 55 57.2	5 43 30.4	4 37 45.6
15...	14 30 38.9	12 15 10.6	10 16 19.4	8 29 55.6	6 52 54.1
20...	18 52 54.2	16 00 46.4	13 28 04.2	11 10 12.8	9 03 41.7
25...	22 54 35.3	19 31 31.7	16 29 04.4	13 42 43.8	11 08 54.3
30...	26 33 54.2	22 45 37.7	19 17 43.2	16 06 07.6	13 07 27.4
35...	29 50 15.2	25 42 03.4	21 52 53.4	18 19 21.1	14 58 26.4
40...	32 43 56.7	28 20 26.8	24 13 54.4	20 21 38.1	16 41 07.5
45...	35 15 51.8	30 40 55.4	26 20 27.6	22 12 27.6	18 14 56.0
50...	37 27 13.4	32 43 56.7	28 12 31.2	23 51 31.2	19 39 26.5
55...	39 19 21.7	34 30 09.7	29 50 15.2	25 18 40.4	20 54 20.5
60...	40 53 36.2	36 00 18.8	31 13 57.1	26 33 54.2	21 59 26.0
65...	42 11 10.5	37 15 08.5	32 23 57.7	27 37 16.1	22 54 35.3
70...	43 13 09.0	38 15 20.3	33 20 38.8	28 28 52.5	23 39 44.5
75...	44 00 25.3	39 01 30.2	34 04 20.6	29 08 50.7	24 14 51.7
80...	44 33 41.2	39 34 07.3	34 35 20.3	29 37 18.0	24 39 56.5
85...	44 53 26.8	39 53 32.9	34 53 50.8	29 54 19.8	24 54 59.1
90...	45 00 00.0	40 00 00.0	35 00 00.0	30 00 00.0	25 00 00.0

Long.	Lat. 70°.	Lat. 75°.	Lat. 80°.	Lat. 85°.	Lat. 90°.
0...	0 00 00.0	0 00 00.0	0 00 00.0	0 00 00.0	0 00 00.0
5...	1 49 01.0	1 20 16.1	0 52 49.6	0 26 12.8	0 00 00.0
10...	3 36 59.2	2 39 50.4	1 45 13.6	0 52 13.4	0 00 00.0
15...	5 22 53.5	3 58 01.7	2 36 46.7	1 17 49.8	0 00 00.0
20...	7 05 45.5	5 14 10.3	3 27 04.2	1 42 50.2	0 00 00.0
25...	8 44 41.0	6 27 38.4	4 15 42.3	2 07 03.0	0 00 00.0
30...	10 18 50.8	7 37 50.7	5 02 18.1	2 30 17.2	0 00 00.0
35...	11 47 31.1	8 44 14.6	5 46 30.3	2 52 22.0	0 00 00.0
40...	13 10 04.2	9 46 20.7	6 27 58.9	3 13 07.4	0 00 00.0
45...	14 25 57.9	10 43 42.9	7 06 25.5	3 32 24.1	0 00 00.0
50...	15 34 45.8	11 35 58.1	7 41 33.5	3 50 03.3	0 00 00.0
55...	16 36 06.4	12 22 46.6	8 13 08.0	4 05 57.1	0 00 00.0
60...	17 29 42.9	13 03 51.5	8 40 55.9	4 19 58.3	0 00 00.0
65...	18 15 22.0	13 38 59.0	9 04 46.1	4 32 00.9	0 00 00.0
70...	18 52 54.2	14 07 57.8	9 24 29.0	4 41 59.5	0 00 00.0
75...	19 22 12.2	14 30 39.0	9 39 56.9	4 49 49.6	0 00 00.0
80...	19 43 11.1	14 46 55.8	9 51 03.9	4 55 27.9	0 00 00.0
85...	19 55 47.6	14 56 43.7	9 57 45.8	4 58 51.8	0 00 00.0
90...	20 00 00.0	15 00 00.0	10 00 00.0	5 00 00.0	0 00 00.0

LAMBERT'S AZIMUTHAL EQUIVALENT PROJECTION—CENTER ON THE EQUATOR.

Radial distance in units of the earth's radius, ρ. $\rho = 2 \sin \frac{\zeta}{2}$.

Long.	Lat. 0°	Lat. 5°	Lat. 10°	Lat. 15°	Lat. 20°	Lat. 25°	Lat. 30°
°							
0	0	0.087239	0.174311	0.261052	0.347296	0.432879	0.517638
5	0.087239	0.123258	0.194626	0.274772	0.357444	0.440774	0.523966
10	0.174311	0.194626	0.245575	0.312246	0.386221	0.463597	0.542460
15	0.261052	0.274772	0.312246	0.366025	0.429713	0.499147	0.571811
20	0.347296	0.357444	0.386221	0.429713	0.483689	0.544700	0.610249
25	0.432879	0.440774	0.463597	0.499147	0.544700	0.597673	0.655918
30	0.517638	0.523966	0.542460	0.571811	0.610249	0.655918	0.707107
35	0.601412	0.606573	0.621760	0.646158	0.678600	0.717770	0.762356
40	0.684040	0.688288	0.700847	0.721190	0.748537	0.781957	0.820470
45	0.765367	0.768874	0.779277	0.796225	0.819192	0.845774	0.880486
50	0.845237	0.848125	0.856712	0.870764	0.889918	0.913714	0.941626
55	0.923497	0.925858	0.932885	0.944423	0.960223	0.979962	1.003263
60	1.000000	1.001901	1.007567	1.016894	1.029712	1.045797	1.064883
65	1.074599	1.076095	1.080557	1.087917	1.098060	1.110836	1.126057
70	1.147153	1.148287	1.151673	1.157267	1.164995	1.174754	1.186425
75	1.217523	1.218332	1.220748	1.224745	1.230276	1.237280	1.245677
80	1.285575	1.286089	1.287628	1.290169	1.293696	1.298169	1.303546
85	1.351180	1.351426	1.352306	1.353376	1.355065	1.357210	1.359795
90	1.414214	1.414214	1.414214	1.414214	1.414214	1.414214	1.414214

Long.	Lat. 30°	Lat. 35°	Lat. 40°	Lat. 45°	Lat. 50°	Lat. 55°	Lat. 60°
0	0.517638	0.601412	0.684040	0.765367	0.845237	0.923497	1.000000
5	0.523966	0.606573	0.688288	0.768874	0.848125	0.925858	1.001901
10	0.542460	0.621760	0.700847	0.779277	0.856712	0.932885	1.007567
15	0.571811	0.646158	0.721190	0.796225	0.870764	0.944423	1.016894
20	0.610249	0.678600	0.748537	0.819192	0.889918	0.960223	1.029712
25	0.655918	0.717770	0.781957	0.845774	0.913714	0.979962	1.045797
30	0.707107	0.762356	0.820470	0.880486	0.941626	1.003263	1.064883
35	0.762356	0.811159	0.863126	0.917358	0.973098	1.029712	1.085799
40	0.820470	0.863126	0.909038	0.957418	1.007567	1.058881	1.110836
45	0.880486	0.917358	0.957418	1.000000	1.044491	1.090340	1.137055
50	0.941626	0.973098	1.007567	1.044491	1.083351	1.123665	1.164995
55	1.003263	1.029712	1.058881	1.090340	1.123665	1.158457	1.194330
60	1.064883	1.085799	1.110836	1.137055	1.164995	1.194330	1.224745
65	1.126057	1.143514	1.162975	1.184149	1.206936	1.230932	1.255938
70	1.186425	1.199861	1.214906	1.231386	1.249123	1.267932	1.287626
75	1.245677	1.255378	1.266280	1.278269	1.291228	1.305027	1.319539
80	1.303546	1.309775	1.316797	1.324547	1.332952	1.341939	1.351426
85	1.359795	1.362796	1.366188	1.369943	1.374029	1.378411	1.383056
90	1.414214	1.414214	1.414214	1.414214	1.414214	1.414214	1.414214

LAMBERT'S AZIMUTHAL EQUIVALENT PROJECTION—CENTER ON THE EQUATOR—Continued.

Radial distance in units of the earth's radius, ρ. $\rho=2 \sin\frac{\zeta}{2}$—Continued.

Long.	Lat. 60°	Lat. 65°	Lat. 70°	Lat. 75°	Lat. 80°	Lat. 85°	Lat. 90°
°							
0.......	1.000000	1.074599	1.147153	1.217523	1.285575	1.351180	1.414214
5.......	1.001901	1.076095	1.148287	1.218332	1.286089	1.351426	1.414214
10.......	1.007567	1.080557	1.151673	1.220748	1.287626	1.352306	1.414214
15.......	1.016894	1.087917	1.157267	1.224745	1.290169	1.353376	1.414214
20.......	1.029712	1.098060	1.164995	1.230276	1.293696	1.355065	1.414214
25.......	1.045797	1.110836	1.174754	1.237280	1.298169	1.357210	1.414214
30.......	1.064883	1.126057	1.186425	1.245677	1.303546	1.359795	1.414214
35.......	1.085799	1.143514	1.199861	1.255378	1.309775	1.362796	1.414214
40.......	1.110836	1.162975	1.214906	1.266280	1.316797	1.366188	1.414214
45.......	1.137055	1.184199	1.231386	1.278269	1.324547	1.369943	1.414214
50.......	1.164995	1.206936	1.249123	1.291228	1.332952	1.374029	1.414214
55.......	1.194330	1.230932	1.267932	1.305027	1.341939	1.378411	1.414214
60.......	1.224745	1.255938	1.287626	1.319539	1.351426	1.383056	1.414214
65.......	1.255938	1.281713	1.308018	1.334630	1.361333	1.387924	1.414214
70.......	1.287626	1.308018	1.328926	1.350169	1.371574	1.392975	1.414214
75.......	1.319539	1.334630	1.350169	1.366025	1.382069	1.398172	1.414214
80.......	1.351426	1.361333	1.371574	1.382069	1.392728	1.403471	1.414214
85.......	1.383056	1.387924	1.392975	1.398172	1.403471	1.408832	1.414214
90.......	1.414214	1.414214	1.414214	1.414214	1.414214	1.414214	1.414214

LAMBERT'S AZIMUTHAL EQUIVALENT PROJECTION—CENTER ON THE EQUATOR—Continued.

Rectangular coordinates in units of the earth's radius.

Long.	Lat. 0°.		Lat. 5°.		Lat. 10°.		Lat. 15°.	
	x	y	x	y	x	y	x	y
° 0	0	0	0	0.087239	0	0.174311	0	0.261052
5	0.087239	0	0.086991	0.087323	0.086241	0.174476	0.084992	0.261297
10	0.174311	0	0.173812	0.087571	0.172313	0.174972	0.169813	0.262032
15	0.261052	0	0.260302	0.087990	0.258051	0.175804	0.254295	0.263265
20	0.347296	0	0.346294	0.088582	0.343285	0.176979	0.338266	0.265002
25	0.432879	0	0.431623	0.089353	0.427851	0.178510	0.421558	0.267277
30	0.517638	0	0.516124	0.090310	0.511581	0.180411	0.504001	0.270093
35	0.601412	0	0.599638	0.091464	0.594311	0.182701	0.585428	0.273485
40	0.684040	0	0.682000	0.092826	0.675879	0.185404	0.665670	0.277488
45	0.765367	0	0.763056	0.094411	0.756122	0.188550	0.744560	0.282142
50	0.845237	0	0.842647	0.096237	0.834881	0.192172	0.821934	0.287499
55	0.923497	0	0.920622	0.098326	0.911995	0.196312	0.897621	0.293617
60	1.000000	0	0.996827	0.100703	0.987311	0.201021	0.971458	0.300570
65	1.074599	0	1.071115	0.103398	1.060670	0.206359	1.043276	0.308444
70	1.147153	0	1.143342	0.106449	1.131919	0.212397	1.112907	0.317341
75	1.217523	0	1.213365	0.109901	1.200903	0.219222	1.180179	0.327383
80	1.285575	0	1.281044	0.113806	1.267469	0.226937	1.244912	0.338721
85	1.351180	0	1.346245	0.118231	1.331607	0.235695	1.306926	0.351527
90	1.414214	0	1.408832	0.123257	1.392729	0.245576	1.366025	0.366025

	Lat. 15°.		Lat. 20°.		Lat. 25°.		Lat. 30°.	
0	0	0.261052	0	0.347296	0	0.432879	0	0.517638
5	0.084992	0.261297	0.083240	0.347617	0.080981	0.433272	0.078211	0.518096
10	0.169813	0.262032	0.166306	0.348581	0.161785	0.434451	0.156241	0.519473
15	0.254295	0.263265	0.249026	0.350199	0.242235	0.436429	0.233908	0.521780
20	0.338266	0.265002	0.331226	0.352484	0.322153	0.439222	0.311030	0.525038
25	0.421558	0.267277	0.412733	0.355457	0.401363	0.442855	0.387426	0.529273
30	0.504001	0.270093	0.493374	0.359147	0.479684	0.447361	0.462910	0.534523
35	0.585428	0.273485	0.572975	0.363589	0.556939	0.452782	0.537297	0.540832
40	0.665670	0.277488	0.651364	0.368827	0.632946	0.459168	0.610397	0.548258
45	0.744560	0.282142	0.728365	0.374912	0.706066	0.465622	0.682022	0.556868
50	0.821934	0.287499	0.803803	0.381911	0.780484	0.475097	0.751972	0.566744
55	0.897621	0.293617	0.877502	0.389897	0.851641	0.484802	0.820046	0.577981
60	0.971458	0.300570	0.949282	0.398961	0.920800	0.495801	0.886036	0.590691
65	1.043276	0.308444	1.018962	0.409211	0.987761	0.508217	0.949722	0.605007
70	1.112907	0.317341	1.086352	0.420776	1.052313	0.522193	1.010871	0.621083
75	1.180179	0.327383	1.151257	0.433805	1.114235	0.537905	1.069235	0.639100
80	1.244912	0.338721	1.213472	0.448481	1.173287	0.555553	1.124542	0.659270
85	1.306926	0.351527	1.272775	0.465022	1.229210	0.575380	1.176491	0.681843
90	1.366025	0.366025	1.328926	0.483690	1.281713	0.597672	1.224745	0.707107

$x = \rho \sin \alpha, \; y = \rho \cos \alpha.$

LAMBERT'S AZIMUTHAL EQUIVALENT PROJECTION—CENTER ON THE EQUATOR—Continued.

Rectangular coordinates in units of the earth's radius—Continued.

Long.	Lat. 30°.		Lat. 35°.		Lat. 40°.		Lat. 45°.	
	x	*y*	*x*	*y*	*x*	*y*	*x*	*y*
°								
0	0	0.517638	0	0.601412	0	0.684040	0	0.765367
5	0.078211	0.518096	0.074923	0.601928	0.071109	0.684605	0.066759	0.765971
10	0.156241	0.519473	0.149660	0.603479	0.142028	0.686305	0.133325	0.767787
15	0.233908	0.521780	0.224026	0.606079	0.212568	0.689152	0.199504	0.770825
20	0.311030	0.525038	0.297835	0.609748	0.282538	0.693167	0.265103	0.775110
25	0.387426	0.529273	0.370897	0.614515	0.351743	0.698379	0.329244	0.779058
30	0.462910	0.534523	0.443023	0.620417	0.419990	0.704826	0.393765	0.787531
35	0.537297	0.540832	0.514021	0.627504	0.487078	0.712559	0.456425	0.795753
40	0.610397	0.548258	0.583694	0.635835	0.552805	0.721635	0.517691	0.805385
45	0.682022	0.556868	0.651842	0.645482	0.616961	0.732126	0.577350	0.816497
50	0.751972	0.566744	0.718257	0.656527	0.679328	0.744114	0.635176	0.829164
55	0.820046	0.577981	0.782723	0.669068	0.739682	0.757694	0.690934	0.843475
60	0.886036	0.590691	0.844341	0.682676	0.797784	0.772979	0.744377	0.859533
65	0.949722	0.605007	0.904904	0.699123	0.853380	0.790097	0.795240	0.877451
70	1.010871	0.621083	0.962126	0.716924	0.906201	0.809194	0.843242	0.897359
75	1.069235	0.639100	1.016411	0.736805	0.955952	0.830435	0.888073	0.919401
80	1.124542	0.659270	1.067459	0.758974	1.002308	0.854010	0.929400	0.943738
85	1.176491	0.681843	1.114934	0.783667	1.044910	0.880132	0.966848	0.970541
90	1.224745	0.707107	1.158456	0.811160	1.083351	0.909039	1.000000	1.000000

	Lat. 45°.		Lat. 50°.		Lat. 55°.		Lat. 60°.	
0	0	0.765367	0	0.845237	0	0.923497	0	1.000000
5	0.066759	0.765971	0.061860	0.845866	0.056398	0.924139	0.050351	1.000635
10	0.133325	0.767787	0.123525	0.847760	0.112600	0.926064	0.100511	1.002542
15	0.199504	0.770825	0.184800	0.850929	0.168412	0.929286	0.149939	1.005727
20	0.265103	0.775110	0.245487	0.855389	0.223635	0.933818	0.199480	1.010205
25	0.329244	0.779058	0.305387	0.861169	0.278071	0.939682	0.247901	1.015991
30	0.393765	0.787531	0.364296	0.868302	0.331516	0.946908	0.295345	1.023106
35	0.456425	0.795753	0.422007	0.876829	0.383762	0.955528	0.341338	1.030750
40	0.517691	0.805385	0.478307	0.886800	0.434595	0.965586	0.386490	1.041432
45	0.577350	0.816497	0.532976	0.898275	0.483798	0.977129	0.429767	1.052708
50	0.635176	0.829164	0.585785	0.911320	0.531139	0.990210	0.471219	1.065441
55	0.690934	0.843475	0.636495	0.926012	0.576381	1.004891	0.510618	1.079673
60	0.744377	0.859533	0.684853	0.942438	0.619275	1.021236	0.547723	1.095445
65	0.795240	0.877451	0.730590	0.960693	0.659555	1.039318	0.582282	1.112802
70	0.843242	0.897359	0.773421	0.980881	0.696939	1.059210	0.614031	1.131788
75	0.888073	0.919401	0.813035	1.003117	0.731128	1.080994	0.642692	1.152445
80	0.929400	0.943738	0.849094	1.027521	0.761799	1.104745	0.667970	1.174806
85	0.966848	0.970541	0.881231	1.054223	0.788602	1.130542	0.689552	1.198901
90	1.000000	1.000000	0.909039	1.083351	0.811160	1.158456	0.707107	1.224745

$x=\rho \sin \alpha,\ y=\rho \cos \alpha.$

LAMBERT'S AZIMUTHAL EQUIVALENT PROJECTION—CENTER ON THE EQUATOR—Continued.

Rectangular coordinates in units of the earth's radius—Continued.

Long.	Lat. 60°.		Lat. 65°.		Lat. 70°.		Lat. 75°.	
	x	y	x	y	x	y	x	y
°								
0.......	0	1.000000	0	1.074599	0	1.147153	0	1.217523
5.......	0.050351	1.000635	0.043698	1.075207	0.036408	1.147710	0.028444	1.218000
10.......	0.100511	1.002542	0.087211	1.077032	0.072644	1.149380	0.056739	1.219429
15.......	0.149939	1.005727	0.130054	1.080079	0.108537	1.152166	0.084733	1.221810
20.......	0.199480	1.010205	0.172940	1.084356	0.143914	1.156072	0.112277	1.225142
25.......	0.247901	1.015991	0.214781	1.089874	0.178601	1.161099	0.139220	1.229422
30.......	0.295345	1.023106	0.255687	1.096644	0.212423	1.167253	0.165411	1.234646
35.......	0.341338	1.030750	0.295462	1.104684	0.245202	1.174540	0.190699	1.240809
40.......	0.386490	1.041432	0.333910	1.114008	0.276761	1.182962	0.214932	1.247906
45.......	0.429767	1.052708	0.370826	1.124640	0.306915	1.192524	0.237959	1.255925
50.......	0.471219	1.065441	0.406007	1.136597	0.334709	1.203229	0.259626	1.264857
55.......	0.510618	1.079673	0.439234	1.149898	0.362271	1.215076	0.279782	1.274684
60.......	0.547723	1.095445	0.470291	1.164563	0.387095	1.228063	0.298274	1.285385
65.......	0.582282	1.112802	0.498947	1.180610	0.409756	1.242180	0.314953	1.296935
70.......	0.614031	1.131788	0.524968	1.198048	0.430061	1.257414	0.329669	1.309303
75.......	0.642692	1.152445	0.548109	1.216887	0.447808	1.273745	0.342275	1.322449
80.......	0.667970	1.174806	0.568115	1.237122	0.462796	1.291138	0.352628	1.336326
85.......	0.689552	1.198901	0.584727	1.258741	0.474823	1.309551	0.360588	1.350874
90.......	0.707107	1.224745	0.597673	1.281713	0.483690	1.328926	0.366025	1.366025

	Lat. 75°.		Lat. 80°.		Lat. 85°.		Lat. 90°.	
°								
0.......	0	1.217523	0	1.285575	0	1.351180	0	1.414214
5.......	0.028444	1.218000	0.019762	1.285937	0.010305	1.351387	0	1.414214
10.......	0.056739	1.219429	0.039407	1.287022	0.020542	1.352150	0	1.414214
15.......	0.084733	1.221810	0.058818	1.288828	0.030638	1.353030	0	1.414214
20.......	0.112277	1.225142	0.077878	1.291350	0.040529	1.354459	0	1.414214
25.......	0.139220	1.229422	0.096471	1.294579	0.050147	1.356283	0	1.414214
30.......	0.165411	1.234646	0.114481	1.298509	0.059427	1.358496	0	1.414214
35.......	0.190699	1.240809	0.131794	1.303128	0.068301	1.361083	0	1.414214
40.......	0.214932	1.247906	0.148297	1.308420	0.076708	1.364033	0	1.414214
45.......	0.237959	1.255925	0.163878	1.314370	0.084588	1.367329	0	1.414214
50.......	0.259626	1.264857	0.178427	1.320956	0.091882	1.370953	0	1.414214
55.......	0.279782	1.274684	0.191837	1.328156	0.098534	1.374885	0	1.414214
60.......	0.298274	1.285385	0.204003	1.335940	0.104491	1.379104	0	1.414214
65.......	0.314953	1.296935	0.214824	1.344276	0.109706	1.383581	0	1.414214
70.......	0.329669	1.309303	0.224204	1.353126	0.114135	1.388292	0	1.414214
75.......	0.342275	1.322449	0.232051	1.362449	0.117736	1.393206	0	1.414214
80.......	0.352628	1.336326	0.238279	1.372193	0.120476	1.398291	0	1.414214
85.......	0.360588	1.350874	0.242811	1.382308	0.122324	1.403512	0	1.414214
90.......	0.366025	1.366025	0.245576	1.392729	0.123257	1.408832	0	1.414214

$x = \rho \sin \alpha$, $y = \rho \cos \alpha$.

APPENDIX.

After the manuscript of this publication had been sent to the printer it was suggested that another kind of latitude might be of use in some cartographic and geodetic applications. This idea was accordingly developed and it was decided to add it as an appendix so that no change of the earlier text would be necessary.

DEFINITION OF RECTIFYING LATITUDE.

A sixth kind of latitude that is of some use in applications may be defined in the following way: If a sphere is determined such that the length of a great circle upon it is equal in length to a meridian upon the earth, we may calculate the latitudes upon this sphere such that the arcs of the meridian upon it are equal to the corresponding arcs of the meridian upon the earth.

If M represents an arc of a meridian on the earth, we have

$$dM = \frac{a(1-\epsilon^2)d\varphi}{(1-\epsilon^2 \sin^2 \varphi)^{3/2}}.$$

The development of this formula is given in full in "General Theory of Polyconic Projections," United States Coast and Geodetic Survey Special Publication No. 57, pages 9 and 10.

If ω denotes the latitude upon the sphere of radius r, the differential element of the meridian will be given in the form

$$dm = r d\omega.$$

The arc of this meridian from the equator to latitude ω is therefore given in the form

$$m = r\omega.$$

On the earth the arc of the meridian from the equator to latitude φ becomes

$$M = a(1-\epsilon^2)\int_0^{\varphi} \frac{d\varphi}{(1-\epsilon^2 \sin^2 \varphi)^{3/2}}.$$

If the arc on the sphere is to be equal to this arc on the earth, we must have as the definition of ω

$$r\omega = a(1-\epsilon^2)\int_0^{\varphi}\frac{d\varphi}{(1-\epsilon^2\sin^2\varphi)^{3/2}}\,.$$

DEVELOPMENT OF $\varphi-\omega$ IN TERMS OF φ.

In order to develop this expression in a Fourier series we must first set $\sin^2\varphi = \frac{1}{2}(1-\cos 2\varphi)$ and we get

$$\begin{aligned}(1-\epsilon^2\sin^2\varphi)^{-3/2} &= \left[\frac{1}{2}(2-\epsilon^2+\epsilon^2\cos 2\varphi)\right]^{-3/2}\\ &= \left[\frac{1}{4}(4-2\epsilon^2+\epsilon^2 e^{2i\varphi}+\epsilon^2 e^{-2i\varphi})\right]^{-3/2}\\ &= \frac{8}{[1+(1-\epsilon^2)^{1/2}]^3}\left\{\frac{4-2\epsilon^2}{[1+(1-\epsilon^2)^{1/2}]^2}\right.\\ &\quad\left.+\frac{\epsilon^2}{[1+(1-\epsilon^2)^{1/2}]^2}(e^{2i\varphi}+e^{-2i\varphi})\right\}^{-3/2}\\ &= \frac{8}{[1+(1-\epsilon^2)^{1/2}]^3}\{1+n^2+ne^{2i\varphi}+ne^{-2i\varphi}\}^{-3/2}\end{aligned}$$

in which

$$n = \frac{1-(1-\epsilon^2)^{1/2}}{1+(1-\epsilon^2)^{1/2}}.$$

Finally we get

$$(1-\epsilon^2\sin^2\varphi)^{-3/2} = \frac{8}{[1+(1-\epsilon^2)^{1/2}]^3}(1+ne^{2i\varphi})^{-3/2}\,(1+ne^{-2i\varphi})^{-3/2}$$

Since n is less than unity, the quantities in the last two parentheses in the right-hand member may be developed by the binomial theorem into convergent series, and in this way we get

$$(1+ne^{2i\varphi})^{-3/2} = 1-\frac{3}{2}ne^{2i\varphi}+\frac{15}{8}n^2e^{4i\varphi}-\frac{35}{16}n^3e^{6i\varphi}+\frac{315}{128}n^4e^{8i\varphi}-\dots.$$

and

$$\begin{aligned}(1+ne^{-2i\varphi})^{-3/2} &= 1-\frac{3}{2}ne^{-2i\varphi}+\frac{15}{8}n^2e^{-4i\varphi}-\frac{35}{16}n^3e^{-6i\varphi}\\ &\quad+\frac{315}{128}n^4e^{-8i\varphi}-\dots\dots\end{aligned}$$

If we multiply these two series and replace $e^{2is\varphi}+e^{-2is\varphi}$ by its equivalent $2\cos 2s\varphi$, we get

$$(1+ne^{2i\varphi})^{-3/2}(1+ne^{-2i\varphi})^{-3/2}=\left(1+\frac{9}{4}n^2+\frac{225}{64}n^4+\cdots\cdots\right)$$
$$-\left(3n+\frac{45}{8}n^3+\cdots\cdots\right)\cos 2\varphi+\left(\frac{15}{4}n^2\right.$$
$$\left.+\frac{105}{16}n^4+\cdots\cdots\right)\cos 4\varphi-\left(\frac{35}{8}n^3+\cdots\cdots\right)\cos 6\varphi$$
$$+\left(\frac{315}{64}n^4+\cdots\cdots\right)\cos 8\varphi-\cdots\cdots\cdots.$$

From the definition of n we obtain the values

$$1-\epsilon^2=\left(\frac{1-n}{1+n}\right)^2$$

and

$$\frac{8}{[1+(1-\epsilon^2)^{1/2}]^3}=(1+n)^3.$$

By substituting these values in the original integral we get

$$r\omega=a(1-n)(1-n^2)\int_0^\varphi\left[\left(1+\frac{9}{4}n^2+\frac{225}{64}n^4+\cdots\cdots\right)\right.$$
$$-\left(3n+\frac{45}{8}n^3+\cdots\cdots\right)\cos 2\varphi+\left(\frac{15}{4}n^2+\frac{105}{16}n^4\right.$$
$$\left.+\cdots\cdots\right)\cos 4\varphi-\left(\frac{35}{8}n^3+\cdots\cdots\right)\cos 6\varphi$$
$$\left.+\left(\frac{315}{64}n^4+\cdots\cdots\right)\cos 8\varphi-\cdots\cdots\right]d\varphi,$$

or after integration

$$r\omega=a(1-n)(1-n^2)\left[\left(1+\frac{9}{4}n^2+\frac{225}{64}n^4+\cdots\cdots\right)\varphi-\left(\frac{3}{2}n\right.\right.$$
$$\left.+\frac{45}{16}n^3+\cdots\cdots\right)\sin 2\varphi+\left(\frac{15}{16}n^2+\frac{105}{64}n^4+\cdots\cdots\right)\sin 4\varphi$$
$$\left.-\left(\frac{35}{48}n^3+\cdots\cdots\right)\sin 6\varphi+\left(\frac{315}{512}n^4+\cdots\right)\sin 8\varphi-\cdots\right].$$

The value of r may now be determined by the condition that ω and φ are to become $\frac{\pi}{2}$ at one and the same time. This condition gives for r the value

$$r=a(1-n)(1-n^2)\left(1+\frac{9}{4}n^2+\frac{225}{64}n^4+\cdots\cdots\right).$$

With this value of r we get

$$\varphi-\omega=\frac{\frac{3}{2}n+\frac{45}{16}n^3+\cdots\cdots}{1+\frac{9}{4}n^2+\frac{225}{64}n^4+\cdots\cdots}\sin 2\varphi$$

$$-\frac{\frac{15}{16}n^2+\frac{105}{64}n^4+\cdots\cdots}{1+\frac{9}{4}n^2+\frac{225}{64}n^4+\cdots\cdot}\sin 4\varphi$$

$$+\frac{\frac{35}{48}n^3+\cdots\cdots}{1+\frac{9}{4}n^2+\frac{225}{64}n^4+\cdots\cdots}\sin 6\varphi$$

$$-\frac{\frac{315}{512}n^4+\cdots\cdot}{1+\frac{9}{4}n^2+\frac{225}{64}n^4+\cdots}\sin 8\varphi+\cdots$$

or approximately in terms of n up to the fourth order inclusive,

$$\varphi-\omega=\left(\frac{3}{2}n-\frac{9}{16}n^3\right)\sin 2\varphi-\left(\frac{15}{16}n^2-\frac{15}{32}n^4\right)\sin 4\varphi+\frac{35}{48}n^3\sin 6\varphi$$
$$-\frac{315}{512}n^4\sin 8\varphi.$$

The latitude ω may be called the rectifying latitude, since it can be used in the computation of arcs of the meridian. The length of the meridian on the earth from the equator to a given latitude φ_1 is given by the formula

$$M=r\omega_1$$

in which ω_1 is the rectifying latitude corresponding to the geodetic latitude φ_1. The meridional arc between the latitudes φ_1 and φ_2 is accordingly given by the expression

$$M=r(\omega_2-\omega_1).$$

The radius of curvature in the meridian (ρ_m) is equal to the value of the expression $r\frac{d\omega}{d\varphi}$. Accordingly we get as the approximation for this quantity

$$\rho_m = r - r\left(3n - \frac{9}{8}n^3\right)\cos 2\varphi + r\left(\frac{15}{4}n^2 - \frac{15}{8}n^4\right)\cos 4\varphi - \frac{35}{8}rn^3\cos 6\varphi + \frac{315}{64}rn^4\cos 8\varphi.$$

DEVELOPMENT OF $\varphi-\omega$ IN TERMS OF ω.

If $f(\omega) = \left(\frac{3}{2}n - \frac{9}{16}n^3\right)\sin 2\omega - \left(\frac{15}{16}n^2 - \frac{15}{32}n^4\right)\sin 4\omega + \frac{35}{48}n^3\sin 6\omega - \frac{315}{512}n^4\sin 8\omega,$

we shall have by application of Lagrange's development

$$\varphi = \omega + \frac{1}{1!}f(\omega) + \frac{1}{2!}\frac{d}{d\omega}[f(\omega)]^2 + \frac{1}{3!}\frac{d^2}{d\omega^2}[f(\omega)]^3 + \frac{1}{4!}\frac{d^3}{d\omega^3}[f(\omega)]^4 + \cdots\cdots$$

By raising to the required power and reducing by aid of the reduction table on page 88, we get the approximations

$$[f(\omega)]^2 = \frac{9}{8}n^2 - \frac{207}{512}n^4 - \frac{45}{32}n^3\cos 2\omega - \left(\frac{9}{8}n^2 - \frac{31}{16}n^4\right)\cos 4\omega + \frac{45}{32}n^3\cos 6\omega - \frac{785}{512}n^4\cos 8\omega,$$

$$[f(\omega)]^3 = \frac{81}{32}n^3\sin 2\omega - \frac{405}{128}n^4\sin 4\omega - \frac{27}{32}n^3\sin 6\omega + \frac{405}{256}n^4\sin 8\omega,$$

$$[f(\omega)]^4 = \frac{243}{128}n^4 - \frac{81}{32}n^4\cos 4\omega + \frac{81}{128}n^4\cos 8\omega.$$

By differentiating these expressions we obtain the values

$$\frac{d}{d\omega}[f(\omega)]^2 = \frac{45}{16}n^3\sin 2\omega + \left(\frac{9}{2} \quad -\frac{31}{4}n^4\right)\sin 4\omega - \frac{135}{16}n^3\sin 6\omega + \frac{785}{64}n^4\sin 8\omega,$$

$$\frac{d^2}{d\omega^2}[f(\omega)]^3 = -\frac{81}{8}n^3 \sin 2\omega + \frac{405}{8}n^4 \sin 4\omega + \frac{243}{8}n^3 \sin 6\omega - \frac{405}{4}n^4 \sin 8\omega,$$

$$\frac{d^3}{d\omega^3}[f(\omega)]^4 = -162n^4 \sin 4\omega + 324n^4 \sin 8\omega.$$

By substituting these values in the Lagrange development, we get

$$\begin{aligned}\varphi = \omega &+ \left(\frac{3}{2}n - \frac{9}{16}n^3\right) \sin 2\omega - \left(\frac{15}{16}n^2 - \frac{15}{32}n^4\right) \sin 4\omega \\ &+ \frac{35}{48}n^3 \sin 6\omega - \frac{315}{512}n^4 \sin 8\omega + \frac{45}{32}n^3 \sin 2\omega \\ &+ \left(\frac{9}{4}n^2 - \frac{31}{8}n^4\right) \sin 4\omega - \frac{135}{32}n^3 \sin 6\omega + \frac{785}{128}n^4 \sin 8\omega \\ &- \frac{27}{16}n^3 \sin 2\omega + \frac{135}{16}n^4 \sin 4\omega + \frac{81}{16}n^3 \sin 6\omega \\ &- \frac{135}{8}n^4 \sin 8\omega - \frac{27}{4}n^4 \sin 4\omega + \frac{27}{2}n^4 \sin 8\omega.\end{aligned}$$

By collecting similar terms we obtain the approximation

$$\begin{aligned}\varphi - \omega = &\left(\frac{3}{2}n - \frac{27}{32}n^3\right) \sin 2\omega + \left(\frac{21}{16}n^2 - \frac{55}{32}n^4\right) \sin 4\omega \\ &+ \frac{151}{96}n^3 \sin 6\omega + \frac{1097}{512}n^4 \sin 8\omega.\end{aligned}$$

TABULATION OF THE DEVELOPMENT.

For convenience of reference we shall give the general approximations in terms of n and then the numerical values of the coefficients for the Clarke Spheroid of 1866.

$$\begin{aligned}\varphi - \omega = &\left(\frac{3}{2}n - \frac{9}{16}n^3\right) \sin 2\varphi - \left(\frac{15}{16}n^2 - \frac{15}{32}n^4\right) \sin 4\varphi \\ &+ \frac{35}{48}n^3 \sin 6\varphi - \frac{315}{512}n^4 \sin 8\omega.\end{aligned}$$

$\log n = 7.22991610 - 10.$

$\varphi - \omega = 525.''3298 \sin 2\varphi - 0.''5575 \sin 4\varphi + 0.''0007 \sin 6\varphi.$

$$\varphi - \omega = [2.7204320] \sin 2\varphi - [9.74623 - 10] \sin 4\varphi + [6.867 - 10] \sin 6\varphi.$$

$$\varphi-\omega=\left(\frac{3}{2}n-\frac{27}{32}n^3\right)\sin 2\omega+\left(\frac{21}{16}n^2-\frac{55}{32}n^4\right)\sin 4\omega$$
$$+\frac{151}{96}n^3\sin 6\omega+\frac{1097}{512}n^4\sin 8\omega.$$

$$\varphi-\omega=525''.3295\sin 2\omega+0''.7805\sin 4\omega+0''.0016\sin 6\omega.$$

$$\varphi-\omega=[2.7204318]\sin 2\omega+[9.89236-10]\sin 4\omega$$
$$+[7.201-10]\sin 6\omega.$$

$$r=a\,(1-n)\,(1-n^2)\left(1+\frac{9}{4}n^2+\frac{225}{64}n^4+\cdots\cdot\right)$$

$\log r=6.80396212.$

LATITUDE TRANSFORMATION.

Geodetic to rectifying.

Geodetic latitude. φ	Geodetic minus rectifying. φ−ω	Rectifying latitude. ω	Geodetic latitude. φ	Geodetic minus rectifying. φ−ω	Rectifying latitude. ω
° ′	′ ″	° ′ ″	° ′	′ ″	° ′ ″
0 00	0 00.000	0 00 00.00	22 30	6 10.907	22 23 49.09
0 30	0 09.149	0 29 50.85	23 00	6 17.334	22 53 42.67
1 00	0 18.295	0 59 41.70	23 30	6 23.646	23 23 36.35
1 30	0 27.435	1 29 32.56	24 00	6 29.842	23 53 30.16
2 00	0 36.568	1 59 23.43	24 30	6 35.920	24 23 24.08
2 30	0 45.689	2 29 14.31	25 00	6 41.877	24 53 18.12
3 00	0 54.796	2 59 05.20	25 30	6 47.713	25 23 12.29
3 30	1 03.887	3 28 56.11	26 00	6 53.425	25 53 06.58
4 00	1 12.958	3 58 47.04	26 30	6 59.011	26 23 00.99
4 30	1 22.008	4 28 37.99	27 00	7 04.471	26 52 55.53
5 00	1 31.032	4 58 28.97	27 30	7 09.801	27 22 50.20
5 30	1 40.029	5 28 19.97	28 00	7 15.001	27 52 45.00
6 00	1 48.996	5 58 11.00	28 30	7 20.069	28 22 39.93
6 30	1 57.930	6 28 02.07	29 00	7 25.004	28 52 35.00
7 00	2 06.828	6 57 53.17	29 30	7 29.803	29 22 30.20
7 30	2 15.687	7 27 44.31	30 00	7 34.466	29 52 35.53
8 00	2 24.506	7 57 35.49	30 30	7 38.991	30 22 21.01
8 30	2 33.280	8 27 26.72	31 00	7 43.376	30 52 16.62
9 00	2 42.009	8 57 17.99	31 30	7 47.621	31 22 12.38
9 30	2 50.688	9 27 09.31	32 00	7 51.724	31 52 08.28
10 00	2 59.316	9 57 00.68	32 30	7 55.683	32 22 04.32
10 30	3 07.889	10 26 52.11	33 00	7 59.498	32 52 00.50
11 00	3 16.405	10 56 43.60	33 30	8 03.167	33 21 56.83
11 30	3 24.862	11 26 35.14	34 00	8 06.690	33 51 53.31
12 00	3 33.257	11 56 26.74	34 30	8 10.064	34 21 49.94
12 30	3 41.588	12 26 18.41	35 00	8 13.290	34 51 46.71
13 00	3 49.851	12 56 10.15	35 30	8 16.366	35 21 43.63
13 30	3 58.044	13 26 01.96	36 00	8 19.290	35 51 40.71
14 00	4 06.166	13 55 53.83	36 30	8 22.063	36 21 37.94
14 30	4 14.213	14 25 45.79	37 00	8 24.684	36 51 35.32
15 00	4 22.183	14 55 37.82	37 30	8 27.150	37 21 32.85
15 30	4 30.073	15 25 29.93	38 00	8 29.463	37 51 30.54
16 00	4 37.882	15 55 22.12	38 30	8 31.621	38 21 28.38
16 30	4 45.606	16 25 14.39	39 00	8 33.623	38 51 26.38
17 00	4 53.244	16 55 06.76	39 30	8 35.469	39 21 24.53
17 30	5 00.794	17 24 59.21	40 00	8 37.158	39 51 22.84
18 00	5 08.252	17 54 51.75	40 30	8 38.689	40 21 21.31
18 30	5 15.616	18 24 44.38	41 00	8 40.063	40 51 19.94
19 00	5 22.885	18 54 37.12	41 30	8 41.278	41 21 18.72
19 30	5 30.056	19 24 29.94	42 00	8 42.335	41 51 17.66
20 00	5 37.127	19 54 22.87	42 30	8 43.233	42 21 16.77
20 30	5 44.096	20 24 15.90	43 00	8 43.972	42 51 16.03
21 00	5 50.960	20 54 09.04	43 30	8 44.551	43 21 15.45
21 30	5 57.718	21 24 02.28	44 00	8 44.970	43 51 15.03
22 00	6 04.368	21 53 55.63	44 30	8 45.230	44 21 14.77
22 30	6 10.907	22 23 49.09	45 00	8 45.329	44 51 14.67

$\varphi-\omega=+525''.3298 \sin 2\varphi-0''.5575 \sin 4\varphi+0''.0007 \sin 6\varphi.$

$\varphi-\omega=[2.7204320] \sin 2\varphi-[9.74623-10] \sin 4\varphi+[6.867-10] \sin 6\varphi.$

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LATITUDE TRANSFORMATION—Continued.

Geodetic to rectifying—Continued.

Geodetic latitude. φ	Geodetic minus rectifying. $\varphi-\omega$	Rectifying latitude. ω	Geodetic latitude. φ	Geodetic minus rectifying. $\varphi-\omega$	Rectifying latitude. ω
° ′	′ ″	° ′ ″	° ′	′ ″	° ′ ″
45 00	8 45.329	44 51 14.67	67 30	6 12.022	67 23 47.98
45 30	8 45.269	45 21 14.73	68 00	6 05.482	67 53 54.52
46 00	8 45.048	45 51 14.95	68 30	5 58.831	68 24 01.17
46 30	8 44.668	46 21 15.33	69 00	5 52.069	68 54 07.93
47 00	8 44.127	46 51 15.87	69 30	5 45.200	69 24 14.80
47 30	8 43.427	47 21 16.57	70 00	5 38.225	69 54 21.78
48 00	8 42.567	47 51 17.43	70 30	5 31.147	70 24 28.85
48 30	8 41.548	48 21 18.45	71 00	5 23.967	70 54 36.03
49 00	8 40.370	48 51 19.63	71 30	5 16.688	71 24 43.31
49 30	8 39.034	49 21 20.97	72 00	5 09.312	71 54 50.69
50 00	8 37.539	49 51 22.46	72 30	5 01.841	72 24 58.16
50 30	8 35.886	50 21 24.11	73 00	4 54.278	72 55 05.72
51 00	8 34.076	50 51 25.92	73 30	4 46.625	73 25 13.38
51 30	8 32.110	51 21 27.89	74 00	4 38.884	73 55 21.12
52 00	8 29.986	51 51 30.01	74 30	4 31.058	74 25 28.94
52 30	8 27.708	52 21 32.29	75 00	4 23.148	74 55 36.85
53 00	8 25.274	52 51 34.73	75 30	4 15.158	75 25 44.84
53 30	8 22.687	53 21 37.31	76 00	4 07.090	75 55 52.91
54 00	8 19.946	53 51 40.05	76 30	3 58.946	76 26 01.05
54 30	8 17.052	54 21 42.95	77 00	3 50.729	76 56 09.27
55 00	8 14.006	54 51 45.99	77 30	3 42.442	77 26 17.56
55 30	8 10.810	55 21 49.19	78 00	3 34.086	77 56 25.91
56 00	8 07.464	55 51 52.54	78 30	3 25.664	78 26 34.34
56 30	8 03.969	56 21 56.03	79 00	3 17.180	78 56 42.82
57 00	8 00.327	56 51 59.67	79 30	3 08.635	79 26 51.36
57 30	7 56.537	57 22 03.46	80 00	3 00.032	79 56 59.97
58 00	7 52.602	57 52 07.40	80 30	2 51.374	80 27 08.63
58 30	7 48.523	58 22 11.48	81 00	2 42.664	80 57 17.34
59 00	7 44.301	58 52 15.70	81 30	2 33.904	81 27 26.10
59 30	7 39.937	59 22 20.06	82 00	2 25.096	81 57 34.90
60 00	7 35.432	59 52 24.57	82 30	2 16.245	82 27 43.76
60 30	7 30.788	60 22 29.21	83 00	2 07.351	82 57 52.65
61 00	7 26.006	60 52 33.99	83 30	1 58.418	83 28 01.58
61 30	7 21.088	61 22 38.91	84 00	1 49.449	83 58 10.55
62 00	7 16.035	61 52 43.96	84 30	1 40.447	84 28 19.55
62 30	7 10.849	62 22 49.15	85 00	1 31.414	84 58 28.59
63 00	7 05.531	62 52 54.47	85 30	1 22.352	85 28 37.65
63 30	7 00.083	63 22 59.92	86 00	1 13.266	85 58 46.73
64 00	6 54.507	63 53 05.49	86 30	1 04.157	86 28 55.84
64 30	6 48.804	64 23 11.20	87 00	0 55.028	86 59 04.97
65 00	6 42.975	64 53 17.02	87 30	0 45.882	87 29 14.12
65 30	6 37.024	65 23 22.98	88 00	0 36.723	87 59 23.28
66 00	6 30.951	65 53 29.05	88 30	0 27.552	88 29 32.45
66 30	6 24.758	66 23 35.24	89 00	0 18.373	88 59 41.63
67 00	6 18.448	66 53 41.55	89 30	0 09.188	89 29 50.81
67 30	6 12.022	67 23 47.98	90 00	0 00.000	90 00 00.00

$\varphi-\omega=+525''.3298 \sin 2\varphi-0''.5575 \sin 4\varphi+0''.0007 \sin 6\varphi.$

$\varphi-\omega=[2.7204320] \sin 2\varphi-[9.74623-10] \sin 4\varphi+[6.867-10] \sin 6\varphi.$

LATITUDE TRANSFORMATION—Continued.

Rectifying to geodetic.

Rectifying latitude. ω	Geodetic minus rectifying. $\varphi-\omega$	Geodetic latitude. φ	Rectifying latitude. ω	Geodetic minus rectifying. $\varphi-\omega$	Geodetic latitude. φ
° ′	′ ″	° ′ ″	° ′	′ ″	° ′ ″
0 00	0 00.000	0 00 00.00	22 30	6 12.246	22 36 12.25
0 30	0 09.196	0 30 09.20	23 00	6 18.672	23 06 18.67
1 00	0 18.388	1 00 18.39	23 30	6 24.981	23 36 24.98
1 30	0 27.576	1 30 27.58	24 00	6 31.173	24 06 31.17
2 00	0 36.754	2 00 36.75	24 30	6 37.245	24 36 37.24
2 30	0 45.921	2 30 45.92	25 00	6 43.195	25 06 43.20
3 00	0 55.075	3 00 55.08	25 30	6 49.022	25 36 49.02
3 30	1 04.211	3 31 04.21	26 00	6 54.723	26 06 54.72
4 00	1 13.328	4 01 13.33	26 30	7 00.298	26 37 00.30
4 30	1 22.422	4 31 22.42	27 00	7 05.743	27 07 05.74
5 00	1 31.490	5 01 31.49	27 30	7 11.058	27 37 11.06
5 30	1 40.531	5 31 40.53	28 00	7 16.242	28 07 16.24
6 00	1 49.540	6 01 49.54	28 30	7 21.292	28 37 21.29
6 30	1 58.516	6 31 58.52	29 00	7 26.206	29 07 26.21
7 00	2 07.456	7 02 07.46	29 30	7 30.984	29 37 30.98
7 30	2 16.357	7 32 16.36	30 00	7 35.625	30 07 35.62
8 00	2 25.215	8 02 25.22	30 30	7 40.125	30 37 40.12
8 30	2 34.029	8 32 34.03	31 00	7 44.485	31 07 44.48
9 00	2 42.796	9 02 42.80	31 30	7 48.703	31 37 48.70
9 30	2 51.512	9 32 51.51	32 00	7 52.778	32 07 52.78
10 00	3 00.176	10 03 00.18	32 30	7 56.708	32 37 56.71
10 30	3 08.785	10 33 08.78	33 00	8 00.492	33 08 00.49
11 00	3 17.336	11 03 17.34	33 30	8 04.129	33 38 04.13
11 30	3 25.826	11 33 25.83	34 00	8 07.618	34 08 07.62
12 00	3 34.252	12 03 34.25	34 30	8 10.959	34 38 10.96
12 30	3 42.613	12 33 42.61	35 00	8 14.149	35 08 14.15
13 00	3 50.906	13 03 50.91	35 30	8 17.188	35 38 17.19
13 30	3 59.128	13 33 59.13	36 00	8 20.076	36 08 20.08
14 00	4 07.276	14 04 07.28	36 30	8 22.811	36 38 22.81
14 30	4 15.348	14 34 15.35	37 00	8 25.392	37 08 25.39
15 00	4 23.342	15 04 23.34	37 30	8 27.818	37 38 27.82
15 30	4 31.255	15 34 31.26	38 00	8 30.090	38 08 30.09
16 00	4 39.085	16 04 39.08	38 30	8 32.206	38 38 32.21
16 30	4 46.830	16 34 46.83	39 00	8 34.166	39 08 34.17
17 00	4 54.486	17 04 54.49	39 30	8 35.969	39 38 35.97
17 30	5 02.052	17 35 02.05	40 00	8 37.614	40 08 37.61
18 00	5 09.525	18 05 09.52	40 30	8 39.102	40 38 39.10
18 30	5 16.903	18 35 16.90	41 00	8 40.431	41 08 40.43
19 00	5 24.184	19 05 24.18	41 30	8 41.601	41 38 41.60
19 30	5 31.365	19 35 31.36	42 00	8 42.612	42 08 42.61
20 00	5 38.445	20 05 38.44	42 30	8 43.464	42 38 43.46
20 30	5 45.421	20 35 45.42	43 00	8 44.157	43 08 44.16
21 00	5 52.292	21 05 52.29	43 30	8 44.690	43 38 44.69
21 30	5 59.054	21 35 59.05	44 00	8 45.062	44 08 45.06
22 00	6 05.706	22 06 05.71	44 30	8 45.275	44 38 45.28
22 30	6 12.246	22 36 12.25	45 00	8 45.328	45 08 45.33

$\varphi-\omega=+525''.3295 \sin 2\omega+0''.7805 \sin 4\omega+0''.0016 \sin 6\omega.$

$\varphi-\omega=[2.7204318] \sin 2\omega+[9.89236-10] \sin 4\omega+[7.201-10] \sin 6\omega.$

LATITUDE TRANSFORMATION—Continued.

Rectifying to geodetic—Continued.

Rectifying latitude. ω ° ′	Geodetic minus rectifying. $\varphi-\omega$ ′ ″	Geodetic latitude. φ ° ′ ″	Rectifying latitude. ω ° ′	Geodetic minus rectifying. $\varphi-\omega$ ′ ″	Geodetic latitude. φ ° ′ ″
45 00	8 45.328	45 08 45.33	67 30	6 10.685	67 36 10.68
45 30	8 45.221	45 38 45.22	68 00	6 04.146	68 06 04.15
46 00	8 44.954	46 08 44.95	68 30	5 57.497	68 35 57.50
46 30	8 44.526	46 38 44.53	69 00	5 50.739	69 05 50.74
47 00	8 43.940	47 08 43.94	69 30	5 43.876	69 35 43.88
47 30	8 43.194	47 38 43.19	70 00	5 36.908	70 05 36.91
48 00	8 42.288	48 08 42.29	70 30	5 29.839	70 35 29.84
48 30	8 41.224	48 38 41.22	71 00	5 22.669	71 05 22.67
49 00	8 40.000	49 08 40.00	71 30	5 15.402	71 35 15.40
49 30	8 38.619	49 38 38.62	72 00	5 08.040	72 05 08.04
50 00	8 37.080	50 08 37.08	72 30	5 00.585	72 35 00.58
50 30	8 35.384	50 38 35.38	73 00	4 53.038	73 04 53.04
51 00	8 33.531	51 08 33.53	73 30	4 45.404	73 34 45.40
51 30	8 31.522	51 38 31.52	74 00	4 37.682	74 04 37.68
52 00	8 29.357	52 08 29.36	74 30	4 29.877	74 34 29.88
52 30	8 27.038	52 38 27.04	75 00	4 21.990	75 04 21.99
53 00	8 24.564	53 08 24.56	75 30	4 14.024	75 34 14.02
53 30	8 21.938	53 38 21.94	76 00	4 05.982	76 04 05.98
54 00	8 19.158	54 08 19.16	76 30	3 57.865	76 33 57.86
54 30	8 16.227	54 38 16.23	77 00	3 49.676	77 03 49.68
55 00	8 13.146	55 08 13.15	77 30	3 41.417	77 33 41.42
55 30	8 09.914	55 38 09.91	78 00	3 33.092	78 03 33.09
56 00	8 06.534	56 08 06.53	78 30	3 24.703	78 33 24.70
56 30	8 03.006	56 38 03.01	79 00	3 16.251	79 03 16.25
57 00	7 59.332	57 07 59.33	79 30	3 07.740	79 33 07.74
57 30	7 55.512	57 37 55.51	80 00	2 59.173	80 02 59.17
58 00	7 51.548	58 07 51.55	80 30	2 50.551	80 32 50.55
58 30	7 47.440	58 37 47.44	81 00	2 41.878	81 02 41.88
59 00	7 43.191	59 07 43.19	81 30	2 33.156	81 32 33.16
59 30	7 38.802	59 37 38.80	82 00	2 24.388	82 02 24.39
60 00	7 34.273	60 07 34.27	82 30	2 15.576	82 32 15.58
60 30	7 29.606	60 37 29.61	83 00	2 06.723	83 02 06.72
61 00	7 24.803	61 07 24.80	83 30	1 57.832	83 31 57.83
61 30	7 19.866	61 37 19.87	84 00	1 48.906	84 01 48.91
62 00	7 14.794	62 07 14.79	84 30	1 39.946	84 31 39.95
62 30	7 09.592	62 37 09.59	85 00	1 30.956	85 01 30.96
63 00	7 04.259	63 07 04.26	85 30	1 21.939	85 31 21.94
63 30	6 58.797	63 36 58.80	86 00	1 12.897	86 01 12.90
64 00	6 53.209	64 06 53.21	86 30	1 03.833	86 31 03.83
64 30	6 47.495	64 36 47.50	87 00	0 54.750	87 00 54.75
65 00	6 41.658	65 06 41.66	87 30	0 45.650	87 30 45.65
65 30	6 35.699	65 36 35.70	88 00	0 36.537	88 00 36.54
66 00	6 29.621	66 06 29.62	88 30	0 27.412	88 30 27.41
66 30	6 23.424	66 36 23.42	89 00	0 18.280	89 00 18.28
67 00	6 17.112	67 06 17.11	89 30	0 09.141	89 30 09.14
67 30	6 10.685	67 36 10.68	90 00	0 00.000	90 00 00.00

$\varphi-\omega=+525.''3295 \sin 2\omega+0.''7805 \sin 4\omega+0.''0016 \sin 6\omega.$

$\varphi-\omega=[2.7204318] \sin 2\omega+[9.89236-10] \sin 4\omega+[7.201-10] \sin 6\omega.$