Stability and Motion Control of Ocean Vehicles

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STABILITY AND MOTION CONTROL OF OCEAN VEHICLES

(Organization, Development, and Initial Notes
of a Course of Instruction in the Subject)

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Alfred H. Keil, Chairman

May, 1969

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INTRODUCTION

The purpose of the project, of which this report is a part, was to develop and initiate a course of instruction in the subject area of motions of vehicles in the ocean environment. The motion stability and control of ocean vehicles and the response of these vehicles to the excitation of the environment in many instances determine the success or failure of some vehicles in accomplishing the intended mission. An ocean engineer, whether he is involved in the design and operation of a vehicle to house sensitive instrumentation for oceanographic research, or of an offshore floating oil drilling rig for exploitation of the ocean bottom, should be in a position to (a) understand and apply the hydrodynamic principles involved in designing vehicles and systems, (b) determine how such vehicles or systems respond to the environment, and (c) evaluate whether such vehicles can be expected to operate properly in the environment.

The course is organized so as to familiarize the student with the hydrodynamic principles involved and with the methods of handling engineering problems in the area. Application of this knowledge to a wide variety of ocean vehicle types is handled by a "case study" approach involving presentation and class discussion. The text begins with considerations of the purpose, extent and academic level of the course taking into account the ability, background and interests of the students involved. In the absence of any suitable book which satisfactorily covers the subject material, the major portion of this report was intended as a partial text on the motion and control of ocean vehicles.

The "partial" varies in degree from chapter to chapter. Some chapters are practically full text, some are more condensed in form, some are just notes, and some are mere references to more detailed publications. The time and funds that could be allocated to this project essentially limited this publication to a "partial" textbook on the subject.

GENERAL CONSIDERATIONS IN ORGANIZATION OF THE SUBJECT

Extent of Course

A one-term course consisting of three one-hour lectures per week and a total of six hours preparation time per week.

Purpose of Course

In order to explore, use, and develop the ocean environment, a great variety of special marine vehicles and structures are in use and under design today. It can be anticipated from the efforts expected in ocean engineering in the near future that existing special vehicles will be required in larger quantity, in larger sizes, with more sophisticated abilities, and that many new and novel designs of vehicles to meet newly conceived mission requirements need be created.

The many types of marine "vehicles" run the entire gauntlet from manned and sophisticated automated ships operating just above the water surface, on the water surface, and deeply below the water surface, to unmanned buoys on the surface or on the ocean bottom - vehicles which may be self-propelled, towed, or stationary - vehicles which have payloads of personnel, material, or instrumentation or all in combination.

With great diversity of vehicle types and mission requirements, all vehicles must be able to cope with the many problems imposed by the ocean environment. The designer must sufficiently understand which properties of the ocean play a significant role in affecting mission accomplishment and in what manner. He must then design his system such that the vehicle has acceptable performance or optimum performance under the environmental conditions.

In practically all the vehicles involved, the ability to maintain or to deliberately change by determined amounts, the position, depth, speed, altitude, proximity, and/or relative orientation is critical for meeting mission requirements, whether it is manned or unmanned. This operating capability is directly associated with the stability and control of the vehicle and is a major aspect in the design of such vehicles.

Scope of Subject

The material of the course will deal mostly with the analysis of the dynamics of various vehicle types when subjected to the disturbances of the ocean environment in which they must operate.

In dealing with motion analysis, one must be concerned with the forces acting on the vehicle. In the ocean environment, the forces are hydrodynamic, aerodynamic and gravitational (hydro and aerostatics included) in origin. In a rather simplified concept, the forces can be looked at as arising from (1) orientation and/or motion of the vehicle in a calm fluid medium, (2) excitation forces arising from motion of the fluid medium and/or the existence of close boundaries of the fluid medium and (3) control forces brought about by activation of control surfaces and/or other control devices. In the general case, there are many interaction effects between these simplified categories.

The analysis process includes the use of theoretical and applied fluid mechanics and applied Newtonian dynamics to evaluate the equations of motion of the vehicle with controls in the environment. These equations are the mathematical model of the system and, depending on the particular vehicle and its mission, can be rather simple (linear) or quite sophisticated and complicated. Certain coefficients and functions in these equations can be evaluated by theoretical means (mathematical model) but many must be determined through testing of physical models (as in towing tanks and wind and water tunnels) because of the unrealistic assumptions that need to be made in order to keep the mathematics tractable and solvable for arbitrary vehicle shapes.

In other subjects, such as 13.00 (Principles of Naval Architecture) and 13.03 (Advanced Hydrodynamics of Ship Design) the general dynamical equations are rigorously developed, and a few typical examples of application to the design of vehicles are indicated, both in simplified linear form and the more sophisticated non-linear forms. From these developments, design principles for hull shape, appendages, and control surfaces are presented with demonstrated applications to surface ship and submarines. In these subjects, there is no time for detailed quantitative evaluation of any submarine or surface ship designs and only a general reference to application to moored and towed bodies.

Student Background and Prerequisite Subjects

In the proposed course, the class is expected to made up of students with rather diverse specific scientific or engineering preparation although all are interested in some aspects of vehicles in the ocean environment. Civil engineering, aero and astronautics, physics, oceanography, mathematics, mechanical engineering, electrical engineering and other fields in addition to naval architecture may have been the undergraduate goals of the students. It is therefore necessary to keep departmental subjects which are to be considered pre-requisites to a minimum, in order to prevent undue restriction on students of disciplines different from that of naval architecture. This consideration results in the subjects 13.00 Principles of Naval Architecture and 13.10 Ship Structures, being designated as pre-requisites because in these subjects the foundational concepts of applied hydrostatics and hydrodynamics of marine vehicles are developed together with the necessary technical nomenclature and vocabulary. Subject 13.03, Advanced Hydromechanics of

Ship Design, covers the pre-requisite material at a higher level with more detail, sophistication and extension and this subject (13.03), by itself can be considered as satisfying not only pre-requisite requirements, but perhaps as much as 50% content of the subject. Hence, the taking of both these subjects is not recommended.

Additional Items

Three hours per week during one semester allows time for about forty lectures and three one hour written exercises. A final exam, either written or oral, is given at the conclusion of the course. There are some reading assignments and problem assignments.

Initially, the basic physical fundamentals are reviewed along with pertinent dynamic analysis, fluid dynamic concepts and several methods of approach. Subsequent lectures deal with the applications of this material to a variety of ocean vehicle types along a "case study" approach. In the absence of a separate subject involving mooring and towing, several lectures on mooring and towing are included in this course.

OCEAN VEHICLE MISSIONS AND SYSTEMS

Vehicles are required to accomplish certain missions in the sea environment and such missions may be part of overall broader missions for which a given system is designed to a accomplish. Each vehicle is an engineering system in itself; therefore it becomes a subsystem of the larger system and is designed specifically to meet certain operational requirements necessitated by the particular missions assigned to such a vehicle. Within each vehicle there may be further subsystems such as:

a. the power plant

b. the control system

c. the cargo handling system

d. aircraft launching and retrieval system and sonar search systems in military type vehicles.

Each "system" should be designed to "operate" in the environment satisfactorily or "optimally" (within the total system envelope). Hence, the concept of "operational analysis" becomes a significant effort in designing vehicles and/or choosing between alternate vehicles or systems. It becomes necessary to "measure" competing systems or vehicles (in being or in concept) as to how well they can meet mission requirements and this imposes on the designer the need to evaluate quantitatively (more than qualitatively) the expected operational performance.

Certainly for any vehicle contemplated to cover a broad mission , there must be a breakdown of these missions into more specific detailed technical requirements with an allocation into primary and/or secondary aspects of the broad-mission. For example the mission of a cargo ship (or fleet) may be to carry goods between two ports as quickly and economically as possible. A primary aspect would involve the ship operation in the open sea environment with regard to speed made good and another primary aspect would involve the cargo handling system with regard to quick port turnaround. A possible secondary aspect might involve the rudder system with regard to negotiating a canal or difficult channel in order to reach one of the ports.

We therefore have OPERABILITY evaluation as a "necessary" condition in ocean vehicle design together with the associated requirements of being "quantitative" in the evaluation process. But operability is not a "sufficient" condition in that the vehicle must have the capability of surviving the extreme conditions that the environment (in the ocean) may impose. Hence, the concept of SURVIVABILITY. The severity of the environment may be well beyond the level at which "operation" ceases.

For example, a deep ocean oil drilling rig may be able to drill (i.e. operate as a drilling vehicle) until the roll (or pitch) angle reaches 2 degrees because of binding of the bit in the casing and this operationally limiting condition may come about in say sea state 5. However, the rig is out in the ocean and although operations have ceased, must be able to withstand (survive in) severe storm seas peculiar to the geographical location, say sea state 9. Another example is that of the aircraft carrier which may cease to be operational (i.e. unable to launch or retrieve planes) in say sea state 6 or 7 but must be able to survive in typhoons at sea.

Survivability must be assured not only in the operational environment and condition but the vehicle must survive in all aspects of what we shall designate as "adjacent" conditions.

These adjacent conditions are conditions which the system (vehicle) must go through in getting to and out of principal operational conditions. Examples of adjacent conditions of concern are:

a. possible capsizing of a ship at launching
b. instability of hydrofoil boat with foils retracted.
c. getting the "flip" ship into position
d. surface retrieval of deep submergence and oceanographic research submarines.
e. getting a drill rig to location

In some of the adjacent conditions we sometimes look for more than "survivability" in that we may consider some of these adjacent conditions as partial operating conditions and hence seek "efficiency" in the operation. For example, the flip ship must be towed to the desired geographical position. In this towing condition it must survive any expected severe environment, yet it could be designed to also give small drag force and be efficient in this adjacent mode.

To deal with both the operability and survivability of the vehicle in the environment we need to predict both the environmental conditions and the resulting vehicle response with regard to certain "averages" (considering operability) and "extremes" (considering survivability) and these predictions must be essentially quantitative. The vehicle designer and perhaps the operator must know or have methods of determining vehicle responses to the various environmental conditions. He needs to set some kind of quantitative measure of mission specification. He combines these many items for the process of evaluation and comparison of alternate vehicles or systems.

The material content to be presented in the course are aimed at the following purposes.

1. To indicate how, why, to what extent and on what

vehicles, motion stability and control are important factors in meeting mission requirements.

2. To explain methods of describing quantitatively the the environmental phenomena important to the type of vehicle and mission under consideration.
3. To introduce methods of predicting vehicle response towards the end of establishing some level of operational performance of the vehicle.
4. To indicate how controls and control systems affect vehicle operation and meeting mission requirements.
5. Indicate the value of "mathematical" and "physical" modelling in the prediction of performance.
6. To introduce the student to all "aspects" of the control system - going back beyond the control device to the power mechanism, the control sensors, the control computer and the human operator in the system.
7. To introduce the importance of survivability of the vehicle and the control system.
8. To apply the above concepts to case studies of several diverse types of ocean vehicles encompassing a variety of mission requirements. This is done in anticipation of developing a capability of handling any new type of vehicle.

A Listing (Not Complete) of Current Ocean Vehicle Types

<u>Surface Displacement-Commercial</u>

1. Cargo - general container
break bulk
2. Cargo bulk
   a. Oil
   b. Ore
   c. Wheat
   d. Sulphur, liquified gas

3. Passenger (large)
4. Ferry
5. Fish factory
6. Fish trawlers
7. Tugs
8. Flip Ship
9. Research ship
10. Oil drilling rigs
11. Barges
12. Etc.

Military (surface-displacement)

1. Aircraft carrier
2. Cruiser - Guided missile
3. Destroyers D.E.
Fleet
4. Service vessels - oilers
replenishment
5. Communication & Command Ships
6. Patrol
7. Landing craft
8. Etc.

Near Surface

1. Hydrofoil boat
2. Hovercraft or GEMS
3. Torpedo - set shallow
4. Submarine - rocket launching
5. Planing boat
6. Etc.

Surface Buoys

1. Free floating
2. Slack cable
3. Taught cable
4. Etc.

Submerged

1. Torpedo
2. Submarine
3. Oceanographic Research
4. Search vehicles
5. Instrument packages
6. Submerged moored buoys
7. Underwater Habitat
8. Etc.

BASIC HYDRODYNAMIC CONSIDERATIONS

The lecture material deals principally with the dynamics and hydrodynamics of ocean vehicles, both surfaced and submerged. Since we wish to apply the physics of fluid flow - in our case, water - to properly develop the engineering system that a vehicle is, the physical nature of the phenomenon will be stressed in the analysis and application of hydrodynamics to ocean vehicle design. The process of designing an engineering system to efficiently meet mission requirements calls for both qualitative and quantitative information about the various phenomena affecting the system.

Understanding the Physical Phenomena Involved

In order to understand the physical behavior of fluids, one resorts to general observations, observations under controlled conditions in the laboratory and the formulation of mathematical models to represent the phenomenon - i.e. describe the phenomenon by a mathematical theory and subsequent equations.

The word "models" has been mentioned. Models are representations or analogies of the actual physical phenomenom and we will be concerned with two types of models - mathematical models and physical models. A theory and/or mathematical equation, such as Newton's law

$$\text{Force} = \frac{d}{dt} \quad \text{(Momentum)}$$

and the differential equations resulting from the dynamical analysis of a specific case, is a mathematical model of the physical phenomenon. Similarly, Archimedes' principle:

$$\text{Buoyant Force} = \text{(Density of fluid)} \times \text{(Volume displaced)}$$

is a mathematical model extensively used in ocean vehicle analysis and design. Symbols and equations represent the phenomenon and solutions of the equations give us quantitative information about the phenomenon.

In physical modelling, a physical system different from the original system of interest is observed for the purpose of obtaining information on the original physical system. Familar examples of physical modelling are the towing of a vehicle model in the towing tank or testing an airplane model in a wind tunnel. Both mathematical and physical types of models play an important part in hydrodynamics. Each type of modelling is used in the area where it is superior to the other. For example, Archimedes' principle and its applications provide a mathematical model in vehicle hydrostatics which gives the desired information conveniently and to the accuracy desired. Hence, it is not necessary to resort to physical model tests to determine the hydrostatic properties of the vehicle. Hydrostatic theory (a mathematical model) is essentially a reflection of gravitational theory which fits reality very well. On the other hand, present theories and mathematical models for determining the resistance or drag of a vehicle have many assumptions which depart from reality. Better information about this phenomenon can be obtained through testing physical models, although some physical modelling in this area still does not yield the accuracy we would like to have.

One type of modelling can aid the other type of modelling in order to give better performance in the area in which the first type is being used. For instance, the use of boundary layer theory can help us to reduce frictional scaling errors in the testing of physical models for resistance force. Similarly, certain types of restricted tests on physical models in the laboratory can be used to simulate and verify the practicality of hydrodynamic theories and mathematical calculations, such as in the field of theortical wave resistance.

For many years numerous large towing tanks and water and wind tunnels have been built throughout the world attesting to the development of physical modelling in fluid mechanics. In recent years large "towing tanks" for testing mathematical models have come into being with accelerated development - i.e. high speed, high capacity electronic computers.

Both types of models can suffer from the inability to completely model the physical situation. In the case of the physical model, an example is the inability to satisfy simultaneously all the necessary parameters describing the problem such as Reynolds' number, Froude number, etc. In the case of mathematical models, the assumptions made to develop the equations represent departures from reality. Modern computers tend to improve the situation somewhat in that certain forms of non-linearity can be considered.

Fundamental Parameters in Hydromechanics

In fluid mechanics the basic concepts, as in any mechanical system, are the conservation of matter and the conservation of energy (at least the accounting for energy).

For an incompressible fluid such as water, the conservation of matter reduces to the conservation of volume and can be expressed in the following vector form.

$$\vec{\nabla} \cdot \vec{U} = 0$$

where $\vec{\nabla}$ is the gradient expressed in cartesian coordinate system (x,y,z) as:

$$\vec{\nabla} = \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z}$$

where $\vec{i}$, $\vec{j}$ and $\vec{k}$ are unit vectors along the x,y, and z axes respectively. $\vec{U}$ is the vector velocity of the water particle and takes the component form

$$\vec{U} = \vec{i}\, u + \vec{j}\, v + \vec{k}\, w$$

where u,v, and w are the velocity components along the x,y, and z axes respectively. $\vec{\nabla} \cdot \vec{U}$ is referred to as the equation of continuity for incompressible fluids and states that the mass (volume) inflow into a given bounded region must equal the mass (volume) outflow in order to conserve matter (volume).

You are also familiar with the Bernoulli equation expressed as $p + \frac{1}{2}\rho U^2 + \rho gz =$ constant, along a streamline

where

- p is the pressure in the fluid at a position along a streamline
- ρ is the mass density of the fluid
- U is the velocity of the fluid
- g is the acceleration constant
- z is the height of the position relative to a reference level.

The above equation holds for steady (non-time variant flow) along the streamline providing there is no energy input or loss (such as caused by frictional, thermal, or other effects). The Bernoulli equation is essentially an expression of the conservation of energy along a streamline. It is derivable from the general Navier-Stokes equations for incompressible flow by integration of the force along a streamline (dropping the viscous and time dependent terms) to give a work-energy relationship.

It is clear from the Bernoulli equation that

- $\frac{1}{2}\rho U^2$ represents the kinetic energy of the fluid
- p represents the potential energy of pressure
- ρgz represents the potential energy of position in a gravitational field.

Hence, one observes in the Bernoulli expression the interchange between potential and kinetic energy as the fluid particle flows along a streamline.

In the case of flow near or on the free surface of a liquid, the pressure along the free surface is a constant (usually atmospheric pressure). The liquid particles in a local raised peak (crest) of the surface would have higher potential energy (due to the gravitational field) than surface particles at a lower height. If such a local raised peak in the liquid surface were allowed to "fall" under the gravitational field, part of the potential energy would be converted to kinetic energy of motion of the liquid particles. As a result, surface waves would propagate outward, and any specific liquid particles would take on at different times alternate forms of kinetic and potential energy or combinations of these forms. Hence, surface waves propagate because they are in a gravitational field and as the waves radiate or move out energy is "conserved" but the form of the energy and the particles involved vary.

It can be seen from the Bernoulli equation that changes in the term ρgz along a streamline, in the absence of any change in p, indicates a change in the potential energy due to the wave formation. If pressures caused by the motion of a body in a liquid are felt near or on the free surface, the pressures tend to move the free surface in a vertical direction thereby generating waves and wave energy through the work done by the pressures against the gravitational field. Hence, the waves generated by a ship moving on the surface of the water represent energy radiating from the ship, which energy was created by the work done by certain pressure (drag) forces acting on the body, (the energy ultimately coming from the ship's power plant).

Since we are discussing energy, its forms of kinetic and potential, its conservation, and its flow, perhaps it is in order to mention at this time an interesting example of energy transfer and flow that appears from the broad physical picture of surface ship motion in a seaway.

Atmospheric pressure differences represent a form of potential energy. These pressure differences cause winds to be generated. Hence, a conversion of potential to kinetic energy. The wind in contact with the sea transmits some of its kinetic energy through friction and pressure at the water surface into the kinetic and potential energy of water waves. The waves propagate and energy flows. The waves excite the ship into motion (such as pitching, heaving and rolling) through the absorption by the ship of energy from the water in both kinetic and potential form. The pitching (or moving ship) then generates and radiates energy in the form of waves. The ship motions increase and reach an "equilibrium amplitude"(in regular waves) when the rate of absorption of energy from the waves equals the rate of energy radiation by the ship due to its motion. This sequence represents an interesting history of energy transfer of importance for surface vehicles.

Let us consider the basic problem of a body moving in water or on the water surface - such as ships, boats, submarines, hydorfoil craft, fish, etc. One wishes to gather as much information as possible about the motion of a body in a fluid - information such as drag (resistance), lift, and trajectories. It is obvious that the phenomena associated with the body in the fluid must depend on the properties of the body, the properties of the motion and the properties of the fluid. Therefore, quantities of interest, such as forces and moments active on the body and the motion of the body itself must be functions of the properties of the body, motion and fluid.

The properties of the body are its size, shape, and mass distribution. The items defined below are sufficient to characterize the body properties (for rigid body treatment and a properly chosen axis system).

L — is the length of the body, furnishing a typical length to characterize size

geom. — characterizes the form, shape, or geometry of the body (including propulsion devices and appendages).

m is the mass of the body.

I is the moment of inertia of the body. (A subscript can be used to denote reference axis, such as I_x is the moment of inertia about an x axis.)

$\vec{R}_G$ is a vector defining the distance of the center of gravity of the body from an arbitrary origin on a coordinate system in the body. $\vec{R}_G = \vec{i}x_G + \vec{j}y_G + \vec{k}z_G$, where $\vec{i}$, $\vec{j}$, $\vec{k}$ are unit vectors along mutually perpendicular axes x, y, z respectively.

(The properties m, I, and $\vec{R}_G$ result from certain measurements of the mass distribution of the body and they can be replaced readily by a defined mass distribution, giving a more general form for the property of mass and allow more responses to be considered such a dynamic structural bending moments).

The properties of the motion are the velocities and accelerations, both linear and angular, of the body and any movable appendages of the body such as control surfaces (for example rudders and propeller shaft). Since body motion in a fluid changes the orientation of the body relative to the fluid, the properties of the orientation of the body in the fluid will be included in the properties of the motion. When a control surface, such as a rudder, is deflected, there results a different body geometry than when the rudder is in its normal undeflected position. In order to avoid handling a new body geometry for each control surface deflection, it is convenient to define body geometry as the condition of undeflected control surface and add the additional parameter of angle of deflection of the control surface. The control surface deflection therefore can be considered as a property of the body. For convenience, since the control surface is a movable part of the body and since its velocity and acceleration are already included under the properties of the motion, it is preferred to group the control surface deflection as a motion property. The items defined below are sufficient to characterize the properties of the motion.

Orientation Relative to the Fluid

x_o , y_o , z_o	is a fixed coordinate system, with origin at the surface of liquid.
x_o and y_o	are mutually perpendicular axes in the horizontal surface of the liquid.
z_o	is an axis perpendicular to the liquid surface (positive direction preferably downward).
ϕ, θ, ψ	are angular rotations roll, pitch and yaw about the x, y, and z axes respectively. θ is the angle that the longitudinal x axis makes with the horizontal plane x_o, y_o.

Body Motion

$\vec{U}$ is the vector linear velocity of the body. $U = \vec{i}\,u + \vec{j}\,v + \vec{k}\,w$, where u, v, w are the components of linear velocity along the x, y, z axes in the body, respectively ($\vec{i}$, $\vec{j}$, $\vec{k}$ are unit vectors along the x, y, z axes).

$\vec{\Omega}$ is the vector angular velocity of the body. $\vec{\Omega} = \vec{i}\,p + \vec{j}\,q + \vec{k}\,r$, where p, q, r are the components of angular velocity of the body about the x, y, z axes respectively.

$\dot{\vec{U}}$ is the vector linear acceleration of the body. The usual convention of signifying the derivative with respect to time by a "dot" above the quantity is used $\dot{\vec{U}} = \frac{d}{dt}(\vec{U})$

$\dot{\vec{\Omega}}$ is the vector angular acceleration of the body.

δ is the deflection of a control surface. If more than one control surface is involved (rudders, diving planes, antiroll fins), then a different subscript can be used to differentiate between the types - i.e. δ_1, δ_2, etc.

$\dot{\delta}$ is the angular velocity of the control surface.

$\ddot{\delta}$ is the angular acceleration of the control surface.

n is the angular velocity of the propeller shaft. If there are more than one or other types of shafts (bow thruster, etc.) then subscripts can be used to differentiate between them.

$\dot{n}$ is the angular acceleration of the shaft.

The properties of the fluid can be characterized sufficiently for our purposes by the following items.

ρ is the mass density of the fluid

μ is the viscosity

τ is the surface tension

g is the acceleration of gravity. It is listed as a property of the fluid because the fluid is in a gravitational field

p is the pressure

p_v is the vapor pressure of the liquid

E is the elasticity of the fluid

(Other physical properties of the fluid such as thermal, magnetic and electrical have not been considered here).

Additional properties of the fluid are the size, shape, and motion of the fluid body. An example is the width, depth and shape of a canal and any fluid currents. The size,shape, velocity and acceleration of the fluid body will be characterized by the following;

W is the width of the fluid body, giving a characteristic size

$geom_f$ characterizes the geometry or shape of the body of fluid

$\vec{U}_w, \vec{\Omega}_w$ velocities linear and angular (relative to fixed spaces) of "fluid body" under consideration.

η surface elevation of the fluid.

$\dot{\vec{U}}_w, \dot{\vec{\Omega}}_w$ acceleration of fluid

The simple statement of "motion of body in a fluid" has become, as a result of fundamental analysis, the following lengthy function

$$\left.\begin{matrix}\text{Forces}\\ \text{Moments}\\ \text{Etc.}\end{matrix}\right\} = f\left\{ \overbrace{L,\ \text{geom},\ m,\ \vec{R}_G,\ I}^{\text{Prop. of Body}} ;\ \overbrace{\overbrace{x_o,\ y_o,\ z_o,\ \phi,\theta,\psi,}^{\text{Orientation}};\ \overbrace{\vec{U},\ \vec{\Omega},\ \dot{\vec{U}},\ \dot{\vec{\Omega}},\ n,\ \dot{n},\ \delta,\dot{\delta},\ddot{\delta},\ \text{etc}}^{\text{Dynamics}}}^{\text{Properties of Motion}} \right.$$

$$\left. \overbrace{;\rho,\ \mu,\ \tau,\ g,\ p,\ p_v,\ E,\ ---;\ W,\ \text{geom}_f,\ \vec{U}_w,\ \vec{\Omega}_w,\ \text{Etc.}}^{\text{Prop. of Fluid}}\ : \right\}$$

In order to measure quantities of interest to him, man has devised a system of measurement based on units and dimensions of his own definition. In this process, several different concepts of dimensions and units have evolved for measuring the same physical quantity. Force is measured in grams or pounds, length is measured in meters, feet, miles, etc. The function expressed above, represents a physical relationship concerning physical quantities involved in a natural phenomenon. It is expected, in attempting to find quantitative information about this natural law, that the only absolute measurement in nature be used. This natural measurement quantity is the non dimensional ratio of two of the same physical quantities, (resulting in significant measurements such as - this mass is twice as large as another mass, or this length is three times as long as some other length). A dimensionless quantity will have the same magnitude of measurement no matter what man made definition of units are used. Another advantage of employing dimensionless quantities in analyzing a physical law is that the number of parameters or variables (if these are expressed as physical quantities) can be reduced by the number of basic defined units involved. The reduction is usually by three in number since physical quantities usually can be expressed in the basic units of mass, length, and time. (The π theorem of dimensional analysis is referred to here). It can be readily seen that the quantities ρ, L, and U can be considered as representative of mass, length and time in hydrodynamic considerations.

One notices in the function that geometry of the body is a parameter already in non-dimensional form. It can be surmised that, if the basic physical relationship of a body in a fluid is best expressed through non-dimensional parameters, then other geometrical relationships such as geometry of the flow field, the fluid acceleration field, and thereby the geometry of the force field become basic quantities in the relationship and become significant items for observation and measurement.

Forces experienced by a body in a fluid are felt at the surface of the body either as forces normal (perpendicular) to the body surface (pressure force) or tangential to the body surface (shear forces). From the Bernoulli equation, one observes that the term $\frac{1}{2}\rho U^2$ has the dimension of a pressure, called the "dynamic pressure" and readily reflects the inertial properties of the fluid. The dynamic pressure force felt at a certain arbitrary element of body surface depends on the dynamic pressure at the element and can be expressed as $\frac{1}{2}\rho U^2 \gamma_1$ (Area). The geometrical parameter γ_1 takes into account both the dependence of local velocity on geometry and the geometrical dependence of the direction of the pressure force (normal to the body surface).

Similarly, the friction force on an element of body surface can be characterized as:

$$\text{Friction force} = \mu \frac{U}{L} \gamma_2 \quad \text{(area)}$$

since (1) according to Newton's law of viscosity, the viscous force is defined as $\mu \frac{\partial u}{\partial y}$ (area), where μ is the coefficient of viscosity and $\frac{\partial u}{\partial y}$ is the velocity gradient normal to the surface and (2) the geometrical parameter γ_2 accounts for the geometry of the body as it determines the velocity field and the directional components of the friction force.

In a similar fashion, the surface tension forces experienced at a point on the body surface at the line of contact of two fluids can be expressed by

$$\text{Surface tension force} = \tau L \gamma_3$$

where τ is the surface tension coefficient, L is the body length and γ_3 is a geometrical parameter determining the actual length of the line of contact and the directional components of the force.

When one treats gravitational forces as derived from the existence of a gravitational field in the fluid, pressure terms, such as ρgz in the Bernoulli equation, become significant. (Also, recall that water waves are a consequence of just these gravitational effects). The forces from gravitational considerations exerted on a given surface element of the body can be expressed as

$$\text{Force} = \rho gz \quad \text{(area)}\gamma_4^1 \quad = \rho g . z/L \ .L \ \text{(area)}\gamma_4^1$$

$$\text{Force} = \rho g \quad L\text{(area)} \quad \gamma_4$$

where γ_4 is a parameter depending on geometry which takes into account the effect of the geometry on the body immersed in the fluid (including immersion caused by the surface wave formation) and the directional components of the force.

The forces, described above, originate from different physical properties of the fluid. Since the natural measurement is the ratio of two of the same physical quantities, a choice is made to use the dynamic force on a surface element as a basis for comparison - i.e. the force contributions from the various sources are to be divided by the dynamic force $\frac{1}{2}\rho U^2$ (area) γ_1, in order to give the dimensionless ratios desired.

Temporarily, the general function for the hydrodynamic phenomena dealing with the motion of a body in a fluid, will be written including only the fluid properties of density (ρ), viscosity (μ), gravity (g), surface tension (τ), the body properties of size (L) and geometry (geom.), and just the linear velocity U. The simplified function becomes:

$$\text{Force} = f\,(L,\ \text{geom.};\rho,g,\ \mu,\tau,U)$$

$$\downarrow$$

$$\begin{pmatrix}\text{Drag}\\ \text{Lift}\\ \text{etc.}\end{pmatrix}$$

If resort is made to changing the dimensional physical parameters to reflect the measurement of the forces associated with each parameter by ratio with the dynamic force, the parameters take the following form:

$$\frac{\text{Body Force}}{\text{Dynamic Force}} = \frac{\text{Force}}{\frac{1}{2}\rho U^2\ (\text{area})\gamma_1}$$ = Force coefficient, such as drag coefficient and lift coefficient

$$\frac{\text{Frictional force}}{\text{Dynamic force}} = \frac{\mu \frac{U}{L}\ (\text{Area})\gamma_2}{\frac{1}{2}\rho U^2 (\text{Area})\gamma_1} = \frac{\mu}{\rho UL}\ \frac{(2\gamma_2)}{(\gamma_1)} = \frac{\mu}{\rho UL}\ (\gamma_5)$$

$$\frac{2\ \gamma_2}{\gamma_1} = \gamma_5 \quad = \text{a factor depending on geometry of the body}$$

$$\frac{\text{Gravitational force}}{\text{Dynamic force}} = \frac{\rho g L\ (\text{Area})\gamma_4}{\frac{1}{2}\ \rho U^2 (\text{Area})\gamma_1} = \frac{gL}{U^2}\ \left(\frac{2\gamma_4}{\gamma_1}\right) = \frac{gL}{U^2}\ (\gamma_6)$$

$$\frac{\text{Surface Tension forces}}{\text{Dynamic forces}} = \frac{\tau L \gamma_3}{\frac{1}{2}\rho U^2 (\text{Area})\gamma_1} = \frac{\tau\ L\ \gamma_3}{\frac{1}{2}U^2\ L^2\ \gamma_7\ \gamma_1} =$$

$$\frac{\tau}{\rho U^2 L}\ \frac{(2\gamma_3)}{\gamma_7\ \gamma_1} = \frac{\tau}{\rho\ U^2\ L}\ (\gamma_8)$$

The area of a body can readily be defined by a length squared multiplied by an appropriate non-dimensional factor depending on the geometry. Hence, in the case above (Area) $= L^2\ \gamma_7$. In non dimensional form, the function becomes

$$\text{Force coefficient} = \frac{\text{Force}}{\frac{1}{2}\rho U^2 (\text{Area})} = f_1\ \left(\text{geom.}\ \frac{\mu}{\rho UL},\ \frac{gL}{U^2},\ \frac{\tau}{\rho U^2 L}\right)$$

The γ factors have been omitted since they are factors depending on the geometry of the body, and the geometry parameter is already included in the function. One notices that the number of parameters have been reduced by three in going from dimensional to non dimensional form. The π theorem of dimensional analysis predicts this reduction in that the three basic units of mass, length, and time are used in defining the physical quantities involved in the function. It can also be seen that the quantities ρ, U, and L (which reflect these basic units) have essentially been used to form the non dimensional parameters. With the ratio of forces established as a function of body geometry, it is clear that the non dimensional form of the parameters define the geometry of the force field on the body and in the fluid and thereby determine the geometry of the acceleration field and velocity field.

Since a function of a variable x can be written as some other function of $\frac{1}{x}$ or $\frac{1}{\sqrt{x}}$,

$$f_1 \quad (x) \quad = f_2 \left(\frac{1}{x}\right) \quad = f_3\left(\frac{1}{\sqrt{x}}\right)$$

then, the function for the force coefficient can be written in the form

$$\text{Force coefficient} = f_2 \left(\text{geom}, \frac{\rho UL}{\mu}, \frac{U}{\sqrt{gL}}, \frac{\rho U^2 L}{\tau}\right)$$

These non dimensional parameters usually are referred to by the name of the person who first demonstrated their importance in fluid mechanics.

$\frac{\rho UL}{\mu}$ is the Reynolds' number

$\frac{U}{\sqrt{gL}}$ is the Froude number

$\frac{\rho U^2 L}{\tau}$ is the Weber number

The additional parameters that appear in the general functional relationship, can readily be reduced to non dimensional form through the use of the parameters ρ, U and L. These are

$$\frac{p-p_v}{\frac{1}{2}\rho U^2} = \text{cavitation number}$$

$\frac{E}{\rho U^2}$, or its inverse form $\frac{\rho U^2}{E}$ = Cauchy no. Since the velocity of a pressure wave, c, in a medium can be expressed as

$$c = \sqrt{\frac{E}{\rho}}\text{, then } \frac{\rho U^2}{E} = \frac{U^2}{c^2} = \left(\frac{U}{c}\right)^2 = (\text{Mach. no.})^2$$

(sound in a fluid is a pressure wave)

$\frac{nL}{U}$, $\frac{\Omega L}{U}$, $\frac{\dot{\delta} L}{U}$ = Strouhal number, or frequency number, or speed coefficient, or reduced frequency.

$$\frac{\dot{U}L}{U^2}, \quad \frac{\dot{\Omega}L}{U^2}, \quad \frac{\dot{n}L}{U^2}, \quad \frac{\ddot{\delta}L^2}{U^2},$$

$$\frac{m}{\rho L^3}, \quad \frac{I}{\rho L^5}, \quad \frac{\vec{R}_G}{L}, \quad \frac{\vec{R}_o}{L}, \quad \frac{W}{L}$$

$$(\vec{R}_o = \vec{i}_o x_o + \vec{j}_o y_o + \vec{k}_o z_o)$$

ϕ, θ, ψ, δ and $(\text{geom.})_f$ are already non dimensional quantities. In non dimensional form, the general function becomes;

$$\left.\begin{array}{l}\text{Force Coeff's}\\ \text{Moment Coeff's}\\ \text{Etc.}\end{array}\right\} = f\Big\{\overbrace{\text{geom}, \frac{m}{\rho L^3}, \frac{I}{\rho L^5}, \frac{\vec{R}_G}{L}}^{\text{Body}}, \overbrace{\frac{\vec{R}_o}{L}, \phi, \theta, \psi}^{\text{Orientation}}$$

$$\overbrace{\frac{\vec{\Omega}L}{U}, \frac{\dot{\vec{U}}L}{U^2}, \frac{\dot{\vec{\Omega}}L^2}{U^2}, \frac{nL}{U}, \frac{\dot{n}L^2}{U^2}\ \delta, \frac{\dot{\delta}L}{U}, \frac{\ddot{\delta}L^2}{U^2}}^{\text{Dynamics}};$$

$$\underbrace{;\ \rho\frac{UL}{\mu}, \frac{U}{\sqrt{gL}}, \frac{\rho U^2 L}{\tau}\ \frac{p-p_v}{\frac{1}{2}\rho U^2}, \frac{U}{c}, \frac{W}{L},;\qquad (\text{geom})_f, \frac{\vec{U}_w}{U}, \vec{\Omega}_w L\ ---;}_{\text{Fluid}}\Big\}$$

If the thermal, electrical and magnetic properties of the fluid had been taken into account, then additional parameters, useful in the area of thermodynamics of compressible flow and magnetohydrodynamics, would have also appeared in the function.

The physical phenomenon which we are investigating has now been reduced to its basic natural measurement parameters. In most cases the liquid is assumed infinite in depth, length, and width, unless otherwise specified and without currents. This assumption removes $\frac{W}{L}$, $(\text{geom.})_f$, $\vec{U}_w$ and $\vec{\Omega}_w$ as parameters in the functional expression. (Also $\dot{\vec{U}}_w$ and $\dot{\vec{\Omega}}_w$)

In several cases in the analysis of the hydrodynamic forces on a vehicle, it is convenient to refer to these forces as arising from a particular type of flow situation and or fluid property. These are:

1. Inertial reaction forces primarily involving fluid density.

2. Viscous forces primarily in the form of friction and involving the viscosity μ.
3. Wave forces arising from the generation of gravity waves for vehicles near to or on the surface.
4. Circulation forces arising from the addition of a fluid rotation about the body to the body velocity. This gives rise to the "lift" forces on airfoils and hydrofoils.

The property of viscosity in the form of friction plays a part in setting up conditions in which certain inertial and circulatory forces come into play, but are not subsequently involved in any major manner. One case is where the viscous boundary layer separates from the body thereby changing the velocity, pressure distribution, and the resulting forces. Separation results from the fluid having insufficient energy to negotiate the pressure rise at the stern, having lost energy to friction as it flowed along the body. In the absence of friction (ideal flow) and away from any free surface, the fluid particles have just enough kinetic energy to form a stagnation point at the stern end.

The other case is where a body (hydrofoil) with a sharp trailing edge cannot support an infinite velocity around this edge in the presence of even the smallest viscosity. A starting vortex is shed from this edge creating a counter vortex or circulation about the foil of sufficient magnitude so that the flow smoothly leaves the trailing edge with no flow around the edge. Since the amount of circulation necessary to have the flow leave the trailing edge is calculable, then the lift forces are calculable from circulation theory.

$$L = \rho \Gamma U \quad \text{per unit span}$$

where:

L is the lift force
ρ is the density
U is the velocity
Γ is the circulation

$\Gamma = \oint \vec{U} \cdot d\vec{\ell}$ indicates the integral around a closed loop, containing the foil, of the velocity component along the path.

Examples of inertial forces are water impinging on a flat plate and the acceleration of a body in a fluid. Examples of viscous forces are the frictional resistance of a flat plate and eddy drag of a circular cylinder. Examples of wave forces are the wave drag of a ship and the heave damping forces of a surface vehicle. Examples of circulation forces are lift forces on hydrofoils and screw propeller thrust.

EQUATIONS OF MOTION AS DERIVED FROM DYNAMIC ANALYSIS

The vehicles operating in the ocean environment move about in the environment as a result of their own propulsive and control systems and as a result of the excitations or restraints caused by the environment. In this course, we are concerned only with the motions of the vehicle as rigid body dynamics in evolving the equations of motion. These equations become mathematical models of the vehicle dynamics - modelling the stability, control, and motion responses to the environmental excitations. The vehicle (as a rigid body) has a shape, size, and mass distribution which we assume does not change in time. Any significant change in shape of a given body, such as caused by a control surface deflection can be handled conveniently as a separate subsystem in the mathematical model and any significant change in mass distribution, such as occurs on an antirolling tank, can also be handled by a coupled subsystem.

With this concept in mind, the general dynamical equations which are developed pertain to all ocean vehicles. Straight forward Newtonian mechanics are used but it is wise to develop the equations in a frame of reference (axes system) most convenient for use in solving ocean vehicle motion problems. Since the vehicle moves under the forces acting upon it, and since these forces depend significantly on the geometry of the vehicle, it is highly desirable to choose an axis system fixed in the body so that the body geometry, described in this frame of reference, does not change with time. It is also convenient to have the flexibility of choosing the location and origin of these axes so that a choice can be made, taking into account the particular vehicle involved, so as to minimize the complexity of the solution of the dynamical problems.

For example, practically all God made (animal, fish, bird) and man made mobile vehicles have a symmetry in geometry, i.e. port and starboard symmetry. By choosing to place two orthogonal axes in this plane of symmetry (the centerline plane), one can more easily express the geometry and the resulting forces in terms of this axis system. Also, since the distribution of mass within the body need not produce inertial symmetries identical to geometrical symmetries, it is convenient to allow a choice of origin location which in general need not be located at the center of gravity of the vehicle. None of the above flexibilities in origin choice compromises or restricts the general validity of the equations of motions.

Further reduction in the complex nature of the equations can be brought about by choosing an orthogonal axis system parallel to the principal axes of inertia so as to eliminate products of inertia in the motion equations. For practically all ocean vehicles, with extremely few exceptions, a longitudinal axis (x axis) in the centerline plane, a downward (toward keel) axis (z axis) perpendicular to the x axis in the centerline plane, and a transverse axis (y axis) perpendicular to the centerline plane satisfy this requirement. For any vehicle (extreme exception) which has a very peculiar and significantly large assymetrical mass distribution, one would have to include the products of inertia.

The x, y, z axis form an orthogonal right hand system of axis fixed in the vehicle. The axes and the associated components of the pertinent physical quantities are defined below.

x axis longitudinal axis (in the plane of symmetry) positive forward. Usually parallel to the keel or calm water line. If upper and lower half of the body are

	symmetrical, then the axis is the intersection of the two planes of symmetry.
y axis	transverse axis, perpendicular to the plane of symmetry, positive to the starboard.
z axis	'downward' axis, in the plane of symmetry (x,z) plane, perpendicular to the x axis, positive downward towards the keel.
$\hat{i},\hat{j},\hat{k}$	unit vectors along the x, y, and z axis respectively.
$\vec{R}$ x,y,z	vector distance of a point from the origin 0, and the corresponding components along the x,y, and z axes. $\vec{R} = \hat{i}\,x + \hat{j}\,y + \hat{k}\,z$
$\vec{R}_G$ x_G, y_G, z_G	vector distance of the center of gravity from the origin. $\vec{R}_G = \hat{i}\,x_G + \hat{j}\,y_G + \hat{k}\,z_G$
$\vec{U}$ u,v,w	velocity of the origin 0 (on the body) and the corresponding components along the x,y, and z axis. $\vec{U} = \hat{i}\,u + \hat{j}\,v + \hat{k}\,w$
$\vec{\Omega}$ p,q,r	angular velocity of the body about the origin and the corresponding components along (about) the axes. $\vec{\Omega} = \hat{i}\,p + \hat{j}\,q + \hat{k}\,r$
I_x, I_y, I_z	moment of inertia of the body about the x, y, and z axes respectively.
$\vec{F}$ X,Y,Z	force acting on the body and the corresponding components along the axes. $\vec{F} = \hat{i}\,X + \hat{j}\,Y + \hat{k}\,Z$.
$\vec{\mathcal{M}}$ K,M,N	moment acting about the origin and the corresponding components about the axes. $\mathcal{M} = \hat{i}\,K + \hat{j}\,M + \hat{k}\,N$.

The components of translational position of the body, x_o, y_o, z_o in an axis system fixed geographically, and the body rotations ϕ, θ, ψ about the x,y, and z axes respectively (in a preferred sequence) have been defined in a previous chapter.

Newton's law of motion for a rigid body can be written as two equations - one a force equation and the second a moment equation provided an origin is taken at the center of gravity and the axis system is fixed in space (relative to the stars). The equations are:

$$\vec{F} = \frac{d}{dt}\ \overrightarrow{(\text{momentum})} = \frac{d}{dt}\ (m\vec{U}_G)$$

$$\vec{\mathcal{M}} = \frac{d}{dt}\ \overrightarrow{(\text{angular momentum})}_G = \frac{d}{dt}\ \overrightarrow{(I\Omega)}_G$$

where the subscript G refers to an origin at the center of gravity and m is the mass of the body. For a mass essentially constant in time (for practically all ocean vehicles) one has:

$$\vec{F} = m\frac{d}{dt}\ (\vec{U}_G)$$

Any jet system which discharges original mass for the purpose of producing thrust can handle the above equations by bringing the momentum change from the jet to the left hand side of the equation and treating it as a force. In this case, the m on the right hand side remains a function of time i.e. m (t). We shall assume the mass of the vehicle is essentially constant throughout a given maneuver or trajectory.

For an origin not at the center of gravity of the body and in a system of axes fixed in and moving with the vehicle

$$\vec{U}_G = \vec{U}_a + \vec{\Omega} \times \vec{R}_G$$

where $\vec{U}_a$ is the velocity of the origin in space. However since the origin is on the surface of the earth and the earth rotates, then

$$\vec{U}_a = \vec{U} + \vec{\Omega}_e \times \vec{R}_b$$

where $\vec{U}$ is the geographical velocity of the body, $\vec{\Omega}_e$ is the angular velocity of the earth, and $\vec{R}_b$ is the radius vector from earth's center to the vehicle. The force equation becomes:

$$\vec{F} = m \frac{d}{dt} (\vec{U} + \vec{\Omega}_e \times \vec{R}_b + \vec{\Omega} \times \vec{R}_G)$$

$$\vec{F} = m \frac{d}{dt} (\vec{U} + \vec{\Omega} \times \vec{R}_G) + m [\dot{\vec{\Omega}}_e \times \vec{R}_b + \vec{\Omega}_e \times \dot{\vec{R}}_b]$$

$$\vec{F} = m \frac{d}{dt} (\vec{U} + \vec{\Omega} \times \vec{R}_G) + m [\vec{\Omega}_e \times \vec{U} + \vec{\Omega}_e \times \vec{\Omega}_e \times \vec{R}_b]$$

since $\dot{\vec{\Omega}}_e = 0$ and $\dot{\vec{R}}_b = \vec{U} + \vec{\Omega}_e \times \vec{R}_b$

It is clear that the term $m\vec{\Omega}_e \times \vec{U}$ involves the well recognized Coriolis force and that the term $\vec{\Omega}_e \times \vec{\Omega}_e \times \vec{R}_b$ is the centripetal acceleration due to the rotation of the earth. Since $\vec{\Omega}_e$ is extremely small and $\vec{U}$ is not extremely large we may consider the Cariolis and centrifugal forces as neglible compared to other forces acting on the body. (In addition, for a neutrally buoyant vehicle there is no net effect of centripetal acceleration since the fluid is also under this same acceleration). Hence, we have:

$$\vec{F} = m \frac{d}{dt} (\vec{U} + \vec{\Omega} \times \vec{R}_G)$$

By substituting the following defined quantities in the above expression

$$\vec{U} = \hat{i}\, u + \hat{j}\, v + \hat{k}\, w \qquad \vec{\Omega} = \hat{i}\, p + \hat{j}\, q + \hat{k}\, r$$

$$\vec{R}_G = \hat{i}\, x_G + \hat{j}\, y_G + \hat{k}\, z_G \qquad \vec{F} = \hat{i}\, X + \hat{j}\, Y + \hat{k}\, Z$$

and recognizing that (see Appendix I, page I-5 and 6)

$$\frac{d\hat{i}}{dt} = \hat{i}\, 0 + \hat{j}\, r - \hat{k}\, q$$

$$\frac{d\hat{j}}{dt} = \hat{j}\, 0 + \hat{k}\, p - \hat{i}\, r$$

$$\frac{d\hat{k}}{dt} = \hat{k}\, 0 + \hat{i}\, q - \hat{j}\, p$$

the following force equations result.

$$X = m[\dot{u} + q\,w - r\,v - x_G\,(q^2 + r^2) + y_G(pq - \dot{r}) + z_G(pr + \dot{q})] \quad (1)$$

$$Y = m[\dot{v} + ru - pw - y_G(r^2 + p^2) + z_G(qr - \dot{p}) + x_G(qp + \dot{r})] \quad (2)$$

$$Z = m[\dot{w} + pv - qu - z_G(p^2 + q^2) + x_G(rp - \dot{q}) + y_G(rq + \dot{p})] \quad (3)$$

Considering the apparent forces resulting from the earth's rotation as negligible in the ocean vehicle dynamics problem, the moment equation becomes:

$$\vec{\mathcal{M}}_G = \frac{d}{dt}\,(\overrightarrow{\text{angular momentum}})_G$$

For an origin 0 not at the C.G., the moment expression becomes;

$$\vec{\mathcal{M}} = \vec{\mathcal{M}}_G + \vec{R}_G \times \vec{F} = \vec{\mathcal{M}}_G + \vec{R}_G \times m \frac{d}{dt} \quad (\vec{U}_G)$$

Since $\vec{\mathcal{M}}_G = \frac{d}{dt} \quad \overrightarrow{(\text{ang. mom.})}_G$ and since

$$\overrightarrow{(\text{ang. mom.})}_G = \hat{i}I_x p + \hat{j}I_y q + \hat{k}I_z r - m\vec{R}_G \times (\vec{\Omega} \times \vec{R}_G)$$

for axes parallel to the principal axes of inertia (as shown in Appendix I page I-11), the moment equation becomes:

$$\hat{i}K + \hat{j}M + \hat{k}N = \vec{\mathcal{M}} = \frac{d}{dt} \left[\hat{i}I_x p + \hat{j}I_y q + \hat{k}I_z r - m\vec{R}_G \times \vec{\Omega} \times \vec{R}_G\right]$$
$$+ \vec{R}_G m \times \frac{d}{dt} \quad (\vec{U}_G)$$

The substitution of vector components and the reduction and consolidation of terms are carried out in Appendix I pages I-11 to I-13 with following resulting equations.

$$K = I_x \dot{p} + (I_z - I_y)qr + m[y_G(\dot{w} + pv - qu) - z_G(\dot{u} + ru - pw)] \qquad (4)$$

$$M = I_y \dot{q} + (I_x - I_z)rp + m \quad [z_G \quad (\dot{u} + qw - rv) \; -x_G(\dot{w} + pv - qu)] \qquad (5)$$

$$N = I_z \dot{r} + (I_y - I_x)pq + m \quad [x_G(\dot{v} + ru - pw) \; -y_G(\dot{u} + qw - rv)] \qquad (6)$$

where all of the terms refer to the axis system through the origin O. The flexibility afforded by allowing an axis system fixed in the vehicle at an origin of not necessarily at the C.G. allows us to choose axes and origin best suited to calculate the forces and moments on the body, i.e. X, Y, Z, K, M, and N by taking advantage of geometrical symmetries in the vehicle shape. The price of this flexibility is very small - just the addition of a few simple algebriac terms in the equation.

The terms (qw-rv) ,(ru-pw), and (pv-qu) represent centripetal acceleration and the terms $(I_z-I_y)qr$, $(I_x-I_z)rp$ and $(I_y-I_x)pq$ represent gyroscopic effects. The terms involving x_G, y_G, and z_G represent acceleration of the center of gravity relative to the origin.

As shown earlier the forces and moments are functions of many variables, i.e.,

$$\left.\begin{matrix}\vec{F}\\ \vec{\mathcal{M}}\end{matrix}\right\} f(L, \text{geom}, m, \vec{R}_G, I, \ldots, \vec{R}_o, \phi, \theta, \psi, \dot{\vec{U}}, \vec{U}, \vec{\Omega}, \dot{\vec{\Omega}},$$
$$n, \dot{n}, \delta, \dot{\delta}, \ddot{\delta}, \ldots;$$
$$\rho, \mu, g, \tau, p, p_v, E \ldots; \text{excitation, restraints}, \ldots)$$

Some discussions on the significance of the parameters representing the physical properties of the fluid have been made previously. We now consider how we might analyze the motion of a given body in a given fluid environment. Hence, for a given vehicle in a given ocean environment,i.e. a given size,geom, mass distribution in a liquid of given physical properties, the function becomes:

$$\left.\begin{matrix}\vec{F}\\ \vec{\mathcal{M}}\end{matrix}\right\} f(x_o, y_o, z_o, \phi, \theta, \psi, u, v, w, p, q, r, \dot{u}, \dot{v}, \dot{w}, \dot{p}, \dot{q}, \dot{r}, n, \dot{n}, \delta, \dot{\delta}, \ddot{\delta}, \text{----}$$

$$\text{--- excitations, restraints)}$$

In the above expression n refers to the rotation for all rotating effectors such as main propulsion propeller, thrusters, etc., and δ refers to the deflection of all control surfaces such as rudders, elevators, flaps, propeller blade pitch, anti-rolling fins, etc. ($n \sim \sum_i n_i$ and $\delta \sim \sum_i \delta_i$)

The variables x_o, y_o, z_o, ϕ, θ, and ψ dealing with orientation in space (also relative to the fluid body) clearly represent hydrostatic effects. The hydrostatic forces and moments are readily calculable by theory; however, the nature of the forces as a function of the dynamic variables is not sufficiently calculable at this time, especially for large values of the variables and coupled variables. It is necessary to expand the above function in the form of a Taylor expansion. The expansion need be made about a suitable initial condition and since the equilibrium condition of straight ahead motion at constant speed is the usual operating condition of ocean vehicles (at least in some mode), the expansion is made about the initial conditions of $u = u_o, n = n_o;\ v_o = w_o = p_o = q_o = r_o = \text{----} = \delta_o = \dot{\delta}_o \text{ --} = 0$. The Taylor expansion on the above variables is carried out in Appendix I pages I-14 to 20 indicating the nature of the linear and higher order terms.

Taylor expansion is limited to analytic functions and we assume the hydrodynamic forces are of this nature at least for the linear term and third order term. For second order terms involved in forward drag when $u_o = 0$, or when considering hovering can be handled by second order terms of the form $u|u|$ and $v|v|$ etc.

In considering motion stability we are dealing with the response of the vehicle after some arbitrary infinitesimal disturbance from the equilibrium condition of straight ahead motion to see if the vehicle returns to the original equilibrium condition after the disturbance. Hence, we need retain only the linear term in the Taylor expansion of the variables. Actually, for a dynamically stable vehicle (stability in straight line motion) the linear theory holds for moderate maneuvers and non-linear terms (usually second or third order) become necessary only for tight maneuvers. However, for a dynamically unstable ship higher order terms are necessary to determine maneuvering properties.

Typical coefficient of linear terms in the Taylor expansion take the form of a partial derivative of a force or moment component with respect to a variable evaluated at the original condition. As an example one may have the term

$$\left(\frac{\partial N}{\partial v}\right)_o (v)$$

with the coefficient $\left(\frac{\partial N}{\partial v}\right)_o$ meaning the partial derivative of the yaw moment function N with respect to a disturbance in transverse velocity v taken at the original condition of $u = u_o$, $v = \dot{v} = r = \dot{r} = \ldots = 0$. As indicated in Appendix I, the shorthand notation of

$$\left(\frac{\partial N}{\partial v}\right)_o \equiv N_v$$

is very convenient.

In developing the linearized equations of motion (i.e. mathematical model of the vehicle dynamics) consistency requires that the dynamical response terms on the right hand side of equations be also linearized, since the force and moment expressions on the left hand side of the equation have been linearized. If one remains general with all six degrees of freedom of motion, even the linearized equations are quite extensive. However, for most ocean vehicles the motion analysis can be conventionally separated into motion in the horizontal plane and motion in the vertical plane. For those particular vehicles in certain environments where there is strong coupling between motion in the horizontal and vertical planes, all six degrees of freedom should be handled together.

Motion in the vertical plane involves only vertical motions of the various sectional shapes along the vehicle. Because of the symmetry of port and starboard (both geometrically and inertially, $y_G = 0$), the vertical motion of the sections do not produce any rolling moment, K. However, if the vehicle has an angle of roll ϕ from the upright, (especially a surface vehicle), then this symmetry is disturbed and vertical motions of the section will produce a rolling moment. Since this latter condition requires both an angle of roll ϕ together with some vertical disturbance such as $w, \dot{w}, z_o, q, \dot{q}$, etc., then the function depends on the combination of the two variables and therefore is second order and will not appear in the linear equations.

One cannot make the same argument for motion in the horizontal plane since but few ocean vehicles have deck-keel symmetry and even those vehicles which have deck and keel geometrical symmetry such as a torpedo do not possess deck-keel inertial symmetry since the center of gravity is below the center of symmetry - i.e. $z_G \neq 0$. For the above response, the roll equation is usually associated with the equation for motion in the horizontal plane. Hence, we have the following breakdown:

Horizontal Plane	Vertical Plane
$\left.\begin{matrix} X \\ Y \\ N \\ K \end{matrix}\right\}$ equations	$\left.\begin{matrix} X \\ Z \\ M \end{matrix}\right\}$ equations

It is shown in Appendix I that, because of port and starboard symmetry the terms $X_v = X_{\dot{v}} = X_r = X_{\dot{r}} = X_p = X_{\dot{p}} = X_\phi = X_\delta = 0$ and as a consequence the X equation can be conveniently decoupled from the Y, N, and K equations. The X equation for motion in the vertical plane cannot be decoupled because of deck-keel assymmetries but the K equation can be decoupled because of port-starboard symmetries.

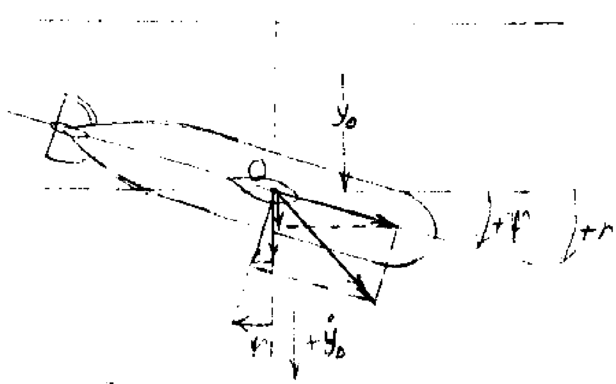
We are now in the position to write the linearized equations of motion for horizontal and vertical motion but again we should take advantage of the flexibility that exists in the choice of which motion variables should be considered as independent. For motion in the horizontal plane, there are four equations; hence, four independent variables can be solved for. For motion in the vertical plane with three equations, three independent motion variables can be chosen. For example if the independent variable is the roll angle $\phi(t)$ then the dependent variables become

$$p = \dot{\phi} = \frac{d\,\phi(t)}{dt} \quad \text{(within linear theory)}$$

$$\dot{p} = \ddot{\phi} = \frac{d^2\,\phi(t)}{dt^2}$$

Similarly, if u(t) is the variable, then $\dot{u}$ becomes a dependent variable, and if $\psi(t)$ is the independent variable, then r and $\dot{r}$ become dependent variables. It remains necessary to reconcile the role played by the variable y_o in the horizontal plane (geographical tranverse displacement) and z_o in the vertical plane (the heave of a surface vehicle or the depth of a submerged vessel) because if these cannot be expressed in some form of relationship to the other variables we will end up with more variables than equations to solve for them. From the following sketches a relationship within linear theory is indicated.

Horizontal Plane



$$\frac{dy_o}{dt} = \dot{y}_o = v \cos \psi + u \sin \psi$$

where $u = u_o + \Delta u$

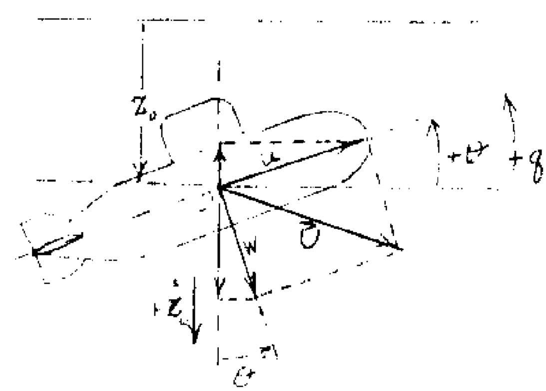
within linear theory for small ψ and Δu

$$\dot{y}_o = v + u_o \psi$$

or $v = \dot{y} - u_o \psi$

$$\dot{v} = \ddot{y}_o - u_o \dot{\psi} = \ddot{y}_o - u_o r$$

Vertical Plane



$$\frac{dz_o}{dt} = \dot{z}_o = w \cos \theta - u \sin \theta$$

within linear theory for small θ

$$\dot{z}_o \doteq w - u_o \theta$$

or $w = \dot{z}_o + u_o \theta$

$$\dot{w} = \ddot{z}_o + u_o \dot{\theta} = \ddot{z}_o + u_o q$$

Here we see that definite relationships exist between the

variables such that either v or y_o, and w or z_o can be chosen as the independent variable. We have the same number of equations as we have independent variables.
In the case of motion in the horizontal plane without roll, there would be three equations involving X,Y, and N and the independent variables chosen would be:

(1) u,v, and r in the case there was (a) no sensitivity to ψ or that no automatic controls with sensitivity to ψ were involved and (b) no sensitivity to transverse displacement y_o (such as would be the case of a towed body or a body moving in a canal or close to another vehicle).

(2) u,v, and ψ if automatic controls sensitive to heading ψ were involved.

(3) u,y_o, and ψ if inherent body sensitivity or control sensitivity to y_o and ψ are involved (particularly for a towed body).

If roll is included with motion in the horizontal plane, then there are four equations and the fourth independent variable becomes the roll angle ϕ .

For motion in the vertical plane (without roll) there are three equations involving X,Z, and M and the three independent variables chosen would be:

(1) u,w, and θ for those vehicles which have no inherent sensitivity to depth disturbance or no controls sensitive to depth disturbance, but have an inherent sensitivity or control sensitivity to trim angle, θ . (Such as torpedo, submarine, and other submersibles sufficiently below the water surface).

(2) u,z_o, and θ for surface vessels, hydrofoil boats, hovercraft, torpedo, and other submersibles operating near the surface wherein vertical displacement relative to the surface can affect body forces; also for those vehicles with

controls sensitive to depth.

The linearized equations of motion can now be written by equating the linearized expressions for the forces and moment functions and equating these expressions to the linearized dynamical response on the right hand side of the equation. The equations for motion in the horizontal plane without roll and without controls, excitation, and restraints are rather simply derived in Appendix I, pages 20-23. When roll is included and the linearized terms for control surface deflection and other control effectors are added, along with linearized exetation and restraint forces, a much more generalized set of equations results. Before writing this set of equations,it is convenient to introduce the operators of the differentiation with respect to time, i.e., $D = \frac{d}{dt}$. This is done in order to clearly isolate the independent variables involved and to put the equations in a form convenient for solution. The linearized equations for motion in the horizontal plane (with no sensitivity to y_o or ψ and with $y_G = 0$) are:

$$[(X_{\dot{u}} - m)D + X_u]\,\Delta u + [X_{\dot{v}}D + X_v]v + [X_{\dot{r}}D + X_r]r + [X_{\dot{p}}D^2 + X_pD + X_\phi]\phi$$

$$= -[\sum X_\delta \delta + \sum X_{\dot{\delta}}\dot{\delta} + \sum X_{\ddot{\delta}}\ddot{\delta} + \sum X_n \Delta n + \sum X_{\dot{n}}\dot{n} + X\text{ (excitation)} \tag{1}$$

$$+ X\text{ (restraints)}]$$

$$[Y_{\dot{u}}D + Y_u]\,\Delta u + [(Y_{\dot{v}} - m)D + Y_v]\,v + [(Y_{\dot{r}} - mx_G)D + (Y_r - mu_o)]r$$

$$+[(Y_{\dot{p}} + mz_G)D^2 + Y_pD + Y_\phi]\phi = -[\sum Y_\delta \delta + \sum Y_{\dot{\delta}}\dot{\delta} + \sum Y_{\ddot{\delta}}\ddot{\delta}$$

$$+ \sum Y_n \Delta n + \sum Y_{\dot{n}}\dot{n} + Y(\text{exc.}) + Y(\text{restraints})] \tag{2}$$

$$[N_{\dot{u}}D + N_u]\ \Delta u + [(N_{\dot{v}} - mx_G)\ D + N_v]\ v + [(N_{\dot{r}} - I_z)D + (N_r - mx_G u_o)]r$$

$$+[N_{\dot{p}}D^2 + N_pD + N_\phi]\phi = -[\sum N_\delta \delta + \sum N_{\dot{\delta}}\dot{\delta} + \sum N_{\ddot{\delta}}\ddot{\delta} + \sum N_n \Delta n \qquad (3)$$

$$+ \sum N_{\dot{n}}\dot{n} + N\ (\text{excitation}) + N\ (\text{restraints})]$$

$$[K_{\dot{u}}D + K_u]\ \Delta u + [(K_{\dot{v}} + mz_G)D + K_v]\ v + [K_{\dot{r}}D + (K_r + mz_G u_o)]\ r$$

$$+[(K_{\dot{p}} - I_x)D^2 + K_pD + K_\phi\ ]\phi \qquad (4)$$

$$= -\ [\ \sum K_\delta \delta + \sum K_{\dot{\delta}}\dot{\delta} + \sum K_{\ddot{\delta}}\ddot{\delta} + \sum K_n\ \Delta n + \sum K_{\dot{n}}\dot{n} + K(\text{excitation})$$

$$+ K(\text{restraints})]$$

In the above equations $\sum X_\delta \delta$ and similar terms refer to all the individual control surfaces and their deflections (mentioned earlier) and the terms of the form $\sum X_n \Delta n$ refer to all rotary control effectors (such as thrusters), including the main propeller. The variable Δn is used to indicate a disturbance from any equlibrium condition of n_o (similar to the use of Δu rather than u in the equations). This point is stressed in detail in Appendix I. The above equations can now be written in the following condensed form of coefficients and variables:

$$a_{11}\Delta u + a_{12}v + a_{13}r + a_{14}\phi = -[\sum X_\delta \delta + \ldots + X(\text{exc.}) + X(\text{restr.})] \qquad (5)$$

$$a_{21}\Delta u + a_{22}v + a_{23}r + a_{24}\phi + -[\sum Y_\delta \delta + \ldots + Y(\text{exc.}) + Y(\text{restr})] \qquad (6)$$

$$a_{31}\Delta u + a_{32}v + a_{33}r + a_{34}\phi = -[\sum N_\delta \delta + \ldots + N(\text{exc.}) + N(\text{restr})] \qquad (7)$$

$$a_{41}\Delta u + a_{42}v + a_{43}r + a_{44}\phi + -[\sum K_\delta \delta + \ldots + K(\text{exc.}) + K(\text{restr})] \qquad (8)$$

If Y_o is desired as an independent variable (such as in the case of towed bodies or operation in a narrow canal) then the equations become, under the restraints of the towline, excluding roll and coupling surge. (See Appendix I pages I-105-113).

$$[(Y_{\dot{v}}-m)D^2 + Y_v D + Y_{yo} - \frac{T}{\ell}]y_o + [(Y_{\dot{r}}-mx_G)D^2 + (Y_r - u_o Y_{\dot{v}})D + Y_\psi - u_o Y_v - T(1 + \frac{x_p}{\ell})]\psi = -[\sum Y_\delta \delta + \sum Y_{\dot{\delta}} \dot{\delta} +---+ Y(\text{exc.})] \quad (9)$$

$$[(N_{\dot{v}}-mx_G)D^2 + N_v D + N_{yo} - \frac{Tx_p}{\ell}]y_o + [(N_{\dot{r}}-I_z)D^2 + (N_r - u_o N_{\dot{v}})D + N_\psi - u_o N_v - Tx_p(1 + \frac{x_p}{\ell})]\psi = -[\sum N_\delta \delta +---+ N(\text{exc.})] \quad (10)$$

Where T is the tension in the towline (in the equilibrium condition of straight ahead motion at constant speed u_o), ℓ is the length of the towline, and x_p is the longitudinal position of the point of the towline attachment to the body (positive forward of the origin). The above equations can be conveniently written in the terms of coefficients and independent variables as:

$$a_{11}y_o + a_{12}\psi = -[\sum Y_\delta \delta +----+ Y(\text{exc.})] \quad (11)$$

$$a_{21}y_o + a_{22}\psi = -[\sum N_\delta \delta +----+ N(\text{exc.})] \quad (12)$$

The linearized equations for motion in the vertical plane, without roll, involving the X,Z, and M equations can be readily developed along simular lines as used for the horizontal plane motion by allowing only for the variables $\Delta u, w, q, \dot{u}, \dot{w}, \dot{q}, \theta, z_o$ plus the control effectors and excitation in the vertical plane. Since for the surface ship, hovercraft, and hydrofoil boat the heave (z_o) is an important parameter and since for submersibles, depth controls are common (sensitive to z_o), the three independent variables are usually chosen as Δu, z_o, and θ. The linearized equations become (using the dependence, developed earlier, of $w = z_o + u_o\theta$, and that $y_G = 0$).

$$[(X_{\dot{u}}-m)D + X_u]\Delta u + [X_{\dot{w}}D^2 + X_w D + X_{z_o}]z_o + [(X_{\dot{q}}-mz_G)D^2 + (X_q + u_o X_{\dot{w}})D + (X_\theta + u_o X_w)]\theta$$
$$= -[\sum X_\delta \delta + \sum X_{\dot{\delta}}\dot{\delta} + \sum X_{\ddot{\delta}}\ddot{\delta} + \sum X_n \Delta n + \sum X_{\dot{n}}\dot{n} + X(\text{exc.}) + X(\text{restr})] \quad (13)$$

$$[Z_{\dot{u}}D + Z_u]\Delta u + [(Z_{\dot{w}}-m)D^2 + Z_w D + Z_{z_o}]z_o + [(Z_{\dot{q}}-mx_G)D^2 + (Z_{\dot{q}} + u_o Z_{\dot{w}})D + (Z_\theta + u_o Z_w)]\theta$$
$$= -[\sum Z_\delta \delta + \sum Z_{\dot{\delta}}\dot{\delta} + \sum Z_{\ddot{\delta}}\ddot{\delta} + \sum Z_n \Delta n + \sum Z_{\dot{n}}\dot{n} + Z(\text{exc.}) + Z(\text{restr.})] \quad (14)$$

$$[(M_{\dot{u}}-mz_G)D + M_u]\Delta u + [(M_{\dot{w}} + mx_G)D^2 + M_w D + M_{z_o}]z_o + [(M_{\dot{q}}-I_y)D^2 + (M_q + u_o M_{\dot{w}})D + (M_\theta + u_o M_w)]\theta$$
$$= -[\sum M_\delta \delta + \sum M_{\dot{\delta}}\dot{\delta} + \sum M_{\ddot{\delta}}\ddot{\delta} + \sum M_n \Delta n + \sum M_{\dot{n}}\dot{n} + M(\text{exc.}) + M(\text{restr.})] \quad (15)$$

The above equations can be put into the convenient form of coefficients and independent variables as follows:

$$a_{11}\Delta u + a_{12}z_o + a_{13}\theta = -[\sum X_\delta \delta + \ldots + X(\text{exc.}) + X(\text{restr.})] \quad (16)$$
$$a_{21}\Delta u + a_{22}z_o + a_{23}\theta = -[\sum Z_\delta \delta + \ldots + Z(\text{exc.}) + Z(\text{restr.})] \quad (17)$$
$$a_{31}\Delta u + a_{32}z_o + a_{33}\theta = -[\sum M_\delta \delta + \ldots + M(\text{exc.}) + M(\text{restr.})] \quad (18)$$

Of course the coefficients a_{11}, a_{12}, etc., in the three sets of equations [(5),(6),(7),(8)],[(11),(12)] and [(16),(17),(18)] refer to different quantities in each set.

The linearized equations developed herein, can be used to handle motion stability problems for most ocean vehicles under the equilibrium condition of forward motion at constant speed. The equations can also handle the maneuvering problems for reasonable maneuvers and motions from the equilibrium condition. For tight maneuvers and large motions, the validity of the linearized equations are degraded and non-linear equations to properly model the phenomena may be required.

SOLUTIONS TO THE LINEARIZED EQUATIONS OF MOTION

The linearized equations of motion as developed, represent a mathematical model of vehicle dynamics. The solution of the equations for the various independent variables as functions of time (and consequently any of the dependent variables), gives a prediction or simulation of the real physical motion responses of the vehicle. Hence, we must go through certain mathematical processes to evolve the solution, but we should realize that the mathematics are a tool and not lose sight of the fact that we are looking for meaningful physical results from the math modelling,which results are to provide sufficiently valid information for engineering use.

There are many ways to solve linear differential equations with constant coefficients (i.e. constant with respect to time) and one such method is the use of the familiar LaPlace transforms. It is assumed that the student has not developed a familiarity with transform methods. Therefore, the more straight forward method of using a differential operator in conjunction with normal algebraic processes are introduced and used for the solution of the linearized equations.

The equations are in the general form of:

$$a_{11} x_1 + a_{12} x_2 + \dots\dots + a_{1n} x_n = A_1(t)$$

$$a_{21} x_1 + a_{22} x_2 + \dots\dots + a_{2n} x_n = A_2(t)$$

$$a_{31} x_1 + \dots\dots\dots\dots\dots\dots\dots\dots = A_3(t)$$

$$a_{n1} x_1 + a_{n2} x_2 + \dots\dots\dots + a_{nn} x_n = A_n(t)$$

where the a_{ij} $(i=1 \rightarrow n, \hat{j}=1 \rightarrow n)$ are coefficients, which in general include the differential operator D and are independent of time. The coefficients are independent of time because they are made up of the various derivatives which are defined by reference to the initial equilibrium condition and not by conditions of the variables during a motion trajectory. The x_j $(j=1 \rightarrow n)$ represent the independent variables and are functions of time i.e. $x_j(t)$. The A_i are, in general, functions of time and represent the various forces and moments exerted on the vehicle by the excitation of the environment and restraints, and the time variation of control effectors. If the control effectors are made "automatic" by controlling them as functions of the independent or dependent motion variables, then the forces and moments on the effectors are not independent functions of time. In the case of automatic controls, the control effects as functions of the motion variables are brought over to the left side of the equation and combined with the coefficients of similar variables. Automatic controls are discussed in more detail later.

A straightforward algebraic solution of the equations for the independent variable x_j is, in determinant form:

$$x_j(t) = \frac{\begin{vmatrix} a_{11} a_{12} \cdots a_{1,j-1} - A_1(t) \; a_{1,j+1} \cdots a_{1,n} \\ a_{21} a_{22} \cdots a_{2,j-1} - A_2(t) \; a_{2,j+1} \cdots a_{2n} \\ \\ a_{n1} a_{n2} \cdots a_{n,j-1} - A_n(t) \; a_{n,j+1} \cdots a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} a_{12} \cdots\cdots a_{1n} \\ a_{21} a_{22} \cdots\cdots a_{2n} \\ \\ a_{n1} a_{n2} \cdots\cdots a_{nn} \end{vmatrix}}$$

the denominator is the determinant of the coefficients and this determinant will in general have terms in several powers of the differential operator D. If m is the highest power of D appearing in the determinant, then the determinant can be written in the form of:

$$|\text{Det.}| = C_m D^m + C_{m-1} D^{m-1} + \ldots\ldots + C_2 D^2 + C_1 D + C_o$$

If the roots of the above equation are designated by σ, then the determinant can be written in the form:

$$|\text{Det.}| = C_m (D-\sigma_m)(D-\sigma_{m-1})(\quad)(\quad)\ldots\ldots(D-\sigma_1)$$

In expanding the determinant of the numerator there will occur terms involving the product of coefficients a_{ij} with the $A_i(t)$ terms. Since the differential operator D may appear in the a_{ij} coefficient and since the $A_i(t)$ terms are functions of time, it is important that in each term of the determinant the coefficient combinations appear in front of the $A_i(t)$ terms so that the proper differentiation of the time function may be carried out. A simple example of this procedure is given in Appendix I, pages I-68 & 69.

If the evaluation of the numerator is carried out in this fashion, then the numerator becomes a function of time, say $F_j(t)$. Hence the solution has been reduced to the form:

$$x_j(t) = \frac{F_j(t)}{C_m (D-\sigma_m)(D-\sigma_{m-1})\ldots\ldots\ldots(D-\sigma_1)}$$

In Appendix I pages I-25 to 33, the significance of the operator $\frac{1}{(D-\sigma)}$ is developed and applied to the solution of $x_j(t)$ for a special case. The operator indicates the following:

$$\left(\frac{1}{D-\sigma}\right)F_j(t) = e^{\sigma t}\int e^{-\sigma t} F_j(t)dt$$

or:

$$\left(\frac{1}{D-\sigma}\right) = e^{\sigma t}\int e^{-\sigma t} (\quad)dt$$

The solution of $x_j(t)$ results from the successive operation with $\frac{1}{D-\sigma}$ as indicated in the following:

$$x_j(t) = \frac{1}{C_m}\left(\frac{1}{D-\sigma_m}\right)\left(\frac{1}{D-\sigma_{m-1}}\right)\cdots\left(\frac{1}{D-\sigma_1}\right)F_j(t)$$

A typical case demonstrating this approach is given in Appendix I pages I-31 to 33 where the $A_i(t)$ are zero and in Appendix I pages I-69 to 71 when $A_i(t)$ is a constant and when it is an exponential function of time. A solution when $A_i(t)$ is a sinusoidal function is demonstrated in the solution of a sample homework problem presented in Appendix B.

CONSIDERATIONS OF MOTION STABILITY

The ability to solve the linearized equations of motion for any excitation, restraint, or control forces which can be expressed as a function of time or as a function of the motion variables allows us to evaluate certain critical properties of the motion responses. In the absence of excitation and the absence of effector action as an independent function of time (i.e. either no effector action or action as a function of the motion variables) the $A_i(t)$ are zero. This is the case for determining the motion stability of the vehicle, since stability may be defined in the following manner.

a. Establish an equilibrium condition of motion.

b. Disturb the equilibrium condition with arbitrary disturbances which are infinitesimal in magnitude.

c. If the vehicle returns to the original equilibrium condition in time, then it is stable in that condition. If it departs from the original equilibium condition or ends up in another condition of equilibrium, then the vehicle is unstable in the original equilibrium condition.

Physically speaking, an unstable system is one which cannot exist in a given equilibrium condition, even in the absence of a disturbance. Hence, the $A_i(t)$ equal to zero represent no excitation or disturbance on the vehicle. Also, the linearized equations of motion are perfectly valid for determining motion stabilities since only small (infinitesimal) values of the motion variables are involved in the dynamic analysis.

If the solution for the independent variables $x_j(t)$ are

are carried out for zero disturbance, i.e. $A_i(t) = 0$, in the manner indicated in the previous chapter, the following results:

$$x_j(t) = C_{1j}e^{\sigma_1 t} + C_{2j}e^{\sigma_2 t} + \ldots + C_{mj}e^{\sigma_m t}$$

where σ_1 to σ_m are the roots of the determinant of the coefficients in the linearized equations of motion. A typical solution of this form is carried out in Appendix I pages I-30 to 33 in which the stability for the equlibrium condition of straight line motion at constant speed is considered. In this case, the linearized variables $\Delta u(t)$, $v(t)$, and $r(t)$ are involved ($j=3$) and there are three characteristic roots of the equations ($m=3$). Since the $x_j(t)$ are motion variables which indicate the departure of that variable from the value of the variable in the equlibrium condition, in order for the vehicle to return to its original equlibrium condition, all the $x_j(t)$ must go to zero as time increases. The C_{ij} in the above equation are constants (resulting from integration) and merely satisfy the initial conditions of the disturbance and are arbitrary for arbitrary disturbances. Hence, independent of any of the coefficients, the only way all motion disturbances will go to zero is for all the σ (1 to m) to be negative if the roots are real and that the real part of all the complex roots are negative.

If one is interested in the actual functions $x_j(t)$ as they change with time after a disturbance of the variables, then C_{ij} coefficients must be determined by the initial conditions of the disturbance. Since there are essentially m integrations involved in the solution of the equations, only m number of C_{ij} can be independently defined, the remaining C_{ij} are dependent and can be determined by substituting back into the equations of motion for the condition t=0, and solving for the C_{ij}.

The relatively simple stability in straight line motion at constant speed (in the horizontal plane) for a vehicle is carried out in Appendix I where the roots are given in terms of the various hydrodynamic coefficients of the vehicle. On pages I-34 through I-52 the nature of the various hydrodynamic coefficients for a surface displacement vehicle is discussed and the development of a criteria for dynamical stability (in straight line motion) is developed. This criteria is as follows:

$$Y_v(N_r-mx_G u_o) - N_v(Y_r-mu_o) > 0$$

Higher degrees of motion stability may be determined by defining more stringent "equilibrium conditions" to which the vehicle must return after an arbitrary disturbance. For example, if we wish the vehicle to return to the original heading angle after the disturbance, the independent variables (without roll) would be Δu, v, and ψ and the order of the determinant of the coefficients would be four, thus giving four roots or four values of σ. For directional stability all four roots must have negative real parts. If we are looking for positional stability, (such as returning to the original depth after a disturbance, returning to the centerline of a canal after a disturbance, or returning to the original transverse position aft a towing vehicle) there will be 5 roots. Due to symmetry of port and starboard the X equation involved in motion in the horizontal plane can be decoupled from the others and one root can be readily factored from the determinant. (This has been shown in previous references to Appendix I). If roll is considered a single degree of freedom then there are two roots involved. If the roll equation is coupled with motion in the horizontal plane, then two additional roots are added. For instance, for directional stability in the horizontal plane with roll, there will be six roots with one of the six roots factorable from the determinant of coefficients.

Routh in his development of dynamics, indicates certain conditions for the coefficients in order for the roots of a polynomial to indicate stability (have negative real parts). For a quadratic of the form:

$$aD^2 + bD + c = 0$$

the conditions are

$$\frac{b}{a} > 0 \text{ and } \frac{c}{a} > 0$$

for the cubic

$$aD^3 + bD^2 + cD + d = 0$$

the conditions are

$$\frac{b}{a} > 0, \ \frac{c}{a} > 0, \ \frac{d}{a} > 0, \qquad \frac{bc - ad}{a^2} > 0$$

for the quartic

$$aD^4 + bD^3 + cD^2 + dD + e = 0$$

the conditions are

$$\frac{b}{a} > 0, \ \frac{c}{a} > 0, \ \frac{d}{a} > 0, \ \frac{e}{a} > 0, \ \frac{bcd - ad^2 - b^2e}{a^3} > 0$$

For general development of these conditions see "Advanced Dynamics of a System of Rigid Bodies", E.J. Routh.

CONSIDERATIONS OF AUTOMATIC CONTROL

It was previously mentioned that control effectors could be moved or positioned as an independent function of time or as a function of some parameter that can be measured by sensors, usually a motion parameter. Gyros and stabilized platforms can measure angular displacement and angular rates, accelerometers can sense acceleration and by integration give linear velocity and displacement; velocities can be measured by logs, pitot tubes, doppler sonar, etc. and depths and distances can be measured by pressure gages and sonar; and so on. For any variable being sensed by instrumentation, the signal can be amplified electrically and/or mechanically and hydraulically to actuate the effectors as a function of the sensing signal.

If the parameter being sensed is a motion variable, then the actuation of the controls is usually a linear function of the motion variable in order to keep the resulting equations linear and readily solvable. However, the general case would allow for non-linear functions which would then be included with the non-linear equations of motion, resulting in a non-linear mathematical model of the vehicle dynamics. Such non-linear models would necessitate computation by numerical methods using automatic computation and preclude any closed solution.

If the control effectors are motivated as independent functions of time, the nature of the solution for the response would be similar to the solution for an excitation as a function of time, as demonstrated in the previous chapter. For the case (usual) where the control actuations are linear functions of (proportional to) the motion parameters (usual case of automatic controls), the following general form pertains:

$$\delta_i(t) = k_{i1}x_1(t) + k_{i2}x_2(t) + \ldots + k_{in}x_n(t) = \sum_{j=1}^{n} k_{ij}x_j(t)$$

where $\delta_i(t)$ is the ith control effector actuation, the $x_j(t)$ are the motion parameters, and the k_{ij} are the proportionality constants. Examples of the $\delta_i(t)$ are rudder deflection, rotary speed of a thruster or change in main propeller revolutions, the displacement of ballast weight, deflection of an anti-rolling fin, pitch change of a propeller etc. Examples of $x_j(t)$ are heading angle, pitch angle, roll angle, angular velocities and accelerations, depth, depth rate, altitude, distance from target, speed error, etc.

Since it takes finite time for a sensor signal to be amplified and the effector to be moved to the called for position, the actual value of δ_i at time t is proportional to the value of x_j measured a short time before, i.e., $x_j(t-\Delta t)$ where Δt is referred to as the time lag. Electrical and electronic times involved in sensing and electrical power amplification are negligible compared to the time necessary to actually move the effectors. Effectors such as ship rudders and roll control fins involve much inertia and the time lag depends on the powering arrangements for their activation.

The above equation should therefore be written as:

$$\delta_i(t) = k_{i1}x_1(t-\Delta t_i) + k_{i2}x_2(t-\Delta t_i) + \ldots + k_{in}x_n(t-\Delta t_i) = \sum_{j=1}^{n} k_{ij}x_j(t-\Delta t_i)$$

Here Δt_i sets up a different time lag for each effector which does not change with the motion variable involved. This is in line with the fact that the x_j are the sensed values involving negligble time and the δ_i involve the electro-mechanical and/or hydraulic powering system of a given effector and involve the same time lag for a given effector.

Since the function $\delta_i(t)$ expressed in terms of $x_j(t + \Delta t_i)$ will be incorporated in the equations of motion variables in terms of $x_j(t)$, it becomes necessary to express $x_j(t-\Delta t_i)$ in terms of $x_j(t)$ for consistency before any solution of the equations, including automatic controls, can be carried out. This is done by a Taylor expansion of the function $x_j(t-\Delta t_i)$ in the variable Δt about the value t. Such a Taylor expansion takes the form of (as demonstrated in a previous chapter):

$$x_j(t-\Delta t_i) = e^{-(\Delta t)_i D}\, x_j(t) =$$

$$[1-(\Delta t)_i D + \frac{(\Delta t_i)^2}{2!}D^2 - \frac{(\Delta t_i)^3}{3!}D^3 - -]x_j(t)$$

It is usual to assume the equivalent time lag Δt_i as small and hence terms in Δt beyond the linear term are usually neglected. Retaining the non-linear terms in Δt do not render the equations non-linear but may have the effect of increasing the order of D in the determinant of coefficients and hence the number of roots involved. With linearized time lag the expression reduces to:

$$x_j(t-\Delta t_i) = x_j(t) - \Delta t_i D x_j(t) = x_j(t) - \Delta t_i \dot{x}_j(t)$$

Hence, under a linear time lag Δt, the control function becomes:

$$\delta_i(t)=\sum_{j=1}^{n} k_{ij}x_j(t)-\Delta t_i \sum_{j=1}^{n} k_{ij}\dot{x}_j(t)$$

The above expression deals with the selected independent variables x_j. In addition to having effectors controlled proportional to the selected independent variables, they may also be controlled and usually are proportial to dependent variables such as $\dot{x}_j$ and $\ddot{x}_j$ and also include a time lag of Δt_i. For example, in automatic pilots for most marine vehicles, the direction or heading of the vehicle is controlled so that one has a rudder deflection of the form $\delta(t)=k_1\psi(t-\Delta t) + k_2\dot{\psi}(t-\Delta t)$. By making the control sensitive to heading rate $\dot{\psi}$(or r) in addition to ψ the control becomes more anticipatory of the error in ψ and therefore makes a better control system. This sensitivity to heading rate is similar to a good skipper "meeting the swinging of the ship". An example of this type of control is developed in Appendix I pages I-76 to I-79 and I-109 to I-112. Hence, the more general control function, with linear time lag would be:

$$\delta_{i(t)}=\sum_{j=1}^{n} k_{ij}x_j(t)-\Delta t_i\sum_{j=1}^{n} k_{ij}\dot{x}_j(t)+\sum_{j=1}^{n} s_{ij}\dot{x}_j(t)-\Delta t_i\sum_{j=1}^{n} s_{ij}\ddot{x}_j(t)+ \text{etc.}$$

$$\delta_i=[\sum_{j=1}^{n} k_{ij} + \sum_{j=1}^{n} (s_{ij}-\Delta t_i k_{ij})D + \sum_{j=1}^{n} (m_{ij}-\Delta t_i s_{ij})D^2]x_j$$

where k_{ij}, s_{ij}, and m_{ij} are the proportional control constants.

The control forces and moments are $X_{\delta_i}\delta_i$, $Y_{\delta_i}\delta_i$ ......, $N_{\delta_i}\delta_i$ and can be obtained by multiplying the control effector displacement by the proper hydrodynamic derivative (or coefficient). Of course, a completely identical mathematical form results if the control effector is in the form of a thruster where $n_i(t)$ is involved instead of $\delta_i(t)$.

When the controls are automatic and are sensitive to the motion variables (and their time derivatives), the resulting forces and moments become functions of these variables and combine with the terms on the left side of the equations of motion and in effect change the a_{ij} coefficients of the selected independent motion variables of interest. Herein lies the general effect of automatic controls - controls can alter the sensitivity of the vehicle to those motion responses to which the vehicle is inherently sensitive and can introduce sensitivity to motion parameters to which the vehicle has no inherent sensitivity. The vehicle with regard to stability and motion response (linear) is characterized by the set of coefficients a_{ij} without controls and by a set of coefficients a'_{ij} with controls.

Time lags tend to degrade system performance and too large a lag can destabalize the vehicle. The magnitude of the lag Δt has to be determined based on the control effector system dynamics, For most marine vehicles, control surface deflection rate ($\dot{\delta}$) primarily determines the magnitude of lag. The concept of a linearized time lag, reduces the complexity of formulating and solving the appropriate mathematical model. In a rigorous treatment of the control system dynamics, several additional linear differential equations describing the dynamics of the electro-hydraulic-mechanical system would have to be set up,and combine with with motion differential equations through a coupling of electrical signals proportional to measured motion response, through the dynamics of rotating machinery, etc., until the control effector response as a function of time evolves. This is a rather complicated set of equations to add onto the equations of motion. If rigor is desired, the control effector manufacturer will have evaluated their system both analytically and experimentally, and should be able to provide information such as frequency response of the system and other pertinent dynamical information.

EXCITATION FROM THE ENVIRONMENT

The solution of the linearized equations for motion response have been set up for any linearized excitation which is expressed as a suitable function of time. Such excitation can be self imposed such as the motion of a control surface (or effector) as a function of time (independent of motion variables*), say oscillating a rudder in sinusoidal fashion at some amplitude and frequency. The excitation can readily come from the environment - from waves and currents, or a vehicle can be subjected to excitation through certain restraints - such as oscillation or time dependent motion of a towing cable to which the vehicle is attached.

For most ocean vehicles, the major excitation source from the ocean environment results from effects of surface waves, with varying currents and subsurface turbulence becoming important for deeply submerged vehicles. We shall discuss briefly the nature of these types of excitation from the ocean environment and indicate methods of evaluating the significant motion responses of the vehicle to the environment. The methods for solving the responses to these excitations can be directly carried over to solving for the vehicle response when the excitation comes from a regular or random (irregular) motion of control effectors or restraining mechanisms.

Ocean wave properties are fairly well defined by gravity - free surface wave theories. These theoretical approaches provide useful engineering data since the

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*This case is covered previously under automatic control results of linear wave theory are quite valid within the valid domain of the linearized equations of motion. The following properties for a train of regular waves in deep water (propagating in a single direction) are: (see "Theory of Seakeeping" - Korvin-Kroukovsky)

Surface elevation:

$$\eta(x,t) = \eta_o \cos\left(\frac{2\pi x}{\lambda} - \omega t + \varepsilon\right)$$

$$\omega = \sqrt{\frac{2\pi g}{\lambda}}$$

where
- η_o is the wave amplitude at the water surface
- x is the distance, along the wave, from a defined origin
- λ is the wave length
- ε phase angle at the origin x, when time $t=0$
- ω is the angular frequency of the wave

Propagation speed or celerity:

$$u_w = \sqrt{\frac{g\lambda}{2\pi}}$$

Wave slope at surface:

$$\frac{\partial \eta}{\partial x} = \frac{-2\pi\eta_o}{\lambda} \sin\left(\frac{2\pi x}{\lambda} - \omega t + \varepsilon\right)$$

Vertical velocity of the surface:

$$\dot{\eta}_s = \frac{\partial \eta}{\partial t} = + \omega\eta_o \sin\left(\frac{2\pi x}{\lambda} - \omega t + \varepsilon\right)$$

Water particle orbital velocity:

$$u_{\bar{o}} = \omega\eta_o \, e^{\frac{-2\pi z_o}{\lambda}}$$

where z_o is the depth below the surface

Constant pressure surface:

$$\eta_1 = \eta_o e^{\frac{-2\pi z_o}{\lambda}} \; ; \; \dot{\eta} = \omega\eta_o e^{\frac{-2\pi z_o}{\lambda}} \sin\left(\frac{2\pi x}{\lambda} - \omega t + \varepsilon\right)$$

Dynamic pressure (excluding hydrostatic pressure):

$$p_d = \rho g \eta_o \, e^{\frac{-2\pi z_o}{\lambda}} \cos\left(\frac{2\pi x}{\lambda} - \omega t + \varepsilon\right)$$

where ρ is the density

Horizontal dynamic pressure gradient:

$$\frac{\partial p}{\partial x} = \frac{-2\pi}{\lambda} \rho g \eta_o e^{\frac{-2\pi z_o}{\lambda}} \sin\left(\frac{2\pi x}{\lambda} - \omega t + \varepsilon\right)$$

Vertical dynamic pressure gradient (excluding hydrostatic pressure):

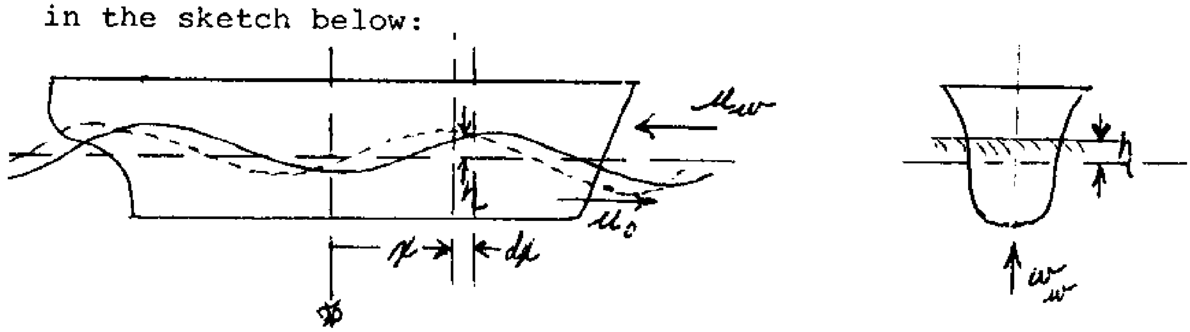
$$\frac{\partial p}{\partial z_o} = \frac{-2\pi}{\lambda} \rho g \eta_o e^{\frac{-2\pi z_o}{\lambda}} \cos\left(\frac{2\pi x}{\lambda} - \omega t + \varepsilon\right)$$

For deep water waves, the orbit of the water particle is circular. This orbit becomes elliptical for shallow water waves and the propagation speed becomes dependent on depth.

In general, when a vehicle is excited by waves, all the properties of wave action can contribute to the make up of the exciting force experienced by the vehicle - hydrostatics of surface elevation and the hydrodynamics of orbital velocity and orbital acceleration field. For example, the surface displacement vehicle is primarily excited by surface wave elevation (hydrostatic plus dynamic pressure gradient) whereas the hydrofoil boats (with completely submerged foils) is primarily excited by the vertical component of the orbital velocity which, in conjunction with forward speed, produces time varying angles of attack on the foil.

"Strip theory" is presently usually used to evaluate the excitation as well as the hydrodynamic coefficients for elongated surface displacement ships. It is essentially a two dimensional theory dealing with what is happening at each ship section. Distortion and/or reflection of the oncoming wave by the presence of the vehicle is neglected. Inadequacies in some three dimensional theories and the potential of other theories to advance the practice beyond "strip theory" are mentioned in the paper, "Recent Developments in Seakeeping Research and Its Application to Design", and in some of the references to this paper listed in Appendix III.

A section of the vehicle (restrained from motion)* at an instant of time in an undistorted wave is indicated in the sketch below:



*By the definition of "linearized" excitation it becomes necessary to analyse the excitation forces by restraining the vehicle to its original equilbrium condition - i.e., the excitation term is developed in the same manner as the hydrodynamic coefficients.

The section undergoes forces (which depend on section shape and frequency) due to the added (or reduced) buoyancy resulting from change of surface elevation, forces due to the vertical component of water orbital velocity w_w, forces due to orbital acceleration $\dot{w}_w$, and forces due to the pressure field or gradient in the liquid. The total force at each section, at that instant due to all wave effects, are integrated to give the force and moment acting on the vehicle at that instant of time. A short time later, when the wave profile has moved relative to the vehicle, (indicated by dotted line in the above sketch), there will be a different force acting at each section. The sectional forces are integrated to give the force and moment acting on the vehicle at this new instant of time. Such a series of calculations are made for a series of time values covering one cycle of excitation - i.e., over the passage of one wave length over the hull. From this information, the amplitude of force and moment excitation and the phasing of these excitations with the wave position along the hull can be determined. For example, the linearized heave and pitch excitation can now be expressed in the following form.*

$$Z(\text{exc.}) = Z_1 \cos (\omega_e t - \varepsilon_Z)$$

$$M(\text{exc.}) = M_1 \cos (\omega_e t - \varepsilon_M)$$

*It is assumed, and experience has been satisfactory, that the excitation is primarily in harmonic form in the fundamental frequency.

where Z_1 and M_1 are the amplitudes of heave force and pitching moment excitation respectively, ε_Z and ε_M are the phase lags for heave and pitch excitation respectively, and ω_e is the angular frequency of excitation (or encounter) and is expressed by:

$$\omega_e = \frac{2\pi}{T_e} = \frac{2\pi}{\lambda}\,[u_w - u_s \cos\alpha] = \frac{2\pi}{\lambda}\,[\sqrt{\frac{g\lambda}{2\pi}} - u_s \cos\alpha]$$

$$\omega_e = \omega - \frac{\omega^2 u_s}{g} \cos\alpha = \omega(1 - \frac{\omega u_s}{g} \cos\alpha$$

where T_e is the period of encounter, u_w is the wave celerity, u_s is the vehicle velocity, λ is the wave length, ω is the wave angular frequency ($\sqrt{\frac{2\pi g}{\lambda}}$), and α is the heading direction of the vehicle relative to the direction of propagation of the wave ($\alpha = 180^\circ$ for head seas).

(If the horizontal component of orbital velocity is large enough so as to have significant effect on the forward velocity of the vehicle relative to the water, then forces, which are functions of speed, such as the lift force on a foil at a given angle of attack, must be computed with this increase in velocity included.)

When the excitation from the environment is due to time varying currents or turbulence in the sea (as differentiated from orbital effects arising from surface waves), the linearized excitation terms in the equations of motion are evaluated in the same fashion as was done with excitation from orbital velocity effects - i.e., there is an amplitude of"turbulence" velocity, there is a wave length of the turbulence disturbance, and there is a frequency of encounter (or excitation) involving the "wake" velocity of the turbulence (mean or D.C. component). The primary source of excitation (linear) of vehicles by turbulence is the combination of the oscillatory (A.C.) components of transverse and/or vertical velocity of the turbulence with the forward speed of the vehicle to produce oscillating (time varying) angles of attack on the vehicle resulting in oscillatory lift forces and moments.

If theoretical means of calculating the excitation in waves proves inadequate, then resort can be made to model tests in regular waves, with a model restrained from motion response, in which the exciting forces and moments are measured. Such restrained model tests are made primarily to check calculation results because, when the goal is the solution of the equations for motion response, one would prefer allowing the model to be unrestrained and to measure the model response, thereby solving the problem by a more valid analogue than the linearized mathematical model.

If one knows the nature of the environment, then the excitation terms in the equations of motion can be evaluated for a given vehicle if certain hydrodynamic properties of the vehicle are calculated or measured through model tests. In any case, the excitation (linearized) is in the form of an amplitude, frequency, and phase and the response can be readily solved for by the techniques discussed in a previous chapter. To ease the calculation of the response, especially in carrying out the integrals involved, the excitation can be expressed in complex exponential form, for example:

$$Z(\text{exc.}) = Z_1 \cos(\omega_e t - \varepsilon_Z) = \text{Real Part } (Z_1 e^{i(\omega_e t - \varepsilon_Z)}) = \text{Real Part } [Z_1 e^{-i\varepsilon_Z} e^{i\omega_e t}]$$

$$Z(\text{exc.}) = \text{Real } [Z_1 e^{-i\varepsilon_Z} (e^{i\omega_e t})]$$

Here $Z_1 e^{-i\varepsilon}$ is independent of time (although possibly frequency and wave length dependent) and differentiation and integration over time in the process of solving for the motion response merely involves operations on the exponential $e^{i\omega_e t}$*.

The process of solving for response to an excitation at a given frequency has been demonstrated. But the ocean environment is not made up of regular waves or single frequency turbulence. The wave conditions in the open ocean and general turbulence are highly irregular and random. The "chaos" of a storm sea needs to be characterized in some systematic quantitative fashion in order to determine the excitation of the environment on the vehicle. This is accomplished by use of energy-density spectra description and techniques.

The irregular elevation of the sea surface or the irregular turbulence velocity component is considered to be composed of many different wave lengths (frequencies) randomly phased. The one property of a seaway sample (once the seaway is in "steady-state") is the energy involved per unit area of sea surface. Hence, the energy density is one quantifying item of the irregularity of the environment. Another important property is how the

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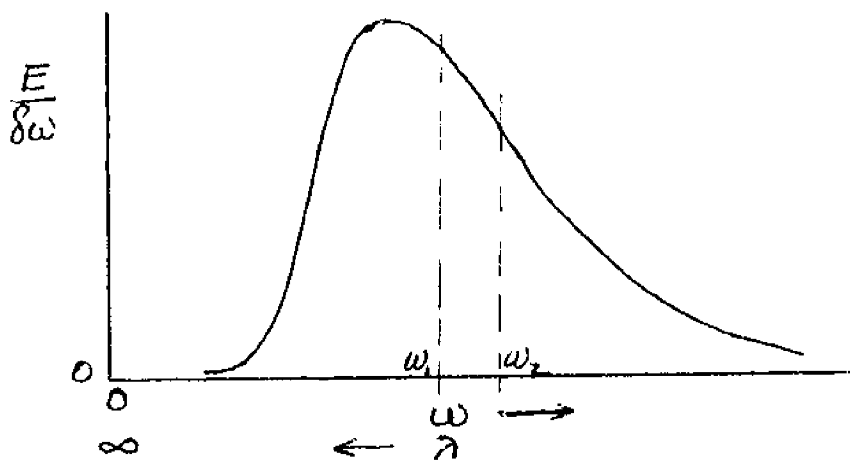
*See solution to problems in Appendices A & B.

energy is distributed among the various wave lengths or frequencies which make up the environment. This distribution is conveniently described by means of an energy-density spectrum, which is a plot of energy-density per unit frequency, $E/\delta\omega$ vs. ω where E is the energy per unit area, $\delta\omega$ is an incremental frequency element, and ω is the frequency (directly related to the wave length). Hence a typical spectrum of wave elevation will appear as shown below:



The shape and size of the spectrum is controlled by what is the largest wave length present (smallest ω since $\omega = \sqrt{\frac{2\pi g}{\lambda}}$) and by the fact that there is a limit on the energy-density that can be associated with small wave lengths (high frequency) before the wave becomes unstable and breaks (white caps). This is connected with the fact that energy of gravity waves is proportional to wave amplitude squared and it is necessary to increase amplitude at constant wave length to increase energy content at a given frequency. Hence, waves of small length (high frequency) can become unstable at small wave amplitudes (as compared to wave amplitudes of the longer waves) which means saturation at low energy content.

The area under the spectrum represents the total energy per unit area of the sea state since:

$$\int_{0}^{\infty} \frac{E}{\delta\omega}\, d\omega = E$$

The area between any two frequencies ω_1 and ω_2 (given band width) represents the energy-density associated with all waves between the two frequencies (or wave lengths) since the area is expressed by:

$$\int_{\omega_1}^{\omega_2} \frac{E}{\delta\omega}\, d\omega$$

Since for gravity waves the energy is proportional to wave amplitude squared, the ordinate of the curve can be $a^2/\delta\omega$ instead of E, where a is the amplitude representation at that frequency. This is a convenient way to characterize the seaway since the excitation on the vehicle depends on wave amplitude η_o with η_o replacing a in the spectral techniques which will be described later. In the case of turbulence, the fact that kinetic energy is proportional to velocity squared allows us to express the ordinate of the spectrum in terms of turbulent velocity amplitude squared (for the given frequency) divided by $\delta\omega$. The spectrum in this form is directly related to the exciting force on the vehicle since it is just this amplitude that determines the angle of attack on the vehicle when superimposed on the forward speed of the vehicle. A similar situation holds for wave orbital velocity spectra in describing the excitation from the environment.

The quantitative (linearized) measure of the irregular random environment is the spectrum and it has been indicated that the ordinate of the spectrum curve can be presented in terms directly related to the excitation of the vehicle. However, the frequency of excitation is in terms of ω_e (frequency of encounter) and not ω (frequency of wave or turbulence). To exploit the spectral presentation for full use in evaluating the excitation from the environment, it is necessary to present the spectrum in terms of ω_e for example, $a^2/\delta\omega_e$ vs. ω_e. In order to do this, the relationship between ω_e and ω is used.

$$\omega_e = \omega(1 - \frac{\omega u_s}{g} \cos \alpha)$$

$$\delta\omega_e = \delta\omega - \frac{2\omega\delta\omega u_s}{g} \cos \alpha = \frac{\delta\omega(1-2\omega u_s \cos \alpha)}{g}$$

It is obvious that the energy density between any two defined wave lengths (or frequencies) in a given spectrum is not changed merely by the manner in which one prefers to describe the situation. Hence,

$$\int_{\omega_1}^{\omega_2} \frac{E}{\delta\omega} \delta\omega = \int_{\omega_{e_1}}^{\omega_{e_2}} \frac{E}{\delta\omega_e} \delta\omega_e$$

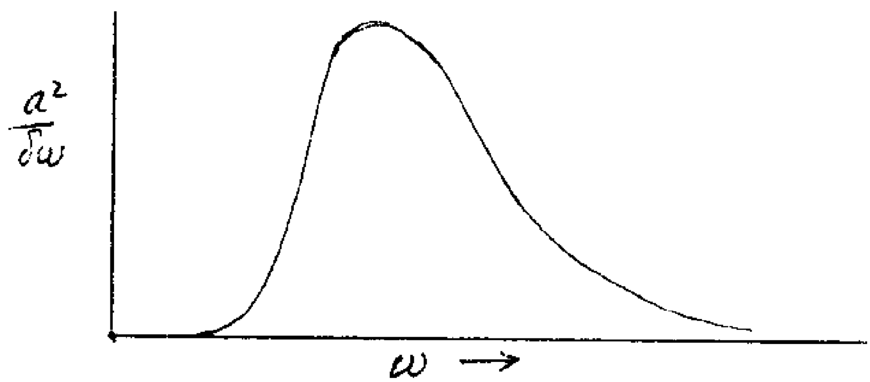
and

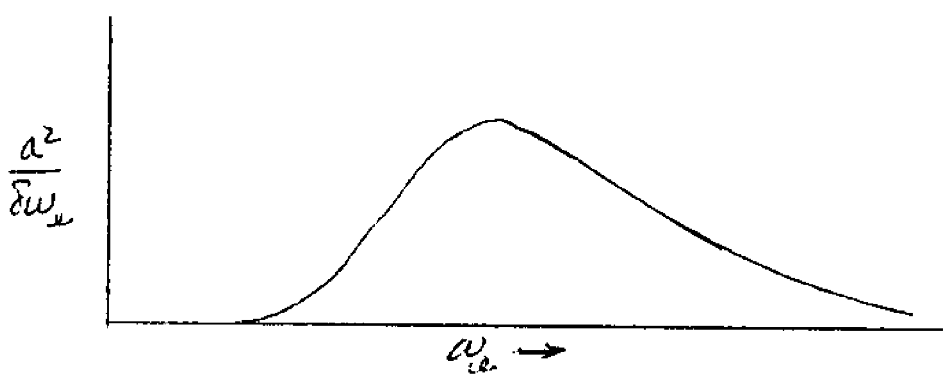
$$\frac{E}{\delta\omega_e} = \left(\frac{E}{\delta\omega}\right) \frac{\delta\omega}{\delta\omega_e}$$

For the gravity wave environment

$$\frac{\delta\omega}{\delta\omega_e} = \frac{1}{1 - \frac{2\omega u_s}{g}\cos\alpha}$$

and the process of converting the spectrum to an ω_e basis consists of multiplying the ordinate at ω by $\delta\omega/\delta\omega_e$ and plotting this product as the ordinate at the value of $\omega_e = (1 - \frac{\omega u_s}{g}\cos\alpha)$. In the case of head seas ($\cos\alpha = -1$), the transformation to ω_e would appear as below:





The area under both curves must be the same.

In following seas ($\cos \alpha = 1$), the term $\delta\omega/\delta\omega_e$ becomes infinite when $\frac{\omega u_s}{g} = 1/2$. Hence, the ordinate plotted at the corresponding ω_e would be infinite. However, since spectral area in a given frequency band width is conserved, the integral over this ordinate is finite. From the expression for ω_e, negative values for ω_e can be obtained for certain values of u_s and α. There are realistic interpretations of what a negative frequency of encounter means but these will not be discussed here.

VEHICLE RESPONSE TO ENVIRONMENTAL EXCITATION

It has been shown that the excitation from the environment can be described by an energy spectrum which essentially indicates the distribution of excitation amplitude over a frequency range. In the equations of motion the $A_i(t)$, at a given excitation frequency, takes the form:

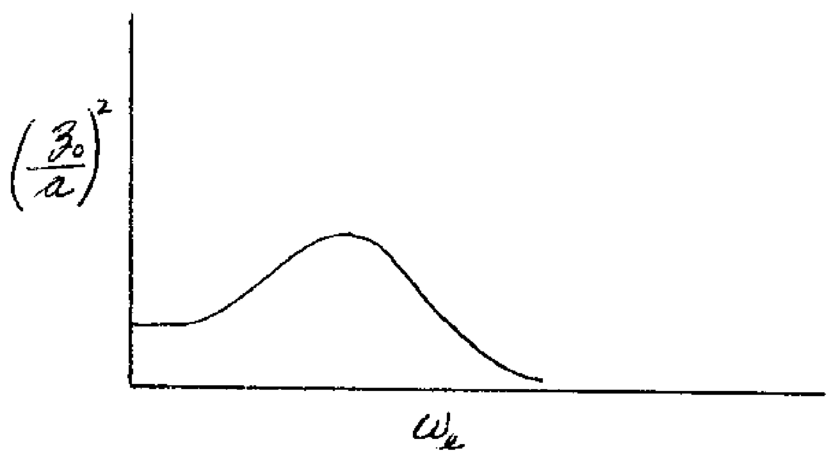
$$A_i(t) = A_{io} e^{-i\varepsilon_i} e^{i\omega_e t}$$

and the solution for the motion response (chosen independent variable) $x_j(t)$ will be in the form:

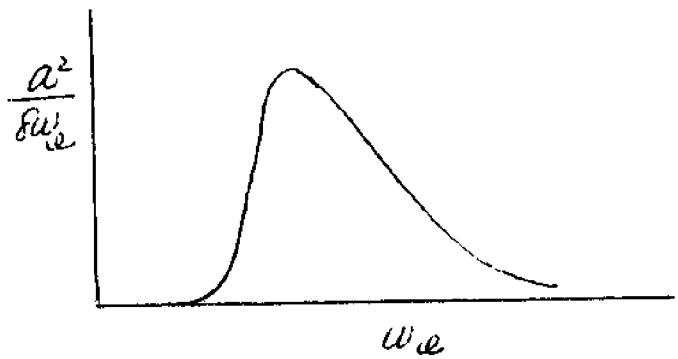
$$x_j(t) = \sum_{k=1}^{m} C_k e^{\sigma_k t} + (x_j)_o e^{-i\varepsilon_j} e^{+ i\omega_e t}$$

This indicates a transient portion and a steady state oscillatory portion of the response. For a stable system the exponential terms go to zero as time goes on (transient) and the steady state response is in terms of an amplitude $(x_j)_o$, and phase lag ε_j, and in the frequency of excitation ω_e. A basic property of linearized equations is that the response amplitude is linearly proportional to the excitation amplitude, hence, one can solve for a response x_j per unit excitation amplitude, or per unit wave amplitude, or per unit turbulence velocity amplitude (since the calculation of the linearized excitation term indicates an excitation force and/or moment proportional to amplitude of the environmental disturbance).

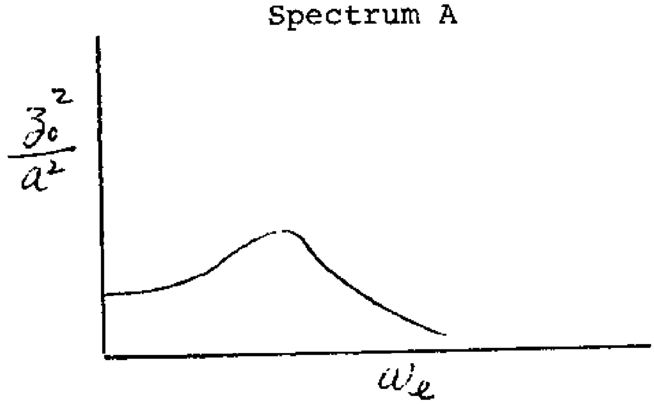
The response x_j per unit amplitude of wave, turbulence velocity, or other pertinent environemntal characteristic can be solved for each frequency of encounter ω_e and a plot of this response or its square vs. ω_e gives a curve of "response operators". For example, if the response x_j is the vertical heave z_o of a surface ship per unit wave amplitude a, the response operator curve would appear as below:



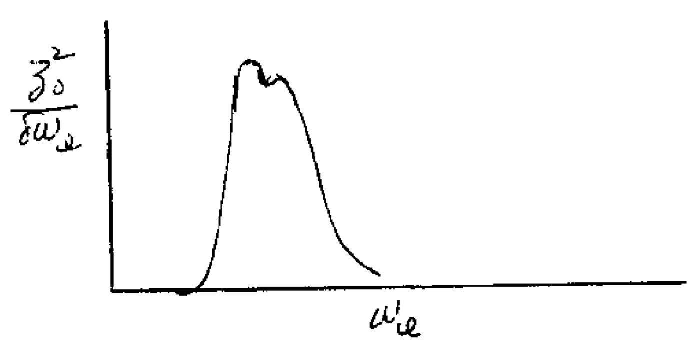
The above curve is analogous to a frequency response characterization of the system. Within the "linearity" assumption of spectral analysis techniques is the assumption of the superposition of linear responses - as was observed when the irregular sea surface could be characterized by a superposition in random phase of a continuous distribution of waves over a frequency (or wave length) range with amplitudes indicated by the characteristic spectrum. In a similar fashion, the irregular motion response of the vehicle to this irregular excitation can also be characterized by a response spectrum. If "linearity" holds then the motion response spectrum resulting from an analysis of the irregular response record should be the same as the "product" of the "excitation" spectrum and the response operator curve. For example, in a seaway characterized by a wave spectrum A (converted to an ω_e reference based on vehicle speed and heading), and a response operator curve B (based on ω_e involving vehicle speed and heading), a response "energy" spectrum C (on a basis of ω_e) is obtained by multiplying the ordinates of A and B at each frequency, as shown below:



Spectrum A



Response Operator B



Response Spectrum C

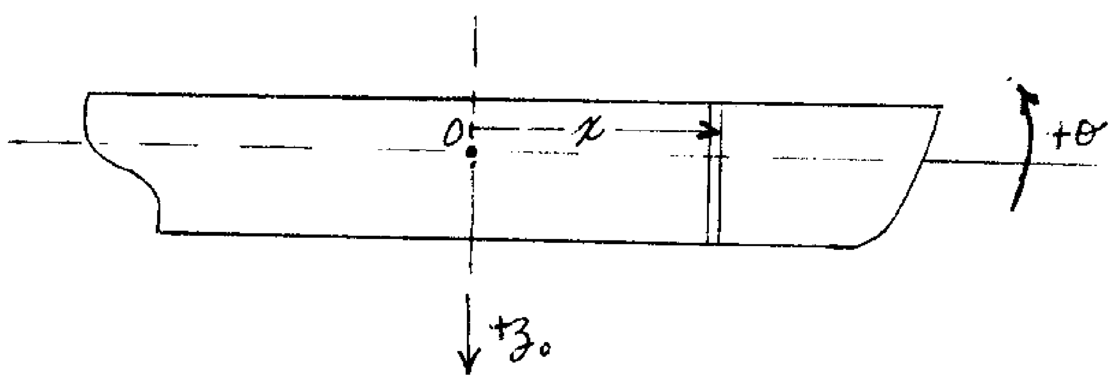
It is clear that

$$\frac{(\text{excitation})^2}{\delta\omega_e} \times \frac{(\text{response})^2}{(\text{excitation})^2} = \frac{(\text{response})^2}{\delta\omega_e}$$

in determining the ordinate of spectrum C. In the particular case above, the heave response z_o has been chosen as the example of a motion response of interest.

There are many motion responses of interest in vehicle design and operational analysis. The responses which limit operations may be different for different vehicles and different missions. Practically all motion responses of concern can be formulated from the solution of the equations of motion, for the chosen independent variables, by combinations of these motion variables.

An example of this would be the vertical motion of any point along a surface ship (for all purposes with length much larger than beam or depth).



The downward displacement (relative to the horizontal surface) of any ship section $(z_o)_x$ located at a distance x from the origin would be:

$$(z_o)_x = (z_o) - x\theta$$

where z_o is the heave of the ship (positive downward) and θ is the pitch angle (positive bow up) in radians. For a given wave length, ship speed, and heading (for which we have obtained a solution of the independent variables z_o and θ)the following holds:

$$(z_o)_x = (z_o)_o e^{i(\omega_e t-\varepsilon_{zo})} - x\theta_o e^{i(\omega_e t-\varepsilon_\theta)}$$

$$(z_o)_x = (z_o)_o \cos(\omega_e t-\varepsilon_{zo}) - x\theta_o \cos\ (\omega_e t-\varepsilon_\theta)$$

The above expression can be reduced by simple trigonometry to:

$$(z_o)_x = (z_{ox})_o \cos\ (\omega_e t-\varepsilon_x)$$

which gives a representation of an ordinate at frequency ω_e for the response operator curve of vertical displacement. Vertical velocity of the section becomes:

$$(\dot{z}_o)_x = -(z_x)_o \omega_e \sin\ (\omega_e t-\varepsilon_x) = (z_{ox})_o \omega_e \cos\ (\omega_e t-\varepsilon_x + \frac{\pi}{2})$$

and the vertical acceleration becomes:

$$(\ddot{z}_o)_x = -(z_x)_o \omega_e^2 \cos\ (\omega_e t-\varepsilon_x) = (z_{ox})_o\ \omega_e^2 \cos(\omega_e t-\varepsilon_x + \pi)$$

Response operator curves for vertical velocity and vertical acceleration can readily be constructed and the corresponding response spectra can be obtained in the manner previously indicated. It should be emphasized here that the phasing of the various responses is an important consideration in developing the amplitudes of the response of interest. Only when the particular response of interest is to be used as response operators in the spectral approach is the phasing considered random and response amplitudes are the primary consideration. A more detailed example of the importance of phasing is in the response of ship motion relative to the water surface (i.e. wave surface and not horizontal surface). In this case, for a given ship section, the vertical motion relative to the surface is:

$$(z_o)_{rel}=(z_o)_o\cos(\omega_e t-\varepsilon_z)-x\theta_o\cos(\omega_e t-\varepsilon_\theta)+\eta_o\cos(\omega_e t-\frac{2\pi x}{\lambda}-\varepsilon_\lambda)$$

where ε_λ is the phasing of the wave relative to the vehicle when t=0. If the wave crest is at the origin (amidships) when t=0, then $\varepsilon_\lambda=0$. The above relative motion response is used in determining keel emergence and water over the bow (when $(z_o)_{rel}$ exceeds the local draft or the freeboard respectively).

Another important response involved in the prediction of "slamming" of ships is the relative velocity of the hull to the water surface at the time of re-entry of the ship section. To obtain $(\dot{z}_o)_{rel}$ one must be careful not to just take the time derivative of $(z_o)_{rel}$ since the amplitude of vertical component of orbital velocity (also velocity of surface elevation) is $\omega\eta$ and not $\omega_e\eta$, whereas the vertical velocity of the ship section is $(\dot{z}_o)_x$. Hence, the relative vertical velocity is given by:

$$-(\dot{z}_o)_{rel}=\omega_e(z_o)_o\sin(\omega_e t-\varepsilon_Z)-\omega_e x\theta_o\sin(\omega_e t-\varepsilon_\theta)+\omega\eta_o\sin(\omega_e t-\frac{2\pi x}{\lambda}-\varepsilon_\lambda)$$

or $$(\dot{z}_o)_{rel}=(\dot{z}_{orel})_o\cos(\omega_e t-\varepsilon_{rel})$$

Other responses of importance to vehicle operation can be expressed in terms of the independent variables solved for in the equations. Some are:

1. Vertical displacement,velocity, and acceleration at the touchdown position on an aircraft carrier in determining the sea states in which planes can operate from the carrier.
2. Vertical displacement and accelerations at the stern of a fishing trawler in determining in what sea states stern trawling fishing is feasible.
3. Relative to water surface motion in determining in what seas hydrofoil or air cushion vehicles would impact the water.
4. Structural bending moment stresses amidships in determining vehicle structural needs.
5. Vertical displacement of propeller to determine the effects of propeller emersion.
6. Accelerations involved in crew, passenger, and cargo safety.
7. The velocity response of the mating hatch of the Deep Submergence Rescue Vehicle from turbulence excitation in determining in what turbulence rescue can be carried out.
8. Etc.

The "quantification" of an irregular response to an irregular excitation is the response spectrum. A more specific and less informative quantity is the "average" value of the response average over the frequency range. For example, the "average" value of wave amplitude from a given spectrum is obtained by integration over frequency

$$\overline{a^2} = \int_0^\infty \frac{a^2}{\delta\omega_e} d\omega_e = \text{area of spectrum}$$

and the root mean square "amplitude" is

$$\text{R.M.S.} = \sqrt{\overline{a^2}} = \sqrt{\text{area}}$$

Similarly, the R.M.S. value of a response is the square root of the area represented by the response spectrum. Hence, the R.M.S. value is a "statistical" average of the irregular or random response.

Other types of "statistical measure" can be obtained from the spectrum if the spectrum is "narrow band" and analysis is made of the "envelope" of the irregular response. Such an analysis gives us statistical averages such as the average value of the $\frac{1}{3}$ highest response, the $\frac{1}{10}$ highest response, or the $\frac{1}{N}$ highest response (as sensed from the envelope of the response). The expression has the form:

$$\overline{\text{Response}}\left(\frac{1}{N}\right) = f(N)\ \sqrt{(\text{area})_e} = f_1(N)\sqrt{\text{area}} = f_1(N)\,(\text{R.M.S.})$$

where $(\text{area})_e$ is the area under the spectrum of the wave response envelope and (area) refers to the area under the wave or response spectrum. The relationship between the two are:

$$\sqrt{(\text{area})_e} = (\text{R.M.S.})_e = \sqrt{2}\ \sqrt{\text{area}} = \sqrt{2}\ \ (\text{R.M.S.})$$

The values of f(N) and f_1(N) as calculated (narrow band spectra) are given below.

N	f(N)	f_1(N)
1	0.886	1.25
3	1.416	2.00
10	1.800	2.55
10^2	2.359	3.34
10^6	3.722	5.27

Since at sea, one observes or senses as wave magnitude not a wave "amplitude", but rather a "wave height", (local trough to crest), the "wave height" magnitude can be obtained by doubling the magnitudes for "amplitude".

The average value of the $\frac{1}{3}$ highest responses (N=3) is often referred to as the significant response and roughly represents what an observer senses as the "average". The above "statistical averages" are an indirect indication of the probability of experiencing a response of given magnitude in a given ocean environment characterized by a spectrum.

The "response operator" curve for a vehicle may be obtained by towing a model of the vehicle in regular waves in a towing tank and measuring the particular responses of interest such as pitch, heave, vertical accelerations, motion relative to water, etc. Since the "physical model" in the towing tank more closely simulates the vehicle than the "mathematical model" represented by the equations of motion, resort to the relatively more expressive model tests is made when doubt exists as to the validity of the mathematical model of the particular vehicle. Tank facilities also exist for testing models in irregular waves representing given spectra and the response spectra is obtained by a spectral analysis (by automatic equipment) of the measured irregular response.

THE HYDRODYNAMIC COEFFICIENTS

The equations of motion include many hydrodynamic derivatives such as N_v, Z_q, K_v, Y_δ, etc. which depend on vehicle shape, size, inertia distribution and the equilibrium condition involved in the analysis. Numerical quantities for these derivatives must be determined for the vehicle under consideration in order to obtain meaningful quantitative information about motion response of the vehicle in the environment.

If reliable theoretical hydrodynamic means are available for calculating the values of the coefficients,then these values are so evaluated. However, in those cases wherein theoretical methods have not been suitably developed, it becomes necessary to obtain the information from special type model tests in a towing tank, water tunnel, wind tunnel, etc., or use the results of a series of model tests which have a systematic shape variation about a parent form. The derivatives involved is the orientation variables x_o, y_o, z_o, ϕ, θ, and ψ (in unrestricted water) are readily calculated by hydrostatic theory and present little problem - such as K_ϕ and M_θ (involving the metacentric height). Similarly, with regard to hydrofoils, circulation and lift theory can be applied to evaluate some of the hydrodynamic and control coefficients. In the case of the vertical motion of a surface vehicle of displacement type,gravity wave hydrodynamic theory gives resonable results for some of the coefficients wherever the forces primarily arrive from the generation of surface waves.

For those vehicle types for which adequate theoretical means are not available, special equipment and techniques have been developed for obtaining the necessary quantitative information by means of model tests. Such facilities and equipment are oscillators, planar motion mechanisms, rotating arm facilities, etc. In Appendix I pages I-34 to 52, some of the coefficients for motion in the horizontal plane are discussed, on pages I-53 to 61 the various techniques of conducting meaningful model tests are described, and on pages I-62 to 66 the use of circulation theory in evaluating some of the coefficients is demonstrated.

One has to keep in mind that the results from model tests are the hydrodynamic coefficients for the model. In order to evaluate the coefficients for the full size vehicle, one must be reasonably satisfied that the scaling laws (hydrodynamic parameters) have been satisfied or if not, that their neglect is of no significant effect. Such items as Reynold's effects (separation), cavitation of the vehicle, control surface, or propeller, Froude effects (free surface), etc., may be of significance in the vehicle coefficients with the possibility that in the model tests, the appropriate hydrodynamic parameters were not satisfied.

For most submerged bodies sufficiently below the free surface, the hydrodynamic coefficients arrive from frictional and eddy drag forces, circulation phenomenon, and buoyancy/ gravity forces. For hydrofoil boats the coefficients are essentially derivable from circulation, surface wave, cavitation, and gravity phenomena. For hovercraft, the significant effects are surface wave, gravity, and aerodynamic. For the normal surface displacement ship in motion in the horizontal plane,frictional, wave making, and eddy drag plus circulation forces are involved. In the case of vertical motion for the surface displacement ship (pitch and heave), hydrostatic-gravitational and surface wave phenemena produce the predominant effects.

Significant research work has been carried out in recent years to develop methods of theoretical calculation for the important hydrodynamic coefficients involved in motion in the vertical plane of surface displacement vehicles. Since the forces arising from free surface wave generation (and hydrostatics) make up practically the total forces involved, reasonably valid calculations for the hydrodynamic coefficients involved in seakeeping motion analysis have been developed. If one were to push down on a floating model and then release it, the vertical motion would die down in about two cycles and water waves would be observed radiating away from the model. One can surmise from this that the damping coefficient Z_w is quite large and primarily results from extraction of energy from the system in the form of radiating surface waves. Hence, a calculation of such a resulting wave radiation would indicate the coefficient Z_w. Similar procedures hold for other important coefficients such as Z_q, M_w, M_q, $Z_{\dot{w}}$, $Z_{\dot{q}}$, $M_{\dot{w}}$, and $M_{\dot{q}}$.

Since vertical motion at the free surface generates waves and since the properties of the waves generated depend on the frequency, the hydrodynamic coefficients become frequency dependent. Hence, terms like Z_w, M_q, $Z_{\dot{w}}$, etc., will be a function of frequency. However, these sort of terms help make up the coefficients a_{ij} in the linearized equations of motion, which coefficients must be considered as independent of time in order to solve for the motion variables in the manner previously demonstrated. The solution of the linearized equations for motion response under a sinusoidal excitation of frequency ω , gives a steady state motion response in the frequency ω , plus transients. This result is demonstrated in the solution to typical problems shown in Appendices A and B. If the motion response is in the same frequency as the excitation, the frequency dependent coefficient takes on the value for that frequency and remains independent of time in solving for the steady state response.

The current state of the art in theoretical calculation of coefficients for vertical motion at the surface and the application to the ship motions problems is pretty well summed up in the paper, "Recent Developments in Seakeeping Research and Its Application to Design", by M. A. Abkowitz, L. A. Vassilopolous, and F. H. Sellars IV, Transactions of the Society of Naval Architects and Marine Engineers, November, 1966. The 142 references given in this paper is rather complete. This reference list is reproduced in Appendix III. Similar information with regard to displacement type vehicles for motion in the horizontal plane is given in "Principles of Naval Architecture - Ship Maneuvering and Control" by P. Mandel, Society of Naval Architects and Marine Engineers, 1967. The bibliography from this chapter is reproduced in Appendix III. Other references in the area are "Theory of Seakeeping", B. V. Korvin Kroukovsky Society of Naval Architects and Marine Engineers, 1961; "Principles of Naval Architecture - The Motion of Ships in Waves" by E. V. Lewis, SNAME, 1967, and "Steering Characteristics of Ships in Calm Water and Waves", by H. Eda and C. L. Crane Jr., Transactions of the SNAME, November 1965.

A complete set of the linear and several non-linear hydrodynamic coefficients (in non-dimensional form) for the Deep Submergence Rescue Vehicle, as measured on a model using a planar motions mechanism, is given in Appendix B in association with a typical problem and solution.

SOME CONSIDERATIONS ON NON-LINEARITIES

Solutions of the linearized equations of motion for the various motion responses are only valid in representing the actual vehicle dynamics if the variables remain reasonably within the range of "linearity". This does not restrict the magnitude of the disturbances, from the equilibrium condition, in a direct quantitative fashion, but rather in a somewhat qualitative fashion. Since the functions of the variables which describe the forces and moments are replaced by linear functions of the variables, the range of linearity (from an applications point of view) is essentially determined by the magnitude of the variable at which the actual curve (of force) significantly departs from the assumed linear curve. Hence, each hydrodynamic derivative (coefficient such as Y_v) has a range in which the linear assumption is acceptable for engineering purposes.

When the linearized equations are solved for motion responses, the range of validity of the solution is not necessarily limited by acceptable range of the least "linear" hydrodynamic coefficient. It may be that for a particular response in a particular maneuver that the contribution of the "least linear" coefficient to the forces involved may be small in comparison to the forces arising from other terms. In this case the range of validity of the solution will be determined essentially by the more important hydrodynamic derivatives. An example of the above situation is maneuvering in the horizontal plane (without roll). In the equations for X,Y, and N, the derivatives which go non-linear the fastest are X_v, X_r, $X_{\dot{v}}$, and $X_{\dot{r}}$, (these are all zero because of port-starboard symmetry as shown previously in Appendix I). These coefficients are important in the X equation which determines the speed loss of the vehicle. If speed loss were the response which is sought for in the solution, then the limit on how tight a turn could be successfully handled would be limited by the linear range of X_v, etc. However, it turns out that the radius of turn resulting from a rudder deflection is dependent on the forward velocity only as the Froude number is affected. Hence, for small changes in speed (wherein Froude effects would be small for a surface vessel), the turning radius calculation by linear theory would remain valid beyond the condition where the X force becomes non-linear. In this particular case, the range of linearity would depend on the linear range of Y_δ, N_δ, Y_v, N_v, Y_r and N_r or on the magnitude of higher order terms such as $N_{\delta rv}$, $N_{r^2\delta}$, etc.

A discussion of such non-linearities and some other important non-linearities, for motion in the horizontal plane,is given in Appendix I pages I-93 to 104. Some of the non-linearities discussed in Appendix I deal with the developing of relatively large angles of attack on the body and/or control surfaces during a maneuver, which angles of attack exceed the range of linear relationship of lift coefficient with angle of attack.

The linearized equations of motion, for the most part, are reasonably valid up to moderately tight turns (in horizontal plane) if the vehicle is dynamically stable and there is a reasonable forward speed. This results from the fact that the significant forces involved depend on "lift" effects which are reasonably linear with angle of attack. In the case of the dynamically unstable ship, the ship cannot exist in the particular equilibrium conditions for a given rudder deflection which are in the unstable range as can be seen from pages I-84 to 87 of Appendix I. Hence, not only is the "range" of linearity small but any condition within the range is unobtainable.

When the equilibrium condition, about which motion or maneuvering takes place, is zero forward speed, such as vertical hovering of a submarine, the lift forces due to circulation disappear and do not contribute to the vertical force Z. The single degree of freedom equation for vertical hovering takes the form

$$(Z_{\dot{w}}-m)\dot{w} + Z_{w|w|}w|w| + Z \text{ (control)} = 0$$

when Z_{z_o} is assumed zero and Z (control) essentially describes a vertical thruster, or a variable ballast system (diving planes are ineffective at zero forward velocity since with $u_o=0$, $Z_\delta = 0$). Here we see that $Z_w=0$ since the Z force is proportional to the square of w, in fact it is equal to the cross-flow drag coefficient multiplied by the $w|w|$ (which is the square of the vertical velocity. The absolute value notation is used to give the function the proper sign when w becomes negative). Even the range of validity of the above equation is limited, since the coefficient $Z_{w|w|}$ is constant beyond a certain Reynold's number because the drag coefficient abruptly changes at the transition from laminar to turbulent flow (or separation).

If the vehicle maneuvers are definitely outside the valid linear range, then resort has to be made to non-linear equations. For predicting tighter maneuvers under significant forward speed, inclusion of important second order or third order terms appears sufficient to model the vehicle dynamics.

On the other hand, for vehicles like the Deep Submergence Rescue Vehicle which experience angles of attack from 0° to 360° in both the vertical and horizontal plane (or both in combination) including backward motion, it appears to be necessary to mathematically fit the force and moment curves over the entire range of variables in order to create a truly valid model. The non-linear equations of motion cannot be solved for motion response in any direct manner as were the linear equations. They can be solved for specific values of the coefficients by electronic computers, usually employing a step by step integration procedure.

The "linear" techniques of calculating vehicle response to the ocean environment contain many types of linearity assumptions and one should recognize those types which are important in determining the validity of the mathematical model. There are "linearity" assumptions of the form.

1. Velocity and accelerations get large so that the forces become non-linear. This can result from large angles of attack or large surface waves generated by vehicle motion.
2. Surface vehicle motions get large so that the hull shape underwater is significantly different from the shape under water in the equilibrium condition. (i.e. hull shape non-linearities).
3. Linearity assumption of superposition in developing the spectral technique.

Motion equations can become non-linear if automatic controls are made sensitive to parameters which take on large values during a maneuver, or if the forces produced are non-linear functions of the control effector deflection or motion. For example, in the case of the Deep Submergence

Rescue Vehicle, there are controls sensitive to heading angle ψ (as projected in horizontal plane) and $\dot{\psi}$ since instrumentation in the form of gyros senses this variable and this variable is important in navigation. Similarly, the effect of roll control tanks depend on vehicle orientation to the gravitational vector. There are other inherent and control forces which depart from linearity when the magnitude of roll, pitch, and heading angles takes on other than small values. In Appendix II , the transformation from axes fixed in the body to axes fixed relative to the horizontal water surface is developed. Page II -4 Appendix II gives the transformation matrix for the axes systems and, on page II -7 of Appendix II , the relationship between the body axes angular velocity components p, q, r and the angular rates of roll, pitch, and yaw (as defined in Appendix II) are given.

TOWING CABLES AND TOWED VEHICLES

The dynamics of a vehicle under tow can be described by adding to the conditions of a free body those restraints, in the form of forces and moments, which the cable or other attachment device places upon the body. Hence, the tension in the cable produces components of force (X,Y,Z) on the body depending on the orientation of the cable relative to the body, and components of moment (K,M,N) depending on the point of attachment of the cable. In a previous chapter, the equations for motion in the horizontal plane for a vehicle under tow were indicated with a more detailed description in pages III-105 to 113 of Appendix III. In this case, the equilibrium condition sets up the tension in the cable and the geometry of the configuration, under a transverse disturbance y_o, sets up the restraining force and moment acting on the body. Hence, in analyzing the motion of towed bodies, one must first establish the configuration of the equilibrium condition before a stability or maneuverability analysis can be carried out.

When submerged bodies are towed from a surface vessel, the configuration of the system is very important because, in addition to putting restraints on the body and affecting its dynamics, the actual geographical position (x_o, y_o, z_o) of the towed body relative to the towing vehicle becomes important for certain operational requirements. There is a great variety of cable-body apparatus used in ocean engineering - moored surface buoys, bottom moored buoys, towed sonar, towed underwater T.V., fish trawling, mine sweeping, general towing, certain oceanographic instrumentation, anchors, etc. Because of this large variety of equipment and because of the need to establish the equilibrium condition before spatial orientation and dynamic problems can be handled, much work has been done in the area of calculating the equilibrium configuration. The methods will be described in rather great detail by including, in this text, two published reports, since there is not much on this subject in the more generally available literature.

The next chapter of this report consists of the full text of the report "Tables for Computing the Equilibrium Configuration of a Flexible Cable in a Uniform Stream" by Leonard Pode, Report number 687, David Taylor Model Basin, March 1951. However, all the tables of the report are not included in the chapter, but only those tables which are necessary for following the various example problems covered in the text of the report.

The tabulated information in the tables is for certain selected values of two parameters of importance in establishing the hydrodynamic characteristics of the particular cable being used. These are the critical angle ϕ_c and the friction factor f as defined below.

$$\cos \phi_c = -\frac{W}{2R} + \sqrt{\left(\frac{W}{2R}\right)^2 + 1} \text{ when } W > 0$$

$$\cos \phi_c = -\frac{W}{2R} - \sqrt{\left(\frac{W}{2R}\right)^2 + 1} \text{ when } W < 0$$

where W is weight in water (i.e., weight minus buoyancy) of the cable per unit length of cable and R is the cross flow drag of the cable per unit length of cable.

$$f = \frac{F}{R}$$

where F is the tangential hydrodynamic force component (along cable axis) on the cable per unit cable length. In the report (DTMB 687), the values of ϕ_c and f which are tabulated are:

$$f = 0.01,\ 0.02,\ 0.03$$
$$\phi_c = 0^\circ \text{ to } 85^\circ \text{ in } 5^\circ \text{ increments}$$

In order to extend the usefulness of the tables to the cases of somewhat more faired cable than the circular cable (larger values of f) and to improve the calculation for the lighter weight cables and/or higher speeds (finer mesh in ϕ_c near $\phi_c = 0$), supplementary tables were published in 1955 in the report "Cable Function Tables for Small Critical Angles", Leonard Pode and Louis Rosenthal, Supplement to Report 687, David Taylor Model Basin, September, 1955. The report covers the range

$$f = 0.01,\ 0.02,\ 0.03,\ 0.10$$
$$\phi_c = 0^\circ \text{ to } 10^\circ \text{ in } 1^\circ \text{ increments}$$

This report is not included in the text.

In the two reports mentioned above, there is an assumption that the force perpendicular to the cable is $R \sin^2\phi$ (where ϕ is the angle of the cable relative to the flow) and a restriction on how f varies with ϕ . These assumptions are acceptable for rather heavy cables of circular cross section. At higher speeds and/or with great lengths of towing cable under water, the drag of circular cables becomes excessive (increased power requirement) and a large depression force is necessary at the bottom of the cable to keep it at depth (both increasing the tension in the cable). Hence, it becomes necessary to abandon the assumptions of a loading function of $R \sin^2\phi$ and also to take into account the actual variation of F (or f) with the cable angle, ϕ .

In order to define the loading and F functions for a specific cable shape, water tunnel, wind tunnel, and towing tank tests have been carried out. Once these functions have been established, a rather simple computer program can be formulated to calculate the shape of the equilibrium configuration and the cable tension. The report "Generalized Hydrodynamic Loading Functions for Bare and Faired Cables in Two-Dimensional Steady-State Cable Configurations", George S. Springston Jr., Naval Ship Research and Development Center, Report number 2424, June 1967, fairly well covers the various loading functions that have been proposed. Because of this valuable information and for the extensive bibliography on the subject which is listed, this report has been included as a chapter in the text.

Once the equilibrium configuration of the cable-vehicle is established, methods for attacking the solution of stability and motion response of the configuration (or specific parts of the system) can be formulated. Take for example the problem of predicting the motion of a submerged vehicle (say housing an oceanographic instrument) attached to the end of a long cable, when the other end of the cable is fixed (at a winch) at the stern of a surface vessel proceeding at some speed in a seaway. The problem (in linear form) can be essentially reduced to the case of the response of the system to a random vertical motion disturbance of the cable (at the surface end) from its equilibrium position. This in turn transmits an irregular time varying tension down the cable to produce excitation forces on the vehicle through the "restraint" forces of the cable. One of the ways to handle a problem of this sort is a "quasi-static" approach in the manner described below.

The "quasi-static" approach in this particular case means that forces in the cable arising from "positional" variables are predominant and that the dynamic forces of damping and inertia of the cable are not significant in comparison for determining the amplitude of the response at the cable end. Whatever phase shift is caused by dynamic effects of the cable is not relevent to the problem since the response will be in spectral form (as is the excitation) and therefore is considered a random phase process. Hence, with the "static" assumption, we are essentially dealing with the "spring constant" of the cable. The cable excitation - restraint force transmission to the vehicle is then modeled as a "spring" connecting the surface vessel with the submerged vehicle, with the end of the spring at the surface forced in a vertical motion characterized by the spectrum which describes the vertical motion of the stern (assuming that the spring has negligible mass and internal damping). The stretching of the spring imposes forces at both ends of the cable - i.e., produces the excitation forces and moments on the submerged vehicle and also produces forces on the towing vessel.

Usually the towing vehicle is a surface vessel of reasonable size, with the towed vehicle comparably small. For surface ships the "spring constant" in heave and pitch are quite large because of hydrostatic considerations, whereas the small submerged vehicle has zero "spring constant" in heave and quite small "spring constant" in pitch (small metacentric height,i.e. vertical distance between center of gravity and center of buoyancy.) Hence, the effect of the submerged vehicle and of cable motion on the ship motion becomes negligible, and the stern response of the surface vessel to the seaway, as a body unrestrained by cable, can be used as the input to the cable excitation at the water surface. On the other hand, if the surface vessel is a small fishing trawler and the towed vehicle is a deep fishing trawl (at stern), then the reaction forces at the cable resulting from cable and trawl motion can significantly effect the response of the surface vessel to the seaway excitation.

In this particular case, as in any case where interaction forces on motions are significant to both vehicles, one has to resort to solving the "two body" problem. The linearized model of the system results in linearized differential equations involving, as variables or unknowns, the motion parameters of each body and the cable. As one might expect, the order of the differential operator D,in the characteristic determinant of the coefficients, will substantially increase to double or more the number associated with a single free body motion analysis. Hence, one expects many roots (equal to the exponential of D) of the determinant, the values of which determine the stability of the total "system" and the motion response of the "system".

The above description of an approach to a motion analysis of an ocean vehicle system was presented not as a complete tackling of the problem, but rather as a demonstration that the mathematical and analytical tools developed can be usefully applied to complicated practical ocean environment situations.

There are some cable dynamics problems, (such as possible flutter or instability of towed faired cables) where it becomes necessary to describe hydrodynamic forces with body axis reference, produced by currents (or flow) in geographical (or fixed axis system). The two reference axes depart from each other by the cable angle ϕ . Since ϕ for many configurations goes beyond the linear range, it becomes necessary to use transformation matrix given in Appendix II . A problem covered in Appendix B, is a simple example of such use of the matrix.

THE DAVID W. TAYLOR MODEL BASIN
WASHINGTON 7, D.C.

TABLES FOR COMPUTING THE EQUILIBRIUM CONFIGURATION OF A FLEXIBLE CABLE IN A UNIFORM STREAM

by

Leonard Pode

March 1951

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NOTATION

x, y	The rectangular coordinates of an arbitrarily chosen point on the cable; see Figure 1 on page 4.
s	The distance along the cable measured positively in the sense of positive progression along the cable; see Figure 1.
ϕ	The angle from the direction of motion to the direction of the tangent to the cable at an arbitrarily chosen point on the cable, the direction of the tangent being taken in the sense of increasing s; see Figure 1.
ϕ'	The difference $\pi - \phi$
ϕ''	The difference $\phi - \pi$
ϕ_0	The value of ϕ at the point chosen as the origin of the coordinate system
ϕ_c	The critical angle of the cable, i.e , the value of the angle ϕ obtained when the cable is freely trailed in the stream
F	The drag per unit length of the cable when the cable is parallel to the stream
R	The drag per unit length of the cable when the cable is normal to the stream
R'	The form drag per unit length of the cable when the cable is normal to the stream; $R' = R - F$
T	The tension in the cable at an arbitrarily chosen point
T_0	The tension in the cable at the point chosen as origin of the coordinate system
W	The weight in water per unit length of the cable
τ	The nondimensional tension, T/T_0
ξ, η	The nondimensional rectangular coordinates; $\xi = \frac{Rx}{T_0}$; $\eta = \frac{Ry}{T_0}$
σ	The nondimensional length of cable, Rs/T_0
f	The ratio F/R
f'	The ratio F/R'
w	The ratio W/R
P	The component of the external forces acting upon an element of cable in the direction of the element
p	The ratio P/R
Q	The component of the external forces, acting upon an element of cable, that is in the direction 90° counterclockwise from the direction of the element
q	The ratio Q/R

TABLES FOR COMPUTING THE EQUILIBRIUM CONFIGURATION OF A FLEXIBLE CABLE IN A UNIFORM STREAM

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ABSTRACT

The general problem of the equilibrium configuration of a flexible cable immersed in a uniform steady stream is treated analytically. It is shown that, when the configuration of the cable lies entirely in a plane, the solution of the differential equations that describe the configuration can be expressed in terms of certain functions which are called the cable functions and are expressed in terms of quadratures. The specific functions that apply to the most general types of configurations assumed by round cables, when neither the weight of the cable nor the tangential drag of the cable can be neglected, are derived and tabulated. The tabulated values of these functions greatly facilitate the determination of the shape and tension of towing or anchoring cables for a large variety of practical problems both in air and water.

INTRODUCTION

The purpose of these tables is to facilitate the determination of the configuration and tensions of a flexible cable moving in a fluid when neither the frictional drag nor the weight of the cable can be neglected. The first part of this report presents a general discussion of cable configurations. This is followed by the derivation of the specific functions which have been tabulated. The appendices of the report describe the numerical methods used in constructing the tables.

The cable functions describe equilibrium configurations assumed by a flexible cable in a parallel, uniform, steady stream when constant forces are applied to the ends of the cable and the entire cable lies in a plane. Such configurations have been studied in previous papers, References 1 through 14.* The problem is treated here in greater generality, both in regard to the forces involved and the types of configurations considered.

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*References are listed on page 30.

BASIC CONSIDERATIONS

The forces that act on an element of cable are threefold in origin:

1. The hydrodynamic force that arises from the flow.
2. The weight of the element of cable in water.
3. The tensions in the cable at the ends of the element.

The component of the external force (the resultant of the hydrodynamic and gravitational forces) that is tangent to the element acts to increase the tension in the cable. Since the cable is flexible the element bends in a manner that results in the balancing of the normal component of the external forces. The shape of the cable configuration and the tensions in the cable may be determined if the external force acting at each element and the tension and direction of the cable at one reference point are known.

The basic assumption in analyzing the configuration of the cable is that the hydrodynamic force that acts on an element of the cable depends only on the angle that the element makes with the stream and is not affected by such matters as the curvature of the cable or the flow at neighboring elements. In other words the specific hydrodynamic force that acts on an infinitely long cylinder is applicable to a small element of cable of the same size and shape and inclined at the same angle to the stream. From this basic assumption immediately follow two important characteristics of the solution of the cable problem. First, as a consequence of this assumption it follows that any section of a known cable configuration is also the solution of a cable problem. Second, consideration of dimensionality also multiplies the information inherent in a single solution. For example, let the dimensions of a known configuration be altered by some scale factor. Then, in most cases, it may be assumed that the hydrodynamic force acting on any element is simply multiplied by the square of this factor and, if the weight of the cable in water and the forces at the ends of the cable are altered in the same manner, the equilibrium of forces is not disturbed. Therefore the shape of the cable is affected only by multiplication by a scale factor.

By finding a cable configuration of most general shape, i.e., covering the widest range of the angle of the cable to the stream and reducing the solution to a nondimensional form, a solution can be obtained which will be applicable to all problems involving the same nondimensional parameters.

RESTRICTION TO PLANE CONFIGURATIONS

All comments up to this point apply when the shape of the cable is either a skew or a plane curve. Also, from the basic assumption that the hydrodynamic force acting upon an element of cable depends only upon the angle between the element of the cable and the stream, it can be demonstrated for both types of configurations that the problem of determining the shape and tensions of the cable can always be reduced to quadratures whatever the law relating the hydrodynamic force to the angle of the cable may be. However, for the present, consideration will be given only to the case of the plane curve. Therefore the restrictions that must be imposed in order to insure that the entire cable will lie in a plane will be discussed.

Because the cable is required to bend in a plane, the external force acting upon any element of cable must lie in the plane of the cable. Conversely, if the external forces on each element of cable and the forces applied to the ends of the cable lie in a plane, the entire cable will lie in the plane of the forces. However, when the hydrodynamic force has a component that is normal to both the direction of motion and the direction of the element of cable, the entire cable will lie in a plane only in unusual cases. Therefore, for the present analysis, it will be required that the hydrodynamic force act in the plane including the direction of motion and the direction of the element of cable. Whenever the cable presents a symmetrical profile to the flow this requirement is fulfilled. Thus a smooth round cable fulfills the requirement but a stranded cable does so only approximately. Fulfillment of this requirement is sufficient to ascertain that the cable will lie entirely in a plane when the weight of the cable is negligible. The plane of the cable will be the plane that includes the direction of motion and the direction of the force applied to one end of the cable. (The force applied at the opposite end of the cable must also lie in this plane in order to obtain an equilibrium configuration.) When the weight of the cable is not negligible the cable must lie in the plane including the direction of gravity and the direction of motion, and the forces applied to the end of the cable must also lie in this plane.

GENERAL INTEGRATION OF THE DIFFERENTIAL EQUATIONS

THE DIFFERENTIAL EQUATIONS

If both the direction of gravity and the law relating the hydrodynamic force to the angle between an element of cable and the stream are specified, the external force acting upon an element of cable is a known function of this angle. Then the components of the force parallel to the element of the cable and normal to the element of the cable may both be written as explicit functions of this angle.

Choose a sense of progression along the cable and let ϕ be the angle measured counterclockwise from the direction of motion to the direction of an element of the cable of length, ds. Let $P(\phi)ds$ and $Q(\phi)ds$ be the tangential and normal components of the external force respectively (where $P(\phi)$ is measured positive in the direction of the element of cable which is taken in the sense of increasing length of cable, s, in accordance with the chosen sense of progression, and $Q(\phi)$ is measured positive in the direction of the positive normal which is taken in the direction 90° counterclockwise from the direction of the element of the cable). Then the equilibrium of the cable element requires

$$dT = -P(\phi)ds \qquad [1]$$

$$Td\phi = -Q(\phi)ds \qquad [2]$$

where T is the tension in the cable and dT and $d\phi$ are the changes in the values of T and ϕ over the length of the element; see Figure 1. Since the forces that act on an element of the cable cannot be affected by the choice of the

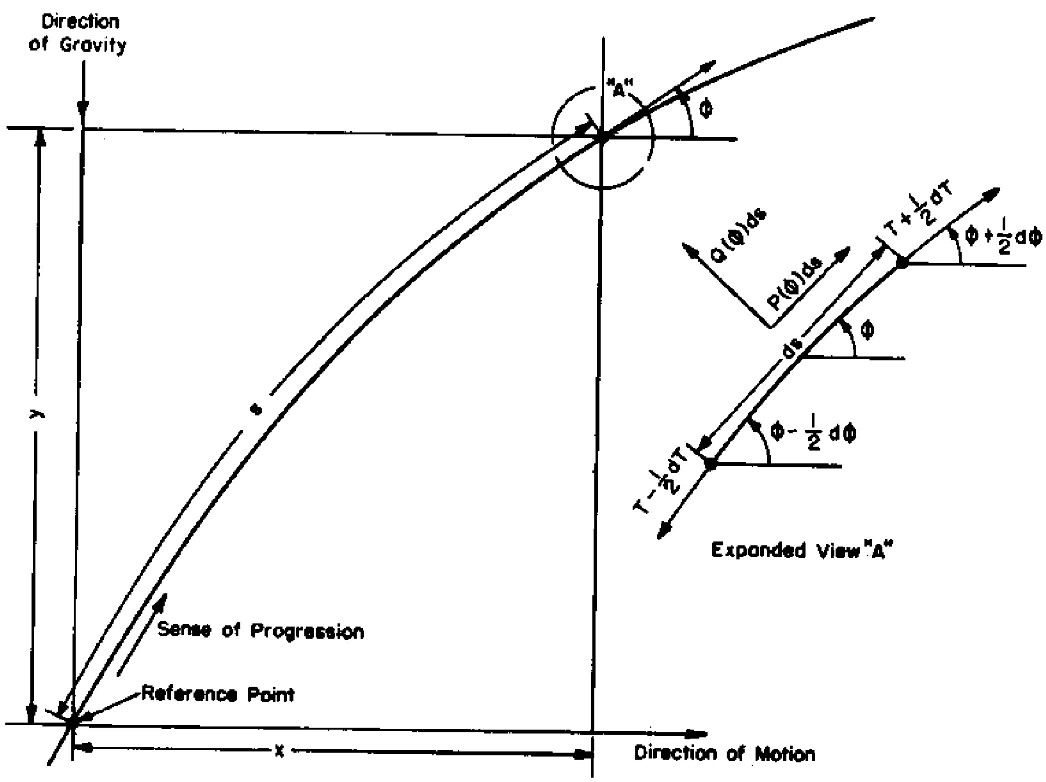

Figure 1 - Coordinate System

sense of progression along the cable the functions $P(\phi)$ and $Q(\phi)$ must satisfy the relations: $P(\phi) = -P(\phi + \pi)$, $Q(\phi) = -Q(\phi + \pi)$.

THE CRITICAL ANGLE

Special interest attaches to the values of the angle $\phi = \phi_c$ which are roots of the equation $Q(\phi_c) = 0$. When the cable is towed by itself, i.e., the cable is simply trailed without a towed body at the end of the cable, the configuration of the cable could be any straight line inclined to the stream at such an angle $\phi = \phi_c$. In order that there be a completely unique solution for this condition it is required that the equation $Q(\phi_c) = 0$ have no more than one root in the range $0 \leq \phi \leq \pi$. Unless this root is of at least order one, Equation [2] would be integrable through all values of ϕ and the most general shape of the cable then would be a spiral of an unlimited number of turns. Such a configuration is in disagreement with observation and would be incongruous in the situation being considered. It is therefore assumed that in general the equation $Q(\phi_c) = 0$ will have only one root and this root will be of order one or greater. This angle will be called the critical angle.

On the basis of the above assumptions regarding the critical angle the following statements can be readily verified:

1. If the angle of the cable is equal to the critical angle anywhere, the angle of the cable is everywhere equal to the critical angle.

2. If the angle of the cable is anywhere different from the critical angle, the angle of the cable is nowhere equal to the critical angle for any finite length of cable.

3. By a suitable choice of the positive sense of progression along the cable the maximum range of the angle of the cable may be restricted to $\phi_c < \phi < \phi_c + \pi$ and in this range the value of Q is always of the same sign so that the curvature of the cable, $d\phi/ds$, is everywhere of the same sign.

4. The angle of the cable approaches the critical angle as the length of the cable is indefinitely increased.

5. When the cable is towed by itself the configuration of the cable is a straight line inclined to the stream at the critical angle.

GENERAL INTEGRATION FROM AN ARBITRARY REFERENCE POINT

The general integration of Equations [1] and [2] may now proceed. Eliminating ds from Equations [1] and [2]

$$\frac{dT}{T} = \frac{P(\phi)}{Q(\phi)}\, d\phi \qquad [3]$$

Now assume that at some point, P_0, on the cable, the tension in the cable T_0 and the angle from the direction of motion ϕ_0 are known. Equation [3] may be integrated from this reference point P_0 along the cable to any arbitrary point P on the cable where the tension is T and the angle is ϕ; thus

$$\frac{T}{T_0} = e^{\int_{\phi_0}^{\phi} \frac{P(\phi)}{Q(\phi)} d\phi} \qquad [4]$$

Using this result in Equation [2]

$$ds = \frac{T_0}{-Q(\phi)} e^{\int_{\phi_0}^{\phi} \frac{P(\phi)}{Q(\phi)} d\phi} \qquad [5]$$

so that the distance along the cable from P_0 to P is given by

$$s = \int_{\phi_0}^{\phi} \frac{T_0}{-Q(\phi)} e^{\int_{\phi_0}^{\phi} \frac{P(\phi)}{Q(\phi)} d\phi} d\phi \qquad [6]$$

The location of the point P in relation to the point P_0 may be found in terms of coordinates x and y, representing a displacement parallel to the direction of motion and displacement perpendicular to the direction of motion respectively. From the geometry, $dx = (\cos\phi)ds$ and $dy = (\sin\phi)ds$; hence

$$x = \int_{\phi_0}^{\phi} \frac{T_0}{-Q(\phi)} e^{\int_{\phi_0}^{\phi} \frac{P(\phi)}{Q(\phi)} d\phi} \cos\phi \, d\phi \qquad [7]$$

$$y = \int_{\phi_0}^{\phi} \frac{T_0}{-Q(\phi)} e^{\int_{\phi_0}^{\phi} \frac{P(\phi)}{Q(\phi)} d\phi} \sin\phi \, d\phi \qquad [8]$$

The question of expressing these results nondimensionally now arises. The tension T is already in nondimensional form in terms of the known tension T_0. For the distances s, x and y, a characteristic unit of length is needed. In general the most convenient unit of length is that length of cable which when entirely normal to the stream has a drag equal to the tension T_0, i.e., T_0/R where R is the drag per unit length when the cable is normal to the stream. Dividing the distances s, x and y by this length the nondimensional values $\sigma = Rs/T_0$; $\xi = Rx/T_0$; $\eta = Ry/T_0$ are obtained. Then letting $p = P(\phi)/R$; $q = Q(\phi)/R$ and using equations [4], [6], [7], and [8], the solution of the cable problem may be written

$$\tau = \frac{T}{T_0} = e^{\int_{\phi_0}^{\phi} \frac{p}{q} d\phi} \quad [9a]$$

$$\sigma = \frac{Rs}{T_0} = \int_{\phi_0}^{\phi} \frac{\tau}{-q} \, d\phi \quad [9b]$$

$$\xi = \frac{Rx}{T_0} = \int_{\phi_0}^{\phi} \frac{\tau \cos \phi}{-q} \, d\phi \quad [9c]$$

$$\eta = \frac{Ry}{T_0} = \int_{\phi_0}^{\phi} \frac{\tau \sin \phi}{-q} \, d\phi \quad [9d]$$

where only nondimensional values are involved and all functions are defined by quadratures.

SHIFTING OF REFERENCE POINT

If it is desired to change the reference point from P_0 to some other point P_1 on the cable where the tension in the cable is T_1 and the angle of the cable is ϕ_1, it is only necessary to replace the 0 subscript in the above equations with the subscript 1 and interpret s_1, x_1, and y_1 as distance along the cable and displacements from the point P_1. The new functions τ', σ', ξ', η' obtained with P_1 as reference point are related to the functions τ, σ, ξ, η obtained with P_0 as reference point by the equations.

$$\tau' = \frac{T}{T_1} = \frac{\tau}{\tau_1} \quad [10a]$$

$$\sigma' = \frac{Rs_1}{T_1} = \frac{\sigma - \sigma_1}{\tau_1} \quad [10b]$$

$$\xi' = \frac{Rx_1}{T_1} = \frac{\xi - \xi_1}{\tau_1} \quad [10c]$$

$$\eta' = \frac{Ry_1}{T_1} = \frac{\eta - \eta_1}{\tau_1} \quad [10d]$$

where τ_1, σ_1, ξ_1, η_1 are respectively the values of the functions τ, σ, ξ, η, for $\phi = \phi_1$. By these equations the shape and tensions in the cable can be determined if the location, tension and angle are known at any point and a table of the functions τ, σ, ξ, η, based on any reference point is available. This statement is true even if the reference point P_0 for the table is hypothetical and does not actually exist in the particular configuration to which the use of the tables is applied.

The set of functions τ, σ, ξ, η, defined by Equations [9a,b,c,d] is referred to as the cable functions. Consideration will now be given to the particular forms assumed by these functions when specific assumptions are made regarding the forces that act upon the cable.

SPECIFIC SOLUTIONS FOR THE CABLE FUNCTIONS

SOLUTIONS NEGLECTING GRAVITY AND THE TANGENTIAL COMPONENT

The simplest situation arises when both the tangential component of the hydrodynamic force and the gravity forces are negligible. The gravity forces may be neglected either when the cable is in fact neutrally buoyant or when the speed of the stream is such that the gravity forces are insignificant in comparison to the hydrodynamic forces. Except when the cable is inclined at very small angles to the stream the tangential component of the hydrodynamic force acting on round or stranded cables is found to be considerably smaller than the normal component. Therefore, when the cable is so short that the change in tension over the length of the cable is small in relation to the forces at the ends of the cable, and when the inclination of the cable to the stream is reasonably large over most of the length of the cable, it is permissible to neglect the tangential component of the hydrodynamic force.

That the normal component of the hydrodynamic force varies with the square of the sine of the angle between the cable and the stream is well established by experimental evidence and supported by theoretical considerations.[2,8,12] Neglecting the gravity forces and the tangential component of the hydrodynamic force and using this variation of the normal component, the forces acting upon an element of cable are as represented in Figure 2a. The tangential component of the force is always zero, i.e., $P(\phi) = 0$ and when the positive sense of progression along the cable is taken in the clockwise direction, the normal component is given by $Q(\phi) = +R \sin\phi\,|\sin\phi|$ where the sign has been arranged to take into account the fact that the normal component will never have a positive projection in the direction of the motion. The critical angle is obviously zero so that the range of the angle ϕ may be taken as $0 < \phi < \pi$. Hence $\sin\phi$ will always be positive and the normal component may be represented by $Q(\phi) = +R \sin^2\phi$. If the point where the cable is normal to the stream is chosen as the reference point and coordinates are chosen as indicated in Figure 1 the cable functions become:

$$\tau = 1 \qquad [11a]$$

$$\sigma = \cot \phi \qquad [11b]$$

$$\xi = \csc \phi - 1 \qquad [11c]$$

$$\eta = \ln \cot \frac{\phi}{2} \qquad [11d]$$

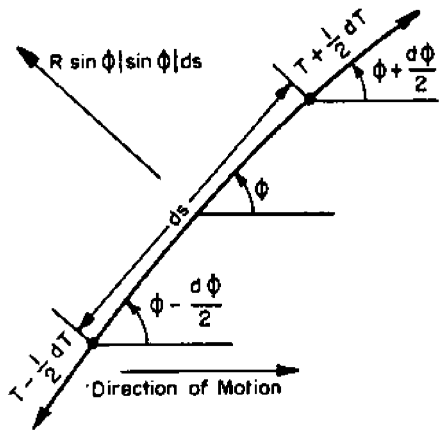

Figure 2a

Assuming sine-squared law for the normal component of the hydrodynamic force and neglecting the tangential component and the weight of the cable.

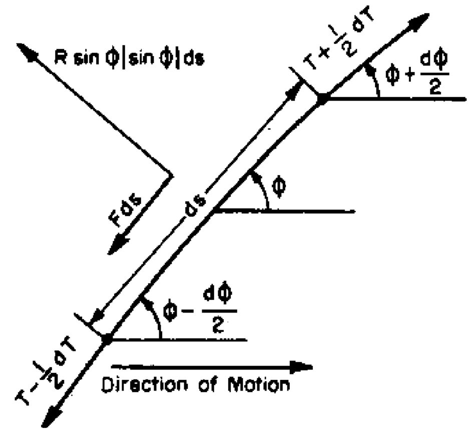

Figure 2b

Assuming a sine-squared law for the normal component of the hydrodynamic force and a constant tangential component and neglecting the weight of the cable.

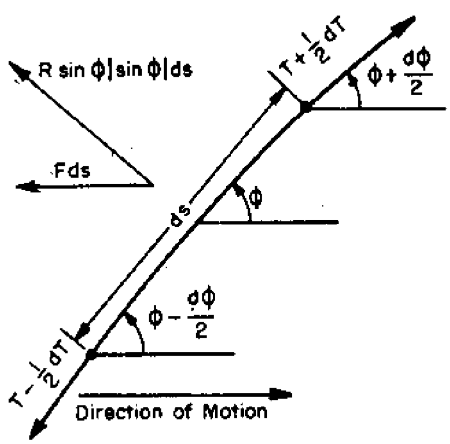

Figure 2c

Assuming a constant frictional drag in the direction of the stream in addition to a sine-squared law for the form drag normal to the cable and neglecting the weight of the cable.

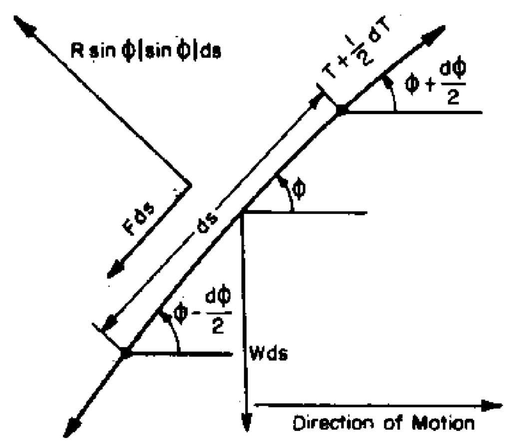

Figure 2d

Assuming a sine-squared law for the normal component of the hydrodynamic force and a constant tangential component and not neglecting the weight of the cable.

Figure 2 - Forces Acting on an Element of Cable

The shape of the cable may be identified as that of a catenary. Eliminating the parameter ϕ between η and ξ one obtains $\xi = \cosh \eta - 1$. It is apparent that the general shape of the cable is symmetrical about a line parallel to the direction of the stream and that the tension is constant throughout the cable. It is also noteworthy that R acts only as a scale factor and does not enter directly in these functions. Therefore, the functions do not change when the speed of the stream varies.

SOLUTION NEGLECTING ONLY GRAVITY

The most serious limitation to the solution given in Equations [11a, b,c,d] is that in many applications the cable will be too long to permit the neglect of the effect of the tangential component of the hydrodynamic force in producing an increase in tension over the length of the cable. Few data are to be had regarding the relation of the tangential component of the hydrodynamic force to the angle between the cable and the stream. This component has been alternately assumed as constant and as varying with the cosine of the angle. When it is assumed that the tangential component is constant (see Figure 2b), all the cable functions are not integrable but ξ and η must be evaluated by numerical integration. This calculation is given by Landweber and Protter[4] for the case when the ratio F/R of the tangential component per unit length to the drag of the cable per unit length when normal to the stream, has the value 0.022. There is, however, a modification, due to Reber[1] of the assumptions regarding the hydrodynamic force under which all the cable functions are explicitly integrable in terms of tabulated functions. Here the hydrodynamic force that acts upon an element of cable is described as consisting of two parts, namely:

1. A profile drag, $R' \sin \phi |\sin \phi|$, per unit length of cable that acts normal to the cable and varies as the square of the sine of the angle that the element makes with the stream.

2. A frictional drag that acts in line with the stream and has a magnitude F per unit length of cable that is independent of the angle that the element makes with the stream.

The forces acting upon an element of cable are then as represented in Figure 2c. Choosing the clockwise sense as the positive sense of progression along the cable $P(\phi) = -F \cos \phi$ and $Q(\phi) = +R' \sin \phi |\sin \phi| + F \sin \phi$. Again the critical angle is zero and the range of ϕ may be restricted to $0 < \phi < \pi$ so that $\sin \phi$ is always positive and $Q(\phi)$ may be written $Q(\phi) = +R' \sin^2 \phi + F \sin \phi$. It is apparent that when F is small and ϕ is large enough so that $\sin \phi$ is not very much less than one, the normal component of the hydrodynamic force has been only slightly changed and the profile drag per unit length of the cable when the cable is normal to the stream, which is given by $R = R' + F$, differs very little from R'.

If the point at which the cable is normal to the stream is chosen as reference point, and coordinates chosen as indicated in Figure 1, then the cable functions integrate to

$$\tau = \frac{1 + f'\csc\phi}{1 + f'} \qquad [12a]$$

$$\sigma = \cot\phi \qquad [12b]$$

$$\xi = \csc\phi - 1 \qquad [12c]$$

$$\eta = \ln\cot\frac{\phi}{2} \qquad [12d]$$

where $f' = F/R'$. Also, by eliminating ϕ between Equation [12a] and Equation [12c] or by direct integration of Equation [1] the additional relationship is obtained

$$\tau = 1 + f\,\xi \qquad [12e]$$

where $f = F/R$. In dimensional form this equation may be written

$$T = T_0 + Fx \qquad [12f]$$

It is seen that the shape of the cable is still that of a catenary, the functions σ, ξ, and η having been unaffected. The only function that has been changed is τ which is also the only function that explicitly involves the parameters F and R'. Because, in general, the ratio $f' = F/R'$ will not change with the speed of the stream, all of the cable functions are again independent of the speed. The cable configuration is again symmetrical about a line parallel to the direction of motion.

SOLUTION NEGLECTING NEITHER GRAVITY NOR THE TANGENTIAL COMPONENT

The present analysis of the cable configuration applies regardless of the density of the fluid in which the cable is moving. When the cable is moving in air the situation might readily arise where the tangential component of the aerodynamic force is negligible but the weight of the cable is not. A treatment of this case for some types of cable configurations is given by Glauert.[2] When the cable is moving at sufficiently low speed in water such a relation of forces may also be obtained but in water this case is less frequent. Furthermore, if the weight of the cable is not neglected, the cable functions (with the exception of τ) are not integrable in terms of tabulated functions, whether the tangential component is or is not neglected, and numerical integrations are necessary. Therefore, whenever the effect of gravitational forces must be considered it is just as well to include in addition the effect of the tangential component of the hydrodynamic force.

It is clear that the gravity forces cannot be ignored when the speed of the stream is sufficiently low so that these forces are not small in comparison to the hydrodynamic force, but even at higher speeds the effect of the weight of the cable may have an important influence upon the shape assumed by the cable. It has been demonstrated that when the weight of the cable is ignored the critical angle is zero. Presently it will be shown that the critical angle is a function of the ratio W/R where W is the weight per unit length of the cable in water and R is, as before, the drag per unit length of the cable when the cable is normal to the stream. When the length of cable is such that a large part of the cable is at an angle close to the critical angle even a relatively small error in the critical angle may introduce a serious error in calculating the depth of the cable. Moreover, certain types of configurations can be realized only by considering the effect of the weight of the cable. For example, the sag in a cable used to tow a float from a surface vessel cannot be found when the effect of the weight of the cable is ignored. Thus situations also arise where neither the inertial forces nor the tangential component of the hydrodynamic force can be ignored.

If, for simplicity, the magnitude of the tangential component of the hydrodynamic force per unit length F is assumed to be constant and the direction of motion is perpendicular to the direction of gravity the forces to be considered acting on an element of cable are as represented in Figure 2d. Again taking progression along the cable as positive in the clockwise sense,

$$P(\phi) = -F\frac{\cos\phi}{|\cos\phi|} - W\sin\phi \qquad [13]$$

$$Q(\phi) = +R\sin\phi\,|\sin\phi| - W\cos\phi \qquad [14]$$

where W is the weight in water per unit length of the cable. The sign of $F\cos\phi/|\cos\phi|$ is proper in order to take into account the fact that the tangential component as well as the normal component of the hydrodynamic force never has a positive projection in the direction of motion.

The critical angle may be assumed to lie in the range $0 \leq \phi_c < \pi$ so that the equation

$$R\sin^2\phi_c - W\cos\phi_c = 0 \qquad [15a]$$

is satisfied. Substituting $\sin^2 \phi_c = 1 - \cos^2\phi_c$ and dividing by R

$$\cos^2\phi_c + \frac{W}{R}\cos \phi_c - 1 = 0 \qquad [15b]$$

Hence

$$\cos \phi_c = -\frac{W}{2R} + \sqrt{\left(\frac{W}{2R}\right)^2 + 1} \qquad [15c]$$

when W is positive as in the case of the negatively buoyant cable and

$$\cos \phi_c = -\frac{W}{2R} - \sqrt{\left(\frac{W}{2R}\right)^2 + 1} \qquad [15d]$$

when W is negative as in the rare case of a positively buoyant cable. The sign of W can be reversed simply by reversing the sign of the direction of gravity so that the configuration of a positively buoyant cable can be obtained from the configuration of a negatively buoyant cable by a reflection in the line of the direction of motion. For the negatively buoyant cable W is positive and the critical angle ranges from zero when $W/R = 0$ to $\pi/2$ when W/R is infinite. Negative values of W would give rise to critical angles in the range $\pi/2 \leq \phi_c \leq \pi$. Since the cable functions for a critical angle in this range can be obtained in a simple manner from the cable functions for the supplementary critical angle, it is only necessary to consider the negatively buoyant cable and the range of critical angles may be restricted to $0 \leq \phi_c \leq \pi/2$. For a given cable, W is constant but R varies with the speed of the stream. Therefore W/R and ϕ_c vary with speed. Hence the cable functions are no longer independent of the speed of the stream.

The cable functions may be written

$$\ln \tau = \int_{\phi_0}^{\phi} \frac{f\frac{\cos\phi}{|\cos\phi|} + w \sin \phi}{-\sin \phi\,|\sin \phi| + w \cos \phi}\, d\phi \qquad [16a]$$

$$\sigma = \int_{\phi_0}^{\phi} \frac{\tau}{-\sin \phi\,|\sin \phi| + w \cos \phi}\, d\phi \qquad [16b]$$

$$\xi = \int_{\phi_0}^{\phi} \frac{\tau \cos \phi}{-\sin \phi\,|\sin \phi| + w \cos \phi}\, d\phi \qquad [16c]$$

$$\eta = \int_{\phi_0}^{\phi} \frac{\tau \sin \phi}{-\sin \phi\,|\sin \phi| + w \cos \phi}\, d\phi \qquad [16d]$$

where $f = F/R$ and $w = W/R$. Again by direct integration of Equation [1]

$$\tau = 1 + f\int_{P_0}^{P} \frac{\cos\phi}{|\cos\phi|}\, d\sigma + w\eta \qquad [16e]$$

If the point where the cable is normal to the stream is chosen as reference point this equation may be written

$$\tau = \tau_0 + f|\sigma| + w\eta \qquad [16f]$$

or in dimensional form

$$T = T_0 + F|s| + Wy \qquad [16g]$$

The best choice of reference point is now not so obvious. In addition to the point where the cable is normal to the stream, i.e., $\phi = \pi/2$, the point where the cable is parallel to the stream, i.e., $\phi = \pi$, is often a useful reference point. For calculating the cable functions it is convenient to divide the integrations into the three quadrants in which the angle ϕ may fall, namely:

Quadrant 1 where $\phi_c < \phi \le \pi/2$

Quadrant 2 where $\pi/2 \le \phi \le \pi$

Quadrant 3 where $\pi \le \phi < \pi + \phi_c$

For Quadrant 1 the point where $\phi = \pi/2$ is the most convenient reference point but for Quadrant 3 the point where $\phi = \pi$ is most convenient. For Quadrant 2 both reference points are equally convenient. Since cable configurations that extend through Quadrants 2 and 3 are more frequent than those extending through Quadrants 1 and 2, the point where $\phi = \pi$ has been used for the reference point of Quadrant 2. With such choice of reference points the cable functions become:

Quadrant 1 $\equiv \phi_c < \phi \le \pi/2$ Reference Point $\phi = \pi/2$

$$\ln\tau = \int_{\pi/2}^{\phi} \frac{f + w\sin\phi}{-\sin^2\phi + w\cos\phi}\, d\phi \qquad [17a]$$

$$\sigma = \int_{\pi/2}^{\phi} \frac{\tau}{-\sin^2\phi + w\cos\phi}\, d\phi \qquad [17b]$$

$$\xi = \int_{\pi/2}^{\phi} \frac{\tau\cos\phi}{-\sin^2\phi + w\cos\phi}\, d\phi \qquad [17c]$$

$$\eta = \int_{\pi/2}^{\phi} \frac{\tau \sin \phi}{-\sin^2\phi + w \cos\phi}\, d\phi \qquad [17d]$$

$$\tau = 1 + f\sigma + w\eta \qquad [17e]$$

Quadrant 2, $\pi/2 \le \phi \le \pi$ Reference Point $\phi = \pi$

$$\ln \tau = \int_{\pi}^{\phi} \frac{-f + w \sin \phi}{-\sin^2\phi + w \cos\phi}\, d\phi \qquad [18a]$$

$$\sigma = \int_{\pi}^{\phi} \frac{\tau}{-\sin^2\phi + w \cos\phi}\, d\phi \qquad [18b]$$

$$\xi = \int_{\pi}^{\phi} \frac{\tau \cos \phi}{-\sin^2\phi + w \cos\phi}\, d\phi \qquad [18c]$$

$$\eta = \int_{\pi}^{\phi} \frac{\tau \sin \phi}{-\sin^2\phi + w \cos\phi}\, d\phi \qquad [18d]$$

$$\tau = 1 - f\sigma + w\eta \qquad [18e]$$

Quadrant 3, $\pi \le \phi < \pi + \phi_c$ Reference Point $\phi = \pi$

$$\ln \tau = \int_{\pi}^{\phi} \frac{-f + w \sin \phi}{+\sin^2\phi + w \cos\phi}\, d\phi \qquad [19a]$$

$$\sigma = \int_{\pi}^{\phi} \frac{\tau}{\sin^2\phi + w \cos\phi}\, d\phi \qquad [19b]$$

$$\xi = \int_{\pi}^{\phi} \frac{\tau \cos \phi}{\sin^2\phi + w \cos\phi}\, d\phi \qquad [19c]$$

$$\eta = \int_{\pi}^{\phi} \frac{\tau \sin \phi}{\sin^2\phi + w \cos\phi}\, d\phi \qquad [19d]$$

$$\tau = 1 - f\sigma + w\eta \qquad [19e]$$

These are the functions which are presented in Tables 1, 3, and 2, respectively. In order to obtain a set of functions based on the same reference point that covers all three quadrants, so that those cable configurations that do extend into all three quadrants may be more readily handled, the functions for Quadrant 1 have been adjusted to the reference point $\phi = \pi$ by means of the relations presented in Equations [10a,b,c,d] and are tabulated in Table 4.

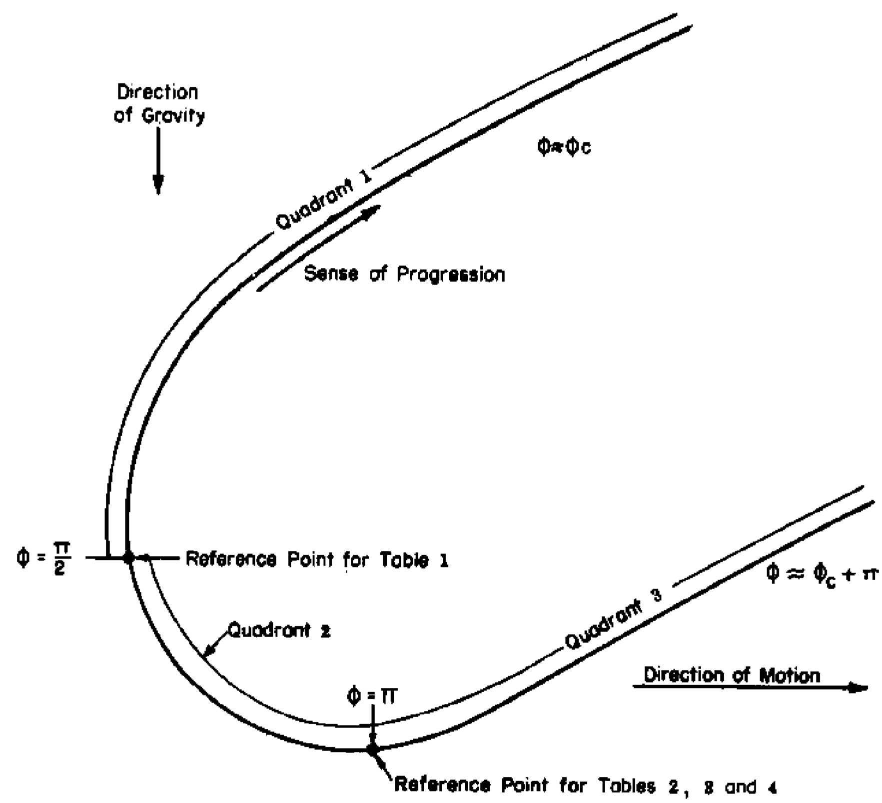

Figure 3 - General Configuration of a Heavy Cable in a Uniform Stream

NUMERICAL EXAMPLES

The following numerical examples have been worked out to illustrate the application of the tables to the solution of cable problems.

EXAMPLE 1. ANCHORING A BUOY

It is desired to anchor a buoy in 3600 feet of water using a 7/16-in. diameter stranded cable. The cable weighs 0.27 lb/ft in water. The drag of the cable when normal to the stream at five knots is 3.9 lb/ft. The buoy has an excess buoyancy of 7300 lb when fully submerged and in this condition in a current of five knots it has dynamic lift of 1800 lb and a drag of 5200 lb. What is the minimum length of cable required to insure that the buoy will never be submerged if the ocean currents are always uniform and less than five knots?

The minimum length of cable required is that with which the buoy would be submerged at the water surface in the extreme condition of a uniform current of five knots. Choose coordinates as shown in Figure 4. The total lift of the body, L, is given as 7300 plus 1800 which equals 9100 pounds. The drag, D, is 5200 pounds. For equilibrium at the point of attachment to the buoy the tension in the cable at this point, T_2, is given by $T_2 = \sqrt{L^2 + D^2} = \sqrt{(9100)^2 + (5200)^2}$ pounds = 10,500 pounds, and the angle of the cable at this point, ϕ_2, is defined by $\tan \phi_2 = \frac{-9100}{5200} = -1.7500$ giving $\phi_2 = 119.75$ degrees. The ratio of the unit weight of the cable W, to the unit drag, R, is given by $\frac{W}{R} = \frac{0.27}{3.9} = 0.0692$. Using this value in Equation [15c] the critical angle ϕ_c is found to be sufficiently close to 15 degrees so that for the purposes of this problem interpolation between critical angles is unnecessary. For the cable being used the value $f = F/R = 0.02$ applies as a sufficiently good approximation. Using Table 3 the values of the cable functions pertaining to the point of attachments are found to be $\tau_2 = 0.9983$; $\sigma_2 = 5.195$, $\eta_2 = 1.474$. If η_1 and τ_1 are the values of η and τ pertaining to the point of contact with the ocean bottom and y is the depth of water, i.e., 3600 feet, we have from Equations [10a] and [10d],

$$\frac{Ry\tau_1}{T_1} = (3.9)(3600)\frac{\tau_1}{T_1} = \eta_2 - \eta_1 = 1.474 - \eta_1 \text{ and } \frac{\tau_1}{T_1} = \frac{\tau_2}{T_2}$$

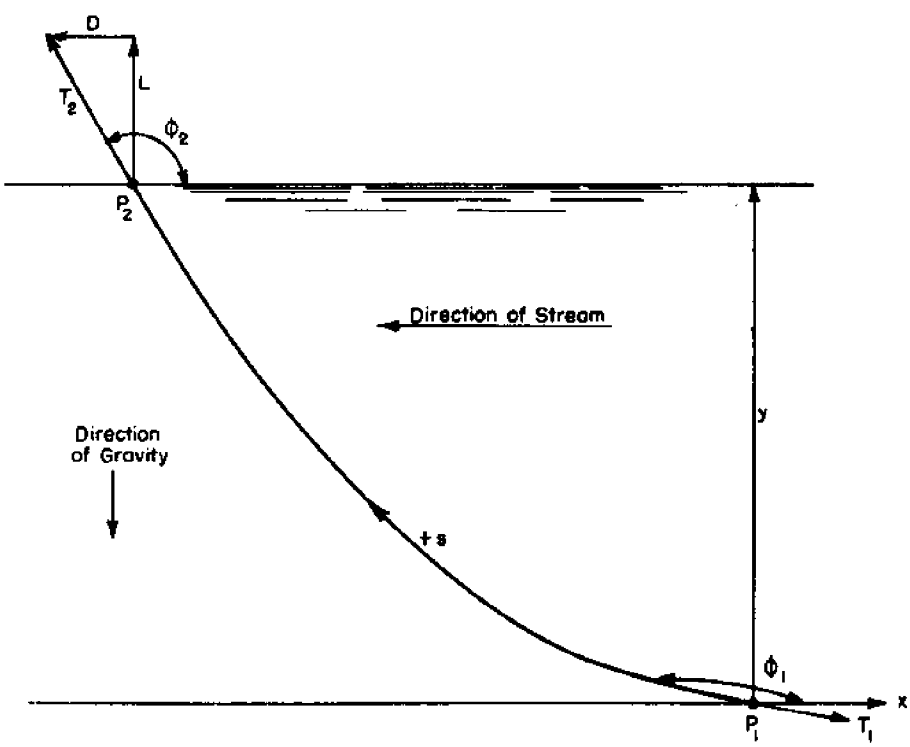

Figure 4 - Cable Configuration for a Moored Buoy

so that

$$\eta_1 = 1.474 - \frac{(3.9)(3600)(0.9983)}{10,500} = 0.139$$

Interpolation in the table then gives ϕ_1 = 171.37° and σ_1 = 1.946 for the value of these functions pertaining to the point of contact with the ocean bottom. The length of the cable can now be determined:

$$s = \frac{T_2}{R}\left(\frac{\sigma_2 - \sigma_1}{\tau_2}\right) = \frac{10,500\ (5.195-1.946)}{3.9 \times 0.9983}\ \text{feet} = 8760\ \text{feet}$$

EXAMPLE 2. TOWING A DEPRESSOR

It is desired to tow a depressor that at operating speed applies a force at its towpoint of 136,000 pounds at an angle of 70° from the direction of the stream. The depressor is to be towed from a cable that weighs ten pounds per foot in water and has a drag of 365 pounds per foot when normal to the stream at operating speed. The ratio of the tangential drag to the normal drag of the cable is known to be 0.022. If a length of 2550 feet of cable is used what is the depth of the depressor and the tension at the upper end of the cable?

Choose coordinates as shown in Figure 5. The critical angle is computed, using Equation [15c]. Thus cos ϕ_c = 0.9865; ϕ_c = 9.43°. Since the functions for this critical angle are not tabulated the problem will be solved

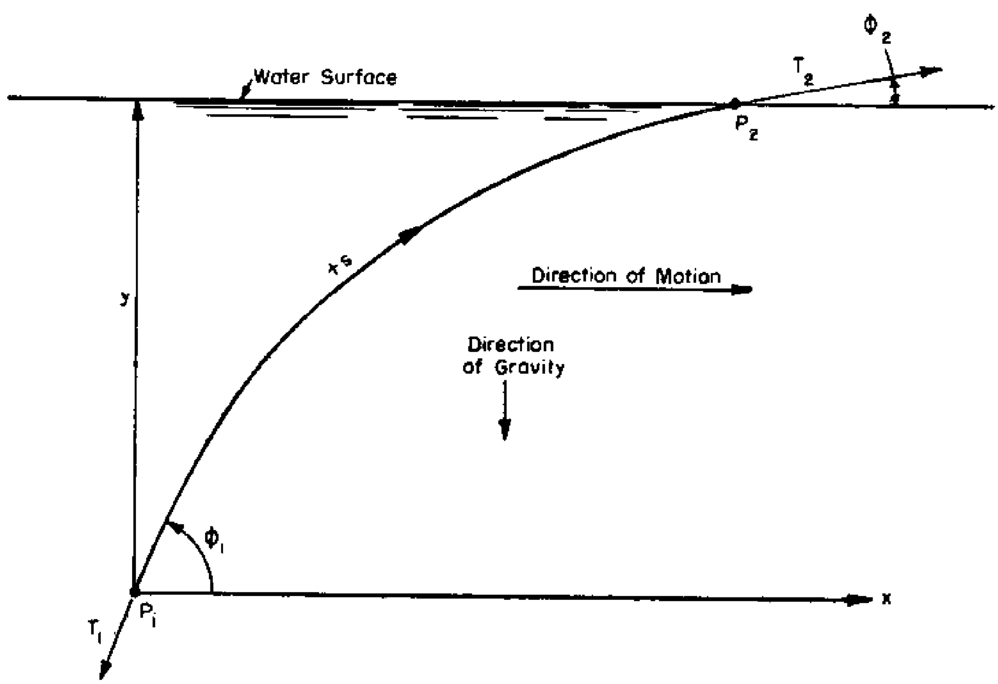

Figure 5 - Cable Configuration for Towing of a Depressor

using the functions for $\phi_c = 5°$ and $\phi_c = 10°$ and the results for $\phi_c = 9.43°$ will be found by interpolation. For the point at the depressor, P_1, $\phi_1 = 70°$, and the values $\tau = 1.0108$, $\sigma_1 = 0.3664$ and $\eta_1 = 0.3588$ are found by interpolation for f = 0.022 in Table 1, using the functions for $\phi_c = 5°$. Then the value of σ_2 for P_2, the point at the upper end of the cable, is

$$\sigma_2 = \frac{Rs\tau_1}{T_1} + \sigma_1 = \frac{(365)(2550)(1.0108)}{136,000} + 0.3664 = 7.2840$$

Now by inverse-interpolation in Table 1, the angle ϕ_2 is found to be $\phi_2 = 9.52°$ and $\eta_2 = 2.8145$ and $\tau_2 = 1.1821$; so that the depth $y = \frac{T_1}{R}\frac{(\eta_2 - \eta_1)}{\tau_1} = 905$ feet and the tension at the upper end

$$T_2 = \frac{\tau_2}{\tau_1} T_1 = \frac{1.1821}{1.0108} \times 136,000 \text{ pounds} = 159,000 \text{ pounds}$$

Following the same procedure but using the functions for $\phi_c = 10°$ we find the depth y = 976 feet and the tension T_2 = 167,000 pounds. Interpolating for $\phi_c = 9.43°$ between these values gives y = 968 feet and T_2 = 166,000 pounds.

EXAMPLE 3. TOWING A SURFACE TARGET

A surface target is towed at a speed of ten knots with a 1 3/8-in. cable. The weight of the cable in water is 3.57 pounds per foot. The general float problem requires a knowledge of the variation of the drag with the displacement of the float and is solved by a method of successive approximations as explained in Reference 5. For the purpose of illustrating the use of the tables the problem will be considerably simplified by assuming that the drag of the target is known to be 20,000 pounds and the cable at the target is known to enter the water at an angle of 40° to the direction of motion. The problem is to locate the lowest point of the cable.

Choose coordinates as shown in Figure 6. Assume

$$\frac{F}{R} = 0.02 \text{ and } R = 1.6\frac{\rho V^2 d}{2} = (1.6)(2.853)(100)\frac{(1.375)}{12} \text{ lb/ft} = 52.3 \text{ lb/ft}$$

The critical angle calculated by Equation [15c] is sufficiently close to 15° so that interpolation is not necessary. The angle ϕ_3 is given as 140°. Hence the tension in the cable at the point of attachment to the target, P_3, is $T_3 = -D \sec \phi_3 = (20,000)(1.305)$ lb = 26,100 lb. Using Table 3 the values of the cable functions pertaining to P_3 are found to be:

$$\sigma_3 = +4.629; \quad \eta_3 = 1.047; \quad \tau_3 = 0.9800; \quad \xi_3 = -4.432$$

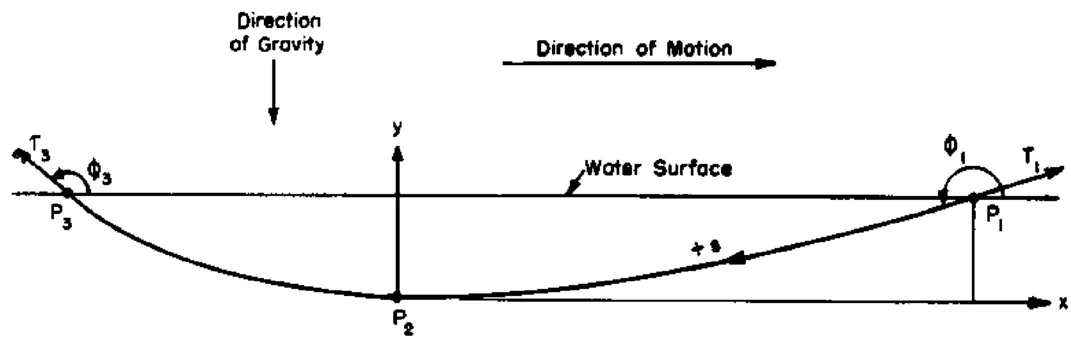

Figure 6 - Configuration of a Cable Towing a Surface Target

Since the point of attachment to the towing vessel, P_1, is in a horizontal plane with P_3; $\eta_1 = \eta_3 = 1.047$. Interpolating in Table 2 it is found:

$$\phi_1 = 193.75°; \quad \sigma_1 = -6.626; \quad \xi_1 = 6.526; \quad \tau_1 = 1.205$$

The tension at the towing vessel is $T_1 = \frac{\tau_1}{\tau_3} T_3 = 32{,}100$ pounds, and the distance from P_1 to P_3 is $\frac{T_1}{R} \frac{(\xi_1 - \xi_3)}{\tau_1} = 5580$ feet. At P_2 the cable has zero slope, i.e., $\phi_2 = 180°$ and the values of the cable functions are: $\tau_2 = 1.000$; $\sigma_2 = \eta_2 = \xi_2 = 0$. The horizontal distance from P_1 to P_2 is thus $\frac{T_1}{R} \frac{(\xi_1 - \xi_2)}{\tau_1} =$ 3320 feet and the depth at P_2 is $\frac{T_1}{R} \frac{(\eta_1 - \eta_2)}{\tau_1} = 533$ feet. The length of the cable is $\frac{T_1}{R} \frac{(\sigma_3 - \sigma_1)}{\tau_1} = 5730$ feet.

EXAMPLE 4. CONFIGURATION OF A STRING IN THE WIND

A string that is 34 in. long is immersed in a uniform horizontal wind with its ends fastened to two points that are one foot apart vertically. When trailed in the wind it has been found that the critical angle is 30°. Assuming F/R = 0.025, what will be the angles of the string at the points of attachment and locate the lowest point on the string and the point farthest downwind.

Choose coordinates as shown in Figure 7. The lower point of attachment is P_1, and upper P_4. The lowest point of the cable is P_2 and the point farthest downwind is P_3. Since P_1 and P_4 are in a vertical line $\xi_4 = \xi_1$ Also $y_4/s_4 = (\eta_4 - \eta_1)/(\sigma_4 - \sigma_1) = 1/2.83 = 0.353$ where y_4 is the vertical distance between P_1 and P_4 and s_4 is the total length of cable, 2.83 feet. To find the values which satisfy these two conditions the following method will be employed. Choosing values of ϕ_4, values of ξ_4, η_4, and σ_4 are obtained

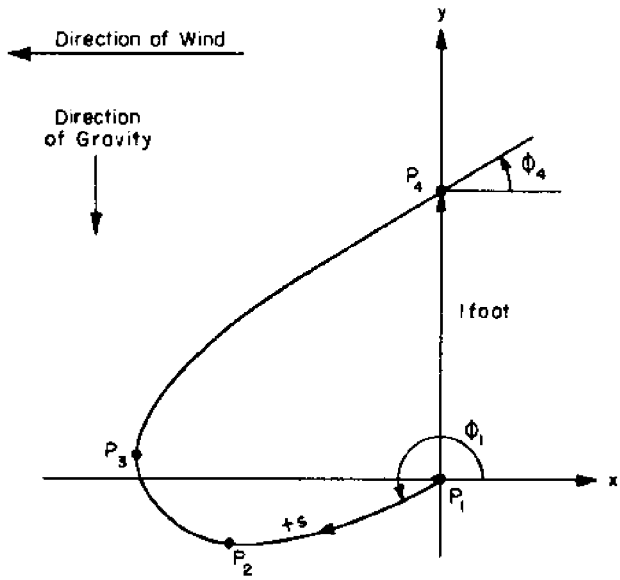

Figure 7 - Configuration of a String in a Stream with Ends Fastened 1 Foot Apart in Direction of Gravity

from Table 4. Then interpolation is made in Table 2 for values which correspond to $\xi_1 = \xi_4$ so that the ratios $(\eta_4 - \eta_1)/(\sigma_4 - \sigma_1)$ can be calculated. Thus the following table is constructed:

ϕ_4	σ_4	η_4	ξ_4	τ_4	ϕ_1	σ_1	η_1	ξ_1	τ_1	$\frac{\eta_4 - \eta_1}{\sigma_4 - \sigma_1}$
32	9.6152	6.0100	2.3318	2.8220	204.98	-2.4388	0.6437	2.3318	1.2468	0.4452
31.5	10.6278	6.5381	3.1958	2.9998	207.47	-3.4044	1.0744	3.1958	1.3952	0.3894
31	11.6405	7.0662	4.0598	3.1775	208.64	-4.3852	1.5385	4.0598	1.5540	0.3449
31.09	11.4562	6.9701	3.9025	3.1452	208.43	-4.2067	1.4540	3.9025	1.5251	0.3530

The values corresponding to $\frac{\eta_4 - \eta_1}{\sigma_4 - \sigma_1} = 0.353$ are given on the last line and were found by interpolation in the table. The ratio $\frac{T_1}{R}$ can now be computed from the relation $\frac{T_1}{R} = \frac{\tau_1 (y_4 - y_1)}{\eta_4 - \eta_1} = \frac{1.5251}{5.5161} = 0.2765$.

The cable functions for P_2 where $\phi_2 = 180°$ are $\sigma_2 = \eta_2 = \xi_2 = 0$ and $\tau_2 = 1$. The cable function for P_3 where $\phi_3 = 90°$ are found either in Table 3 or Table 4 and are $\sigma_3 = 3.0658$; $\eta_3 = 1.6061$; $\xi_3 = -2.2445$; $\tau_3 = 1.3871$. The location of P_2 with reference to P_1 is given by:

$$x_2 = \frac{(\xi_2 - \xi_1)}{\tau_1}\frac{T_1}{R} = \frac{-3.9025}{1.5251}(0.2765) \text{ feet} = -0.71 \text{ feet}$$

$$y_2 = \frac{(\eta_2 - \eta_1)}{\tau_1}\frac{T_1}{R} = \frac{(-1.4540)(0.2765)}{1.5251} \text{ feet} = -0.26 \text{ feet}$$

The location of P_3 with reference to P_1 is given by

$$x_3 = \frac{(\xi_3 - \xi_1)}{\tau_1} \frac{T_1}{R} = \frac{(-2.2445-3.9025)(0.2765)}{1.5251} \text{ feet} = -1.11 \text{ feet}$$

$$y_3 = \frac{(\eta_3 - \eta_1)}{\tau_1} \frac{T_1}{R} = \frac{(1.6061-1.4540)(0.2765)}{1.5251} \text{ feet} = 0.028 \text{ feet}$$

APPENDIX 1

NUMERICAL METHODS USED FOR THE CONSTRUCTION OF THE TABLES

The computations were carried out partly by hand with the use of desk calculators but mainly with the use of International Business Machines computing equipment. Initially IBM equipment installed at the David Taylor Model Basin was used. Before completion of the work, however, this equipment was removed from the Taylor Model Basin and the computations were then completed at the Bureau of Standards Computation Laboratory.

The values of the parameter f for which the cable functions have been evaluated are 0.01, 0.02, and 0.03. Because values of the cable functions are relatively insensitive to changes in this parameter, interpolation for values of f in the range 0.01 to 0.03 should be satisfactory. Past experience indicates that the value of f applicable to any round cable will fall within this range. On the other hand, for a faired cable, values of f ten times as great may be anticipated. The assumptions made above regarding the hydrodynamic force acting on an element of cable may not apply with sufficient accuracy to the case of a faired cable and until sufficient experimental work is done to establish the appropriate laws for the hydrodynamic force acting on a faired cable it is felt that the preparation of tables to cover this case is not justified.

The following values of the parameter ϕ_c and the related parameter w are covered in the tables:

ϕ_c degrees	w	ϕ_c degrees	w
0	0	45	0.707107
5	0.007625	50	0.912936
10	0.030619	55	1.16987
15	0.069350	60	1.50000
20	0.124485	65	1.94358
25	0.197070	70	2.58178
30	0.288675	75	3.60488
35	0.401623	80	5.58512
40	0.539363	85	11.38656

In many problems, interpolation for other values of the critical angle can be avoided by judicious choice of towing speeds. Small values of the critical angle, i.e., in the range of 0 to 5 degrees, occur frequently. It would be desirable to have tables for this range with a smaller interval of ϕ_c and it is hoped that the tables can be extended to include this range in the future. The case of $\phi_c = \pi/2$ can arise only when the speed of towing is reduced to zero and the determination of configuration of the cable is then a simple matter. For the case $\phi_c = 0$ the cable configuration is symmetrical about a line parallel to the direction of motion. Therefore in this case the cable functions are required only for Quadrant 1.

The main difficulty in evaluating the cable functions arises from the divergence of the functions in the region near the critical angle. This divergence is most rapid for $\phi_c = 0$. This case was treated separately and is discussed in Appendix 4. Also for other values of the critical angle special methods were used for values of ϕ within 5° of the critical angle. These methods are described in Appendix 3.

Otherwise the numerical integrations were made using Simpson's One-Third Rule:

$$\int_{x_0}^{x+2h} y \, dx = \frac{h}{3} (y_0 + 4y_1 + y_2)$$

together with a correction term involving fourth differences.[15] The interval chosen was 1°, i.e., h = 0.01745329, and the interval factor h/3 was always applied before the summation process and treated as a part of the integral, i.e., the form $\frac{h}{3} y_0 + 4\left(\frac{h}{3} y_1\right) + \frac{h}{3} y_2$ was used. The starting points for evaluating the functions at even values of ϕ were always at $\phi = 90°$ for Quadrant 1 (Table 1) and at $\phi = 180°$ for Quadrants 2 and 3. The starting points for evaluating the functions at odd values of ϕ were $\phi = 89°$ for Quadrant 1, 179° for Quadrant 2 and 181° for Quadrant 3. To obtain initial values of the functions at the latter starting points separate integrations were made using a smaller interval.

Although the function τ is integrable in closed form, see Appendix 2, the expressions obtained are so complicated that in general it was preferable to compute τ by numerical integration. The formulas given for τ in Appendix 2 were used only for checking purposes and for evaluation of τ in the region close to the critical angle, see Appendix 3. The integrands for computing $\ln \tau$ were evaluated to a minimum of six decimal places. The values of τ obtained by numerical integration were checked with values of τ computed by means of the formula given in Appendix 2 at the extreme points of the integrations, i.e., at $\phi = \phi_c + 5°$ in Quadrant 1, at $\phi = 90°$ in Quadrant 2, and at $\phi = 175° + \phi_c$ in Quadrant 3. The largest difference was found to be 0.00001. The conversion from $\ln \tau$ to τ was made with the use of the W.P.A. Exponential tables,[16] which were transcribed to IBM cards for the purpose.

The integrands for the functions σ, ξ, and η were evaluated to six decimal places. The fourth differences which were needed for computations of the correction to Simpson's Rule served as a check upon the integrands. In addition the equations for τ as a function of σ and η as given by Equations [17e], [18e], and [19e], were used to check the integrations. The deviations of τ from the values computed from these equations exceeds 0.0001 only for a few cases where the critical angle was 75°, 80°, or 85° and the values of the functions were large. The maximum deviation was less than 0.0004. An absolute error of 0.00001 in $\ln \tau$ would cause a relative error in τ and hence in σ, ξ, and η of 0.001 percent. The deviations always indicated that the relative error in the functions was less than this amount.

APPENDIX 2

THE INTEGRATION OF $\ln \tau$

From Equation [17a] for Quadrant 1

$$\ln \tau = \int_{\pi/2}^{\phi} \frac{f+w \sin \phi}{-\sin^2 \phi + w \cos \phi} d\phi \qquad [20]$$

But the denominator of the integrand can be factored, i.e.,

$$-\sin^2 \phi + w \cos \phi = (\cos \phi - \cos \phi_c)(\cos \phi + \sec \phi_c) \qquad [21]$$

since $w = \sec \phi_c - \cos \phi_c$ from [18b].

So that $1/-\sin^2 \phi + w \cos \phi$ can be written

$$\frac{1}{-\sin^2 \phi + w \cos \phi} = \frac{1}{\sec \phi_c + \cos \phi_c} \left[\frac{1}{\cos \phi - \cos \phi_c} - \frac{1}{\cos \phi + \sec \phi_c} \right] \qquad [22]$$

Hence

$$\ln \tau = \frac{1}{\sec \phi_c + \cos \phi_c} \left\{ f \left[\int_{\pi/2}^{\phi} \frac{d\phi}{\cos \phi - \cos \phi_c} - \int_{\pi/2}^{\phi} \frac{d\phi}{\cos \phi + \sec \phi_c} \right] \right.$$

$$\left. + w \left[\int_{\pi/2}^{\phi} \frac{\sin \phi \, d\phi}{\cos \phi - \cos \phi_c} - \int_{\pi/2}^{\phi} \frac{\sin \phi \, d\phi}{\cos \phi + \sec \phi_c} \right] \right\} \qquad [23]$$

The integrals that appear in this equation are listed in Pierce's Tables of Integrals.[17] Upon integration

$$\ln \tau = \frac{1}{\sec \phi_c + \cos \phi_c} \left\{ f \left[\csc \phi_c \ln \left(\frac{\tan \frac{\phi}{2} + \tan \frac{\phi_c}{2}}{\tan \frac{\phi}{2} - \tan \frac{\phi_c}{2}} \right) - 2 \cot \phi_c \tan^{-1} (\tan \frac{\phi_c}{2} \tan \frac{\phi}{2}) \right] \right.$$

$$\left. + w \ln \frac{\cos \phi + \sec \phi_c}{\cos \phi - \cos \phi_c} \right\} \Bigg|_{\pi/2}^{\phi} \qquad [24]$$

From Equation [18a] for Quadrant 2

$$\ln\tau = \int_{\pi}^{\phi} \frac{-f + w \sin\phi}{-\sin^2\phi + w \cos\phi} d\phi = \int_{0}^{\phi'} \frac{-f + w \sin\phi'}{\sin^2\phi' + w\cos\phi'} d\phi' \quad [25]$$

where $\phi' = \pi - \phi$. But

$$\frac{1}{\sin^2\phi' + w\cos\phi'} = \frac{1}{\sec\phi_c + \cos\phi_c}\left[\frac{1}{\cos\phi_c + \cos\phi'} + \frac{1}{\sec\phi_c - \cos\phi'}\right] \quad [26]$$

So that

$$\ln\tau = \frac{1}{\sec\phi_c + \cos\phi_c}\left\{-f\left[\int_0^{\phi'} \frac{d\phi'}{\cos\phi_c + \cos\phi'} + \int_0^{\phi'} \frac{d\phi'}{\sec\phi_c - \cos\phi'}\right]\right.$$

$$\left. + w\left[\int_0^{\phi'} \frac{\sin\phi' d\phi'}{\cos\phi_c + \cos\phi'} + \int_0^{\phi'} \frac{\sin\phi' d\phi'}{\sec\phi_c - \cos\phi'}\right]\right\} \quad [27]$$

Using Pierce's Tables of Integrals[17] to evaluate the integrals

$$\ln\tau = \frac{1}{\sec\phi_c + \cos\phi_c}\left\{f\left[\csc\phi_c \ln\left(\frac{1 - \tan\frac{\phi_c}{2}\tan\frac{\phi'}{2}}{1 + \tan\frac{\phi_c}{2}\tan\frac{\phi'}{2}}\right) - 2\cot\phi_c \tan^{-1}\left(\frac{\tan\frac{\phi'}{2}}{\tan\frac{\phi_c}{2}}\right)\right]\right.$$

$$\left. + w\left[\ln\frac{\sec\phi_c - \cos\phi'}{\cos\phi_c + \cos\phi'}\right]\right\}\Bigg|_0^{\phi'} \quad [28]$$

From Equation [19a] in Quadrant 3

$$\ln\tau = \int_{\pi}^{\phi} \frac{-f + w\sin\phi}{+\sin^2\phi + w\cos\phi} d\phi = \int_0^{\phi''} \frac{f + w\sin\phi''}{-\sin^2\phi'' + w\sin\phi''} d\phi'' \quad [29]$$

where $\phi'' = \phi - \pi$. The integrand in the last expression is exactly the same as that obtained for Quadrant 1 so that in this case

$$\ln\tau = \frac{1}{\sec\phi_c + \cos\phi_c}\left\{f\left[\csc\phi_c \ln\left(\frac{\tan\frac{\phi_c}{2} + \tan\frac{\phi''}{2}}{\tan\frac{\phi_c}{2} - \tan\frac{\phi''}{2}}\right) - 2\cot\phi_c \tan^{-1}\left(\tan\frac{\phi_c}{2}\tan\frac{\phi''}{2}\right)\right.\right.$$

$$\left. + w\ln\left[\frac{\cos\phi'' + \sec\phi_c}{\cos\phi'' - \cos\phi_c}\right]\right\}\Bigg|_0^{\phi''} \quad [30]$$

APPENDIX 3

TECHNIQUES USED IN THE REGION NEAR THE CRITICAL ANGLE

To evaluate the cable functions in the region of Quadrant 1 where $\phi_c + 5° \geq \phi \geq \phi_c + 1°$ (except for $\phi_c = 0$), the following methods were used: The values of τ were not obtained by numerical integration but were computed by use of Equation [24] of Appendix 2. Instead of integrating for the functions η and ξ directly the better behaved functions, $\bar{\eta}$ and $\bar{\xi}$, where

$$\bar{\eta} = \eta - \sin \phi_c \sigma; \quad d\bar{\eta} = \frac{\tau(\sin \phi - \sin \phi_c)}{-\sin^2 \phi + w \cos \phi} d\phi \qquad [31]$$

and

$$\bar{\xi} = \xi - \cos \phi_c \sigma; \quad d\bar{\xi} = \frac{\tau(\cos \phi - \cos \phi_c)}{-\sin^2 \phi + w \cos \phi} d\phi \qquad [32]$$

were first evaluated by numerical integration. Simpson's Rule was used again but the interval of integration was reduced to one-half degree. By eliminating η in [17e] and [31] and solving for σ it is found that

$$\sigma = \frac{\tau - 1 - w\bar{\eta}}{f + w \sin \phi_c} \qquad [33]$$

The values of σ were thus computed from the values of τ and $\bar{\eta}$. Having computed σ the values of η and ξ were found using [31] and [32].

The procedure used for computing the cable functions in the region of Quadrant 3 where $\phi_c + 175° \leq \phi \leq \phi_c + 179°$ was exactly analogous to the procedure described above for Quadrant 1.

Since no independent method of checking was available for these regions the work was checked by repeating the calculation.

APPENDIX 4

TECHNIQUES USED FOR THE CASE $\phi_c = 0$

Although the formulas for the cable functions given in Equations [12a] to [12d] can be applied in many cases where $\phi_c = 0$ the function defined by [16a] to [16d] were also evaluated for $\phi_c = 0$. This was done for two reasons: First because the modification of the law of hydrodynamic force that was made in obtaining the Equations [12a] to [12d] has a significant effect when the value of ϕ is small, and secondly in order to obtain a set of cable functions for $\phi_c = 0$ consistent with those computed for the other critical angles. As mentioned above, because of the symmetry of the cable configuration that obtains for $\phi_c = 0$, in this case the functions need be evaluated only for Quadrant 1. The functions take on particularly simple forms. From Equations [17a] to [17d], with $w = 0$,

$$\ln \tau = -\int_{\pi/2}^{\phi} \frac{f}{\sin^2 \phi}\, d\phi = f \cot \phi \qquad [34a]$$

$$\sigma = -\int_{\pi/2}^{\phi} e^{f \cot \phi} \csc^2 \phi \, d\phi = \frac{e^{f \cot \phi} - 1}{f} \qquad [34b]$$

$$\xi = -\int_{\pi/2}^{\phi} e^{f \cot \phi} \cot \phi \csc \phi \, d\phi \qquad [34c]$$

$$\eta = -\int_{\pi/2}^{\phi} e^{f \cot \phi} \csc \phi \, d\phi \qquad [34d]$$

It is seen that the functions τ and σ are integrable in closed form and hence numerical integrations are not required for computing these functions. However, numerical integrations are required for the functions ξ and η.

The cable functions diverge more rapidly for $\phi_c = 0$ than for the other critical angles. Also the method described in Appendix 2 for reducing the rate of divergence is not applicable. However because of the ease of computing the integrands the use of much smaller intervals of integration is not precluded. For the region $90° \geq \phi \geq 25°$ the functions were computed by numerical integration using Simpson's Rule and a 1° interval in the same manner as for the other critical angles. For ϕ smaller than 25°, τ and σ were computed with the use of Equations [34a] and [34b]. The functions ξ and η were computed by numerical integrations using Simpson's Rule, and the following intervals of ϕ:

Range of ϕ	Interval of ϕ
$25° > \phi > 15°$	0.5°
$15° > \phi > 10°$	0.25°
$10° > \phi > 5°$	0.1°
$5° > \phi > 1.1°$	.05°
$1.1° > \phi > 1°$	.01°

This work was checked by repeating the calculation.

PERSONNEL

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EXCERPTS FROM THE TABULATION OF CABLE FUNCTIONS

$\phi_c = 5°$

Table 1

$\phi°$	τ			σ		
	f = 0.01	f = 0.02	f = 0.03	f = 0.01	f = 0.02	f = 0.03
6	1.1784	1.3517	1.5505	14.9867	16.0926	17.3068
7	1.1347	1.2570	1.3925	10.9586	11.5512	12.1863
8	1.1116	1.2085	1.3140	8.8553	9.2420	9.6528
9	1.0965	1.1775	1.2646	7.4962	7.7742	8.0658
10	1.0855	1.1554	1.2299	6.5251	6.7358	6.9556
11	1.0771	1.1387	1.2038	5.7880	5.9540	6.1263
12	1.0704	1.1255	1.1834	5.2056	5.3399	5.4789
13	1.0649	1.1147	1.1669	4.7316	4.8426	4.9571
14	1.0603	1.1058	1.1532	4.3369	4.4302	4.5262
15	1.0563	1.0981	1.1417	4.0025	4.0820	4.1636
16	1.0528	1.0916	1.1317	3.7148	3.7833	3.8535
17	1.0498	1.0858	1.1231	3.4644	3.5240	3.5850
18	1.0471	1.0807	1.1155	3.2441	3.2964	3.3498
19	1.0446	1.0762	1.1088	3.0486	3.0948	3.1419
20	1.0424	1.0722	1.1027	2.8738	2.9148	2.9566
21	1.0404	1.0685	1.0973	2.7163	2.7530	2.7903
22	1.0386	1.0652	1.0924	2.5736	2.6065	2.6400
23	1.0369	1.0621	1.0879	2.4436	2.4733	2.5035
24	1.0354	1.0593	1.0837	2.3246	2.3515	2.3788
25	1.0340	1.0567	1.0800	2.2151	2.2395	2.2643
26	1.0326	1.0543	1.0764	2.1140	2.1363	2.1588
27	1.0314	1.0521	1.0732	2.0203	2.0406	2.0612
28	1.0302	1.0500	1.0701	1.9331	1.9517	1.9706
29	1.0291	1.0480	1.0673	1.8518	1.8689	1.8861
30	1.0280	1.0462	1.0646	1.7757	1.7913	1.8072
31	1.0270	1.0444	1.0621	1.7042	1.7187	1.7333
32	1.0261	1.0428	1.0598	1.6369	1.6503	1.6638
33	1.0252	1.0412	1.0575	1.5735	1.5858	1.5983
34	1.0243	1.0398	1.0554	1.5135	1.5249	1.5365
35	1.0235	1.0384	1.0534	1.4567	1.4673	1.4780
36	1.0227	1.0370	1.0515	1.4027	1.4125	1.4225
37	1.0220	1.0357	1.0497	1.3514	1.3605	1.3697
38	1.0213	1.0345	1.0479	1.3025	1.3109	1.3195
39	1.0206	1.0334	1.0463	1.2558	1.2636	1.2716
40	1.0199	1.0322	1.0447	1.2111	1.2184	1.2258
41	1.0193	1.0312	1.0432	1.1683	1.1751	1.1820
42	1.0187	1.0301	1.0417	1.1273	1.1336	1.1400
43	1.0181	1.0291	1.0403	1.0879	1.0938	1.0997
44	1.0175	1.0281	1.0389	1.0499	1.0554	1.0610
45	1.0169	1.0272	1.0376	1.0134	1.0185	1.0237
46	1.0164	1.0263	1.0363	.9781	.9829	.9877
47	1.0159	1.0254	1.0351	.9441	.9485	.9530
48	1.0154	1.0246	1.0339	.9111	.9153	.9195
49	1.0149	1.0237	1.0327	.8793	.8831	.8870
50	1.0144	1.0229	1.0316	.8484	.8520	.8556
51	1.0139	1.0222	1.0305	.8184	.8217	.8251
52	1.0134	1.0214	1.0294	.7893	.7924	.7955
53	1.0130	1.0207	1.0284	.7610	.7639	.7668
54	1.0125	1.0199	1.0274	.7334	.7361	.7388

$\phi_c = 5°$

Table 1

$\phi°$	ξ f = 0.01	ξ f = 0.02	ξ f = 0.03	η f = 0.01	η f = 0.02	η f = 0.03
6	13.9649	15.0525	16.2475	3.7443	3.9144	4.0986
7	9.9622	10.5397	11.1592	3.2931	3.4058	3.5254
8	7.8767	8.2509	8.6471	3.0201	3.1063	3.1967
9	6.5325	6.7983	7.0774	2.8199	2.8900	2.9630
10	5.5745	5.7741	5.9824	2.6600	2.7191	2.7803
11	4.8498	5.0053	5.1669	2.5260	2.5769	2.6295
12	4.2791	4.4036	4.5324	2.4101	2.4547	2.5006
13	3.8162	3.9180	4.0230	2.3076	2.3472	2.3878
14	3.4325	3.5170	3.6040	2.2156	2.2510	2.2874
15	3.1086	3.1798	3.2529	2.1319	2.1639	2.1966
16	2.8314	2.8920	2.9541	2.0551	2.0841	2.1139
17	2.5913	2.6434	2.6966	1.9840	2.0105	2.0376
18	2.3812	2.4263	2.4723	1.9178	1.9421	1.9670
19	2.1958	2.2351	2.2752	1.8558	1.8782	1.9010
20	2.0310	2.0654	2.1005	1.7975	1.8182	1.8392
21	1.8835	1.9138	1.9447	1.7423	1.7615	1.7810
22	1.7507	1.7776	1.8049	1.6901	1.7079	1.7259
23	1.6306	1.6545	1.6787	1.6403	1.6569	1.6737
24	1.5215	1.5427	1.5643	1.5929	1.6083	1.6240
25	1.4218	1.4409	1.4602	1.5475	1.5619	1.5765
26	1.3306	1.3477	1.3650	1.5040	1.5175	1.5311
27	1.2467	1.2621	1.2776	1.4622	1.4748	1.4876
28	1.1694	1.1832	1.1972	1.4219	1.4338	1.4457
29	1.0979	1.1104	1.1230	1.3831	1.3942	1.4055
30	1.0316	1.0429	1.0543	1.3456	1.3561	1.3666
31	.9701	.9803	.9906	1.3094	1.3192	1.3291
32	.9127	.9220	.9314	1.2742	1.2835	1.2928
33	.8592	.8676	.8761	1.2402	1.2488	1.2576
34	.8092	.8168	.8246	1.2071	1.2152	1.2235
35	.7624	.7693	.7763	1.1749	1.1826	1.1903
36	.7184	.7248	.7311	1.1435	1.1508	1.1581
37	.6772	.6829	.6887	1.1130	1.1198	1.1267
38	.6384	.6436	.6489	1.0832	1.0897	1.0962
39	.6018	.6066	.6114	1.0542	1.0602	1.0663
40	.5673	.5717	.5761	1.0258	1.0315	1.0372
41	.5348	.5388	.5428	.9980	1.0034	1.0088
42	.5041	.5077	.5113	.9708	.9759	.9810
43	.4750	.4783	.4816	.9441	.9489	.9537
44	.4475	.4505	.4535	.9180	.9225	.9271
45	.4214	.4241	.4269	.8924	.8967	.9009
46	.3967	.3992	.4017	.8673	.8713	.8753
47	.3732	.3755	.3778	.8426	.8463	.8501
48	.3510	.3531	.3551	.8183	.8218	.8254
49	.3299	.3318	.3336	.7944	.7977	.8011
50	.3098	.3115	.3132	.7709	.7741	.7772
51	.2907	.2923	.2938	.7478	.7507	.7537
52	.2726	.2740	.2754	.7250	.7278	.7305
53	.2554	.2567	.2579	.7026	.7051	.7077
54	.2390	.2402	.2413	.6804	.6828	.6852

$\phi_c = 5°$ (continued)

Table 1

$\phi°$	τ f = 0.01	τ f = 0.02	τ f = 0.03	σ f = 0.01	σ f = 0.02	σ f = 0.03
55	1.0121	1.0192	1.0264	.7066	.7091	.7116
56	1.0117	1.0185	1.0254	.6804	.6827	.6851
57	1.0112	1.0178	1.0245	.6549	.6570	.6592
58	1.0108	1.0172	1.0236	.6299	.6319	.6339
59	1.0104	1.0165	1.0227	.6055	.6074	.6092
60	1.0100	1.0159	1.0218	.5817	.5833	.5850
61	1.0096	1.0153	1.0209	.5583	.5598	.5614
62	1.0093	1.0147	1.0201	.5353	.5368	.5382
63	1.0089	1.0141	1.0192	.5129	.5142	.5155
64	1.0085	1.0135	1.0184	.4908	.4920	.4932
65	1.0081	1.0129	1.0176	.4691	.4702	.4713
66	1.0078	1.0123	1.0168	.4478	.4488	.4498
67	1.0074	1.0117	1.0160	.4268	.4277	.4286
68	1.0071	1.0112	1.0153	.4061	.4069	.4078
69	1.0067	1.0106	1.0145	.3857	.3865	.3872
70	1.0064	1.0101	1.0137	.3657	.3663	.3670
71	1.0060	1.0095	1.0130	.3458	.3464	.3470
72	1.0057	1.0090	1.0123	.3263	.3268	.3273
73	1.0054	1.0085	1.0115	.3069	.3074	.3079
74	1.0050	1.0079	1.0108	.2878	.2882	.2886
75	1.0047	1.0074	1.0101	.2689	.2692	.2696
76	1.0044	1.0069	1.0094	.2501	.2504	.2507
77	1.0041	1.0064	1.0087	.2315	.2318	.2321
78	1.0037	1.0059	1.0080	.2131	.2134	.2136
79	1.0034	1.0054	1.0073	.1949	.1950	.1952
80	1.0031	1.0049	1.0067	.1767	.1769	.1770
81	1.0028	1.0044	1.0060	.1587	.1588	.1590
82	1.0025	1.0039	1.0053	.1408	.1409	.1410
83	1.0022	1.0034	1.0046	.1230	.1231	.1231
84	1.0019	1.0029	1.0040	.1052	.1053	.1054
85	1.0015	1.0024	1.0033	.0876	.0876	.0877
86	1.0012	1.0019	1.0026	.0700	.0700	.0700
87	1.0009	1.0014	1.0020	.0524	.0525	.0525
88	1.0006	1.0010	1.0013	.0349	.0349	.0349
89	1.0003	1.0005	1.0007	.0175	.0175	.0175
90	1.0000	1.0000	1.0000	.0000	.0000	.0000

$\phi_c = 5°$ (continued)

Table 1

$\phi°$	ξ			η		
	f = 0.01	f = 0.02	f = 0.03	f = 0.01	f = 0.02	f = 0.03
55	.2234	.2245	.2255	.6586	.6608	.6631
56	.2086	.2095	.2105	.6370	.6391	.6412
57	.1945	.1953	.1962	.6157	.6177	.6196
58	.1811	.1818	.1826	.5947	.5965	.5983
59	.1683	.1690	.1697	.5739	.5756	.5773
60	.1562	.1568	.1574	.5533	.5549	.5564
61	.1447	.1452	.1458	.5329	.5344	.5359
62	.1338	.1342	.1347	.5128	.5142	.5155
63	.1234	.1238	.1242	.4929	.4941	.4953
64	.1135	.1139	.1143	.4731	.4742	.4754
65	.1042	.1045	.1048	.4535	.4546	.4556
66	.0954	.0956	.0959	.4341	.4351	.4360
67	.0870	.0872	.0875	.4149	.4157	.4166
68	.0791	.0793	.0795	.3958	.3966	.3974
69	.0716	.0718	.0720	.3768	.3775	.3783
70	.0646	.0647	.0649	.3580	.3587	.3593
71	.0580	.0581	.0582	.3393	.3399	.3405
72	.0518	.0519	.0520	.3208	.3213	.3218
73	.0459	.0460	.0461	.3023	.3028	.3032
74	.0405	.0406	.0407	.2840	.2844	.2848
75	.0354	.0355	.0356	.2657	.2661	.2664
76	.0307	.0308	.0308	.2476	.2479	.2482
77	.0264	.0265	.0265	.2295	.2298	.2301
78	.0224	.0225	.0225	.2116	.2118	.2120
79	.0188	.0188	.0188	.1936	.1938	.1940
80	.0155	.0155	.0155	.1758	.1760	.1761
81	.0125	.0125	.0125	.1580	.1582	.1583
82	.0099	.0099	.0099	.1403	.1404	.1405
83	.0075	.0075	.0075	.1227	.1227	.1228
84	.0055	.0055	.0055	.1050	.1051	.1052
85	.0038	.0038	.0038	.0875	.0875	.0876
86	.0024	.0024	.0024	.0699	.0700	.0700
87	.0014	.0014	.0014	.0524	.0524	.0524
88	.0006	.0006	.0006	.0349	.0349	.0349
89	.0002	.0002	.0002	.0175	.0175	.0175
90	.0000	.0000	.0000	.0000	.0000	.0000

$\phi_c = 10°$

Table 1

$\phi°$	τ			σ		
	f = 0.01	f = 0.02	f = 0.03	f = 0.01	f = 0.02	f = 0.03
11	1.2033	1.3126	1.4317	9.6870	10.1330	10.6061
12	1.1677	1.2502	1.3385	7.4730	7.7403	8.0202
13	1.1476	1.2159	1.2882	6.2630	6.4515	6.6476
14	1.1337	1.1926	1.2546	5.4513	5.5946	5.7429
15	1.1231	1.1753	1.2298	4.8523	4.9662	5.0835
16	1.1146	1.1615	1.2103	4.3848	4.4779	4.5737
17	1.1076	1.1502	1.1945	4.0059	4.0838	4.1636
18	1.1015	1.1406	1.1812	3.6904	3.7566	3.8244
19	1.0962	1.1324	1.1698	3.4223	3.4793	3.5376
20	1.0915	1.1252	1.1599	3.1907	3.2404	3.2910
21	1.0873	1.1188	1.1511	2.9881	3.0317	3.0762
22	1.0835	1.1130	1.1433	2.8090	2.8475	2.8868
23	1.0801	1.1078	1.1363	2.6491	2.6834	2.7183
24	1.0769	1.1031	1.1299	2.5052	2.5359	2.5671
25	1.0739	1.0987	1.1241	2.3749	2.4025	2.4306
26	1.0712	1.0947	1.1187	2.2561	2.2811	2.3064
27	1.0686	1.0909	1.1138	2.1473	2.1699	2.1929
28	1.0662	1.0874	1.1092	2.0471	2.0677	2.0886
29	1.0639	1.0842	1.1049	1.9545	1.9733	1.9923
30	1.0617	1.0811	1.1008	1.8685	1.8857	1.9031
31	1.0597	1.0782	1.0971	1.7884	1.8042	1.8201
32	1.0577	1.0755	1.0935	1.7136	1.7280	1.7426
33	1.0559	1.0729	1.0901	1.6434	1.6567	1.6701
34	1.0541	1.0704	1.0869	1.5774	1.5897	1.6020
35	1.0524	1.0680	1.0839	1.5152	1.5265	1.5379
36	1.0508	1.0658	1.0810	1.4564	1.4669	1.4774
37	1.0492	1.0636	1.0782	1.4007	1.4104	1.4202
38	1.0477	1.0616	1.0756	1.3479	1.3568	1.3659
39	1.0462	1.0596	1.0731	1.2976	1.3059	1.3143
40	1.0448	1.0577	1.0706	1.2497	1.2574	1.2652
41	1.0435	1.0558	1.0683	1.2040	1.2111	1.2184
42	1.0421	1.0540	1.0661	1.1602	1.1669	1.1736
43	1.0408	1.0523	1.0639	1.1183	1.1245	1.1307
44	1.0396	1.0506	1.0618	1.0781	1.0839	1.0897
45	1.0384	1.0490	1.0598	1.0395	1.0448	1.0502
46	1.0372	1.0474	1.0578	1.0023	1.0073	1.0123
47	1.0360	1.0459	1.0559	.9665	.9711	.9758
48	1.0349	1.0444	1.0540	.9319	.9362	.9405
49	1.0338	1.0430	1.0522	.8985	.9025	.9065
50	1.0327	1.0416	1.0505	.8662	.8699	.8737
51	1.0317	1.0402	1.0488	.8349	.8384	.8419
52	1.0307	1.0389	1.0471	.8046	.8078	.8110
53	1.0296	1.0375	1.0455	.7752	.7781	.7811
54	1.0287	1.0363	1.0439	.7466	.7493	.7521
55	1.0277	1.0350	1.0424	.7187	.7213	.7239
56	1.0267	1.0338	1.0408	.6916	.6940	.6964
57	1.0258	1.0326	1.0394	.6652	.6674	.6697
58	1.0249	1.0314	1.0379	.6395	.6415	.6436
59	1.0240	1.0302	1.0365	.6143	.6162	.6181

$\phi_c = 10°$

Table 1

$\phi°$	ξ f = 0.01	ξ f = 0.02	ξ f = 0.03	η f = 0.01	η f = 0.02	η f = 0.03
11	8.6125	9.0405	9.4950	3.4757	3.5891	3.7085
12	6.4425	6.6954	6.9604	3.0367	3.1147	3.1959
13	5.2610	5.4370	5.6202	2.7756	2.8366	2.8998
14	4.4716	4.6036	4.7404	2.5865	2.6370	2.6891
15	3.8917	3.9952	4.1020	2.4368	2.4799	2.5242
16	3.4411	3.5247	3.6107	2.3120	2.3496	2.3882
17	3.0778	3.1467	3.2174	2.2045	2.2378	2.2718
18	2.7768	2.8346	2.8938	2.1097	2.1395	2.1699
19	2.5226	2.5716	2.6218	2.0247	2.0515	2.0789
20	2.3043	2.3464	2.3894	1.9474	1.9718	1.9967
21	2.1146	2.1510	2.1882	1.8765	1.8988	1.9215
22	1.9476	1.9796	2.0120	1.8109	1.8313	1.8521
23	1.8001	1.8279	1.8563	1.7497	1.7685	1.7877
24	1.6682	1.6927	1.7177	1.6924	1.7098	1.7274
25	1.5496	1.5713	1.5934	1.6383	1.6545	1.6708
26	1.4424	1.4617	1.4813	1.5872	1.6022	1.6174
27	1.3450	1.3622	1.3797	1.5387	1.5526	1.5667
28	1.2561	1.2715	1.2872	1.4925	1.5054	1.5186
29	1.1747	1.1885	1.2026	1.4483	1.4604	1.4726
30	1.0999	1.1123	1.1249	1.4059	1.4173	1.4287
31	1.0309	1.0421	1.0534	1.3653	1.3759	1.3866
32	.9670	.9772	.9874	1.3262	1.3361	1.3461
33	.9076	.9170	.9262	1.2885	1.2978	1.3072
34	.8528	.8611	.8694	1.2521	1.2608	1.2696
35	.8015	.8090	.8166	1.2168	1.2250	1.2333
36	.7537	.7605	.7673	1.1827	1.1904	1.1982
37	.7089	.7151	.7213	1.1496	1.1568	1.1641
38	.6670	.6726	.6782	1.1174	1.1242	1.1311
39	.6276	.6327	.6379	1.0861	1.0925	1.0990
40	.5907	.5953	.6000	1.0557	1.0617	1.0677
41	.5559	.5601	.5644	1.0259	1.0316	1.0373
42	.5231	.5270	.5308	.9970	1.0023	1.0077
43	.4922	.4957	.4992	.9687	.9737	.9787
44	.4631	.4662	.4694	.9410	.9457	.9504
45	.4355	.4384	.4413	.9139	.9183	.9228
46	.4095	.4121	.4147	.8874	.8916	.8957
47	.3848	.3872	.3896	.8614	.8653	.8692
48	.3614	.3636	.3658	.8359	.8396	.8433
49	.3393	.3413	.3432	.8109	.8144	.8178
50	.3183	.3201	.3219	.7864	.7896	.7928
51	.2984	.3000	.3017	.7622	.7652	.7683
52	.2796	.2810	.2825	.7385	.7413	.7442
53	.2616	.2629	.2643	.7151	.7178	.7204
54	.2446	.2458	.2470	.6921	.6946	.6971
55	.2285	.2295	.2306	.6695	.6718	.6741
56	.2131	.2141	.2150	.6472	.6493	.6515
57	.1985	.1994	.2003	.6251	.6272	.6292
58	.1847	.1855	.1863	.6034	.6053	.6072
59	.1716	.1723	.1729	.5820	.5837	.5854

ϕ_c = 10° (continued)

Table 1

$\phi°$	τ			σ		
	f = 0.01	f = 0.02	f = 0.03	f = 0.01	f = 0.02	f = 0.03
60	1.0231	1.0290	1.0351	.5898	.5915	.5932
61	1.0222	1.0279	1.0337	.5657	.5673	.5689
62	1.0213	1.0268	1.0323	.5422	.5436	.5451
63	1.0205	1.0257	1.0310	.5191	.5205	.5218
64	1.0196	1.0246	1.0297	.4965	.4977	.4990
65	1.0188	1.0236	1.0284	.4743	.4754	.4766
66	1.0180	1.0225	1.0271	.4525	.4535	.4545
67	1.0171	1.0215	1.0259	.4311	.4320	.4329
68	1.0163	1.0205	1.0246	.4100	.4108	.4117
69	1.0155	1.0195	1.0234	.3892	.3900	.3908
70	1.0147	1.0185	1.0222	.3688	.3695	.3702
71	1.0140	1.0175	1.0210	.3486	.3492	.3499
72	1.0132	1.0165	1.0198	.3288	.3293	.3298
73	1.0124	1.0155	1.0187	.3091	.3096	.3101
74	1.0117	1.0146	1.0175	.2897	.2901	.2906
75	1.0109	1.0136	1.0163	.2705	.2709	.2713
76	1.0101	1.0127	1.0152	.2516	.2519	.2522
77	1.0094	1.0117	1.0141	.2328	.2331	.2333
78	1.0087	1.0108	1.0130	.2142	.2144	.2146
79	1.0079	1.0099	1.0119	.1957	.1959	.1961
80	1.0072	1.0090	1.0107	.1774	.1776	.1778
81	1.0064	1.0080	1.0097	.1593	.1594	.1595
82	1.0057	1.0071	1.0086	.1413	.1414	.1414
83	1.0050	1.0062	1.0075	.1233	.1234	.1235
84	1.0043	1.0053	1.0064	.1055	.1056	.1056
85	1.0036	1.0044	1.0053	.0878	.0878	.0878
86	1.0028	1.0035	1.0043	.0701	.0701	.0702
87	1.0021	1.0027	1.0032	.0525	.0525	.0525
88	1.0014	1.0018	1.0021	.0350	.0350	.0350
89	1.0007	1.0009	1.0011	.0175	.0175	.0175
90	1.0000	1.0000	1.0000	.0000	.0000	.0000

ϕ_c = 10° (continued)

Table 1

ϕ°	ξ			η		
	f = 0.01	f = 0.02	f = 0.03	f = 0.01	f = 0.02	f = 0.03
60	.1591	.1597	.1603	.5608	.5624	.5640
61	.1473	.1478	.1483	.5399	.5414	.5429
62	.1360	.1365	.1370	.5192	.5206	.5219
63	.1254	.1258	.1262	.4987	.5000	.5013
64	.1153	.1157	.1160	.4785	.4797	.4808
65	.1057	.1061	.1064	.4585	.4595	.4606
66	.0967	.0970	.0973	.4386	.4396	.4406
67	.0881	.0884	.0886	.4190	.4199	.4208
68	.0801	.0803	.0805	.3995	.4003	.4011
69	.0725	.0727	.0728	.3802	.3809	.3817
70	.0653	.0655	.0656	.3611	.3617	.3624
71	.0586	.0587	.0588	.3421	.3426	.3432
72	.0523	.0524	.0525	.3232	.3237	.3242
73	.0464	.0465	.0466	.3045	.3049	.3054
74	.0409	.0409	.0410	.2859	.2863	.2867
75	.0357	.0358	.0359	.2674	.2677	.2681
76	.0310	.0310	.0311	.2490	.2493	.2496
77	.0266	.0266	.0267	.2308	.2310	.2313
78	.0226	.0226	.0226	.2126	.2128	.2130
79	.0189	.0189	.0189	.1945	.1947	.1949
80	.0156	.0156	.0156	.1765	.1767	.1768
81	.0126	.0126	.0126	.1586	.1588	.1589
82	.0099	.0099	.0099	.1408	.1409	.1410
83	.0076	.0076	.0076	.1230	.1231	.1232
84	.0055	.0055	.0055	.1053	.1054	.1054
85	.0038	.0038	.0038	.0877	.0877	.0877
86	.0025	.0025	.0025	.0700	.0701	.0701
87	.0014	.0014	.0014	.0525	.0525	.0525
88	.0006	.0006	.0006	.0350	.0350	.0350
89	.0002	.0002	.0002	.0175	.0175	.0175
90	.0000	.0000	.0000	.0000	.0000	.0000

$\phi_c = 5°$

Table 2

$\phi°-180°$	τ			σ		
	f = 0.01	f = 0.02	f = 0.03	f = 0.01	f = 0.02	f = 0.03
0	1.0000	1.0000	1.0000	.0000	.0000	.0000
1	1.0236	1.0477	1.0723	- 2.3475	- 2.3751	- 2.4031
2	1.0504	1.1026	1.1574	- 4.9698	- 5.0933	- 5.2208
3	1.0844	1.1740	1.2709	- 8.2637	- 8.6051	- 8.9652
4	1.1385	1.2911	1.4643	-13.4267	- 14.3274	- 15.3087

$\phi_c = 10°$

$\phi°-180°$	τ			σ		
	f = 0.01	f = 0.02	f = 0.03	f = 0.01	f = 0.02	f = 0.03
0	1.0000	1.0000	1.0000	.0000	.0000	.0000
1	1.0059	1.0117	1.0175	- .5736	- .5753	- .5769
2	1.0123	1.0240	1.0359	- 1.1627	- 1.1695	- 1.1763
3	1.0193	1.0374	1.0559	- 1.7814	- 1.7973	- 1.8133
4	1.0272	1.0523	1.0780	- 2.4479	- 2.4778	- 2.5083
5	1.0364	1.0694	1.1034	- 3.1884	- 3.2392	- 3.2910
6	1.0475	1.0897	1.1337	- 4.0453	- 4.1270	- 4.2109
7	1.0616	1.1155	1.1722	- 5.0966	- 5.2262	- 5.3602
8	1.0815	1.1516	1.2263	- 6.5175	- 6.7293	- 6.9502
9	1.1158	1.2139	1.3206	- 8.8742	- 9.2660	- 9.6808

$\phi_c = 15°$

$\phi°-180°$	τ			σ		
	f = 0.01	f = 0.02	f = 0.03	f = 0.01	f = 0.02	f = 0.03
0	1.0000	1.0000	1.0000	.0000	.0000	.0000
1	1.0027	1.0052	1.0077	- .2524	- .2527	- .2530
2	1.0057	1.0108	1.0159	- .5078	- .5091	- .5104
3	1.0091	1.0169	1.0247	- .7689	- .7718	- .7748
4	1.0129	1.0234	1.0341	-1.0384	-1.0438	-1.0492
5	1.0173	1.0307	1.0443	-1.3199	-1.3286	-1.3373
6	1.0222	1.0387	1.0555	-1.6175	-1.6305	-1.6437
7	1.0279	1.0478	1.0680	-1.9367	-1.9554	-1.9744
8	1.0346	1.0581	1.0822	-2.2851	-2.3112	-2.3376
9	1.0424	1.0702	1.0986	-2.6735	-2.7091	-2.7454
10	1.0520	1.0846	1.1182	-3.1186	-3.1670	-3.2165
11	1.0640	1.1025	1.1423	-3.6483	-3.7146	-3.7824
12	[illegible]	1.1259	1.1739	-4.3156	-4.4081	-4.5032
13	1.1[illegible]1	1.1599	1.2196	-5.2402	-5.3762	-5.5170
14	1.1[illegible]44	1.2203	1.3012	-6.8133	-7.0420	-7.2609

$\phi_c = 5°$

Table 2

$\phi°$-180°	ξ f = 0.01	ξ f = 0.02	ξ f = 0.03	η f = 0.01	η f = 0.02	η f = 0.03
0	.0000	.0000	.0000	.0000	.0000	.0000
1	2.3474	2.3749	2.4029	.0207	.0210	.0214
2	4.9687	5.0922	5.2197	.0900	.0929	.0960
3	8.2594	8.6005	8.9604	.2351	.2478	.2613
4	13.4124	14.3117	15.2915	.5549	.6027	.6551

$\phi_c = 10°$

$\phi°$-180°	ξ f = 0.01	ξ f = 0.02	ξ f = 0.03	η f = 0.01	η f = 0.02	η f = 0.03
0	.0000	.0000	.0000	.0000	.0000	.0000
1	.5736	.5752	.5769	.0050	.0050	.0051
2	1.1625	1.1692	1.1760	.0205	.0206	.0208
3	1.7806	1.7964	1.8124	.0475	.0481	.0486
4	2.4458	2.4757	2.5061	.0883	.0897	.0912
5	3.1840	3.2347	3.2864	.1465	.1496	.1527
6	4.0369	4.1184	4.2020	.2289	.2349	.2412
7	5.0814	5.2105	5.3439	.3482	.3597	.3717
8	6.4900	6.7005	6.9201	.5345	.5568	.5801
9	8.8205	9.2090	9.6203	.8851	.9342	.9865

$\phi_c = 15°$

$\phi°$-180°	ξ f = 0.01	ξ f = 0.02	ξ f = 0.03	η f = 0.01	η f = 0.02	η f = 0.03
0	.0000	.0000	.0000	.0000	.0000	.0000
1	.2524	.2527	.2530	.0022	.0022	.0022
2	.5077	.5090	.5103	.0089	.0089	.0090
3	.7685	.7714	.7744	.0203	.0204	.0205
4	1.0375	1.0429	1.0483	.0368	.0370	.0373
5	1.3181	1.3268	1.3356	.0589	.0594	.0599
6	1.6144	1.6274	1.6406	.0874	.0884	.0893
7	1.9315	1.9502	1.9691	.1236	.1252	.1268
8	2.2770	2.3029	2.3292	.1691	.1717	.1742
9	2.6611	2.6965	2.7325	.2266	.2306	.2346
10	3.1000	3.1481	3.1971	.3002	.3062	.3124
11	3.6209	3.6864	3.7535	.3968	.4062	.4157
12	4.2747	4.3659	4.4598	.5301	.5447	.5597
13	5.1773	5.3110	5.4493	.7308	.7548	.7797
14	6.7064	6.9303	7.1640	1.0995	1.1453	1.1933

$\phi_c = 30°$

Table 2

$\phi°$-180°	τ			σ		
	f = 0.01	f = 0.02	f = 0.03	f = 0.01	f = 0.02	f = 0.03
0	1.0000	1.0000	1.0000	.0000	.0000	.0000
1	1.0008	1.0014	1.0020	- .0605	- .0605	- .0605
2	1.0018	1.0030	1.0043	- .1212	- .1213	- .1214
3	1.0032	1.0050	1.0069	- .1823	- .1825	- .1826
4	1.0049	1.0074	1.0098	- .2439	- .2442	- .2445
5	1.0069	1.0100	1.0131	- .3063	- .3067	- .3072
6	1.0093	1.0130	1.0168	- .3695	- .3702	- .3709
7	1.0121	1.0165	1.0209	- .4339	- .4348	- .4358
8	1.0152	1.0203	1.0253	- .4996	- .5009	- .5021
9	1.0188	1.0245	1.0303	- .5670	- .5686	- .5702
10	1.0227	1.0292	1.0357	- .6362	- .6382	- .6402
11	1.0272	1.0344	1.0417	- .7076	- .7100	- .7126
12	1.0322	1.0402	1.0483	- .7815	- .7845	- .7876
13	1.0378	1.0466	1.0555	- .8584	- .8620	- .8657
14	1.0440	1.0537	1.0635	- .9387	- .9431	- .9475
15	1.0509	1.0616	1.0723	-1.0230	-1.0282	-1.0335
16	1.0587	1.0703	1.0820	-1.1121	-1.1182	-1.1244
17	1.0674	1.0801	1.0929	-1.2068	-1.2140	-1.2212
18	1.0772	1.0910	1.1050	-1.3080	-1.3165	-1.3250
19	1.0883	1.1034	1.1186	-1.4173	-1.4272	-1.4372
20	1.1010	1.1174	1.1341	-1.5362	-1.5479	-1.5596
21	1.1155	1.1335	1.1518	-1.6672	-1.6809	-1.6947
22	1.1325	1.1522	1.1724	-1.8135	-1.8296	-1.8459
23	1.1525	1.1743	1.1965	-1.9795	-1.9987	-2.0180
24	1.1766	1.2009	1.2256	-2.1721	-2.1950	-2.2183
25	1.2064	1.2337	1.2615	-2.4019	-2.4298	-2.4582
26	1.2448	1.2759	1.3077	-2.6874	-2.7221	-2.7574
27	1.2971	1.3335	1.3708	-3.0641	-3.1088	-3.1544
28	1.3762	1.4206	1.4665	-3.6156	-3.6769	-3.7396
29	1.5258	1.5861	1.6487	-4.6261	-4.7237	-4.8240

$\phi_c = 30°$

Table 2

$\phi°-180°$	ξ			η		
	f = 0.01	f = 0.02	f = 0.03	f = 0.01	f = 0.02	f = 0.03
0	.0000	.0000	.0000	.0000	.0000	.0000
1	.0605	.0605	.0605	.0005	.0005	.0005
2	.1212	.1213	.1213	.0021	.0021	.0021
3	.1822	.1824	.1825	.0048	.0048	.0048
4	.2437	.2440	.2443	.0085	.0086	.0086
5	.3059	.3063	.3068	.0134	.0135	.0135
6	.3688	.3695	.3702	.0195	.0196	.0196
7	.4328	.4337	.4347	.0268	.0269	.0270
8	.4980	.4992	.5005	.0354	.0355	.0356
9	.5646	.5662	.5678	.0453	.0455	.0457
10	.6326	.6348	.6368	.0568	.0570	.0572
11	.7030	.7055	.7080	.0698	.0701	.0704
12	.7755	.7785	.7815	.0845	.0849	.0854
13	.8505	.8541	.8578	.1012	.1017	.1023
14	.9286	.9329	.9373	.1199	.1207	.1214
15	1.0103	1.0154	1.0205	.1410	.1420	.1429
16	1.0961	1.1021	1.1082	.1649	.1660	.1672
17	1.1868	1.1939	1.2010	.1917	.1932	.1947
18	1.2834	1.2916	1.2999	.2222	.2241	.2260
19	1.3870	1.3966	1.4063	.2569	.2592	.2616
20	1.4992	1.5104	1.5217	.2966	.2995	.3024
21	1.6219	1.6350	1.6482	.3425	.3461	.3498
22	1.7579	1.7733	1.7889	.3961	.4006	.4052
23	1.9113	1.9295	1.9479	.4597	.4654	.4711
24	2.0879	2.1096	2.1315	.5365	.5437	.5510
25	2.2970	2.3232	2.3498	.6319	.6411	.6505
26	2.5546	2.5868	2.6198	.7549	.7670	.7795
27	2.8917	2.9328	2.9750	.9231	.9398	.9567
28	3.3807	3.4366	3.4939	1.1781	1.2024	1.2273
29	4.2681	4.3560	4.4463	1.6612	1.7029	1.7459

$\phi_c = 15°$

Table 3

180°-φ°	τ			σ		
	f = 0.01	f = 0.02	f = 0.03	f = 0.01	f = 0.02	f = 0.03
0	1.0000	1.0000	1.0000	.0000	.0000	.0000
1	.9976	.9951	.9926	.2510	.2507	.2504
2	.9956	.9906	.9857	.4994	.4981	.4969
3	.9939	.9865	.9792	.7432	.7404	.7377
4	.9925	.9828	.9732	.9807	.9759	.9712
5	.9915	.9795	.9677	1.2106	1.2033	1.1961
6	.9907	.9766	.9626	1.4318	1.4215	1.4114
7	.9903	.9741	.9581	1.6433	1.6298	1.6165
8	.9901	.9719	.9540	1.8447	1.8277	1.8109
9	.9901	.9701	.9504	2.0358	2.0151	1.9947
10	.9904	.9685	.9472	2.2165	2.1920	2.1678
11	.9909	.9673	.9443	2.3870	2.3586	2.3306
12	.9915	.9663	.9419	2.5476	2.5153	2.4835
13	.9922	.9656	.9397	2.6988	2.6625	2.6268
14	.9931	.9651	.9379	2.8409	2.8007	2.7612
15	.9941	.9647	.9363	2.9745	2.9304	2.8872
16	.9952	.9646	.9349	3.1001	3.0522	3.0053
17	.9963	.9645	.9338	3.2182	3.1666	3.1162
18	.9975	.9646	.9328	3.3293	3.2742	3.2202
19	.9988	.9648	.9320	3.4340	3.3754	3.3180
20	1.0001	.9651	.9314	3.5327	3.4706	3.4100
21	1.0014	.9655	.9309	3.6258	3.5605	3.4967
22	1.0027	.9660	.9305	3.7138	3.6452	3.5784
23	1.0041	.9665	.9303	3.7970	3.7254	3.6555
24	1.0055	.9671	.9301	3.8758	3.8012	3.7285
25	1.0069	.9677	.9300	3.9505	3.8730	3.7975
26	1.0083	.9684	.9300	4.0214	3.9412	3.8630
27	1.0097	.9691	.9301	4.0889	4.0059	3.9252
28	1.0111	.9698	.9302	4.1531	4.0675	3.9843
29	1.0126	.9706	.9303	4.2143	4.1262	4.0405
30	1.0140	.9714	.9306	4.2727	4.1822	4.0941
31	1.0154	.9722	.9308	4.3285	4.2356	4.1453
32	1.0168	.9730	.9311	4.3819	4.2867	4.1943
33	1.0182	.9739	.9315	4.4331	4.3357	4.2411
34	1.0196	.9747	.9318	4.4821	4.3826	4.2860
35	1.0209	.9756	.9322	4.5292	4.4276	4.3290
36	1.0223	.9765	.9327	4.5745	4.4709	4.3704
37	1.0237	.9773	.9331	4.6181	4.5125	4.4101
38	1.0250	.9782	.9336	4.6602	4.5526	4.4484
39	1.0264	.9791	.9341	4.7007	4.5913	4.4853
40	1.0277	.9800	.9346	4.7398	4.6286	4.5209
41	1.0290	.9809	.9351	4.7776	4.6647	4.5553
42	1.0303	.9818	.9356	4.8142	4.6996	4.5885
43	1.0317	.9827	.9362	4.8497	4.7334	4.6207
44	1.0329	.9837	.9367	4.8841	4.7661	4.6519

$\phi_c = 15°$

Table 3

$180° - \phi°$	ξ			η		
	f = 0.01	f = 0.02	f = 0.03	f = 0.01	f = 0.02	f = 0.03
0	.0000	.0000	.0000	.0000	.0000	.0000
1	- .2510	- .2507	- .2504	.0022	.0022	.0022
2	- .4993	- .4980	- .4968	.0087	.0087	.0086
3	- .7428	- .7401	- .7373	.0193	.0192	.0191
4	- .9800	- .9752	- .9704	.0338	.0336	.0334
5	-1.2092	-1.2018	-1.1946	.0518	.0514	.0510
6	-1.4293	-1.4190	-1.4089	.0730	.0723	.0716
7	-1.6394	-1.6260	-1.6127	.0969	.0959	.0948
8	-1.8391	-1.8222	-1.8055	.1232	.1217	.1202
9	-2.0281	-2.0075	-1.9872	.1514	.1494	.1473
10	-2.2063	-2.1820	-2.1580	.1813	.1785	.1759
11	-2.3740	-2.3458	-2.3181	.2123	.2089	.2055
12	-2.5314	-2.4993	-2.4678	.2443	.2401	.2360
13	-2.6790	-2.6431	-2.6078	.2770	.2720	.2670
14	-2.8171	-2.7774	-2.7385	.3102	.3042	.2984
15	-2.9465	-2.9030	-2.8605	.3436	.3367	.3299
16	-3.0675	-3.0204	-2.9743	.3772	.3692	.3615
17	-3.1807	-3.1301	-3.0806	.4107	.4017	.3929
18	-3.2868	-3.2327	-3.1798	.4441	.4340	.4242
19	-3.3860	-3.3286	-3.2726	.4773	.4661	.4552
20	-3.4791	-3.4185	-3.3593	.5103	.4979	.4859
21	-3.5663	-3.5026	-3.4404	.5429	.5294	.5163
22	-3.6481	-3.5815	-3.5165	.5751	.5604	.5462
23	-3.7250	-3.6555	-3.5877	.6069	.5911	.5757
24	-3.7973	-3.7250	-3.6546	.6383	.6213	.6048
25	-3.8653	-3.7904	-3.7175	.6693	.6511	.6334
26	-3.9293	-3.8519	-3.7766	.6999	.6804	.6616
27	-3.9896	-3.9099	-3.8322	.7299	.7093	.6894
28	-4.0466	-3.9645	-3.8846	.7596	.7378	.7166
29	-4.1004	-4.0161	-3.9341	.7888	.7658	.7435
30	-4.1512	-4.0646	-3.9808	.8175	.7933	.7699
31	-4.1993	-4.1108	-4.0249	.8459	.8204	.7959
32	-4.2448	-4.1544	-4.0666	.8738	.8472	.8214
33	-4.2880	-4.1957	-4.1061	.9012	.8734	.8466
34	-4.3289	-4.2348	-4.1435	.9283	.8993	.8714
35	-4.3677	-4.2719	-4.1790	.9550	.9248	.8957
36	-4.4046	-4.3072	-4.2126	.9813	.9500	.9197
37	-4.4397	-4.3406	-4.2446	1.0072	.9747	.9434
38	-4.4730	-4.3725	-4.2750	1.0328	.9991	.9667
39	-4.5047	-4.4027	-4.3038	1.0580	1.0232	.9896
40	-4.5349	-4.4315	-4.3313	1.0829	1.0470	1.0123
41	-4.5637	-4.4589	-4.3575	1.1075	1.0704	1.0346
42	-4.5911	-4.4851	-4.3824	1.1318	1.0935	1.0567
43	-4.6172	-4.5100	-4.4061	1.1557	1.1163	1.0784
44	-4.6422	-4.5337	-4.4287	1.1794	1.1388	1.0999

ϕ_c = 15° (continued)

Table 3

180°-φ°	τ			σ		
	f = 0.01	f = 0.02	f = 0.03	f = 0.01	f = 0.02	f = 0.03
45	1.0342	.9846	.9373	4.9174	4.7979	4.6821
46	1.0355	.9855	.9379	4.9499	4.8287	4.7115
47	1.0368	.9864	.9384	4.9814	4.8587	4.7400
48	1.0380	.9873	.9390	5.0120	4.8879	4.7678
49	1.0393	.9882	.9396	5.0419	4.9163	4.7948
50	1.0405	.9891	.9402	5.0710	4.9440	4.8211
51	1.0418	.9900	.9408	5.0995	4.9710	4.8468
52	1.0430	.9909	.9414	5.1272	4.9974	4.8718
53	1.0442	.9918	.9420	5.1543	5.0231	4.8963
54	1.0454	.9927	.9426	5.1808	5.0483	4.9202
55	1.0467	.9936	.9433	5.2068	5.0730	4.9437
56	1.0479	.9945	.9439	5.2323	5.0971	4.9666
57	1.0490	.9954	.9445	5.2572	5.1208	4.9890
58	1.0502	.9963	.9451	5.2817	5.1440	5.0111
59	1.0514	.9972	.9458	5.3057	5.1668	5.0327
60	1.0526	.9981	.9464	5.3293	5.1892	5.0539
61	1.0538	.9990	.9470	5.3525	5.2112	5.0748
62	1.0549	.9999	.9477	5.3754	5.2329	5.0953
63	1.0561	1.0007	.9483	5.3979	5.2542	5.1155
64	1.0572	1.0016	.9489	5.4201	5.2752	5.1354
65	1.0584	1.0025	.9496	5.4419	5.2959	5.1551
66	1.0595	1.0034	.9502	5.4635	5.3163	5.1744
67	1.0607	1.0043	.9509	5.4848	5.3365	5.1935
68	1.0618	1.0051	.9515	5.5058	5.3564	5.2124
69	1.0629	1.0060	.9522	5.5267	5.3761	5.2310
70	1.0641	1.0069	.9528	5.5472	5.3956	5.2495
71	1.0652	1.0078	.9534	5.5676	5.4149	5.2677
72	1.0663	1.0087	.9541	5.5878	5.4340	5.2858
73	1.0675	1.0095	.9547	5.6078	5.4529	5.3037
74	1.0686	1.0104	.9554	5.6277	5.4717	5.3214
75	1.0697	1.0113	.9560	5.6474	5.4903	5.3390
76	1.0708	1.0121	.9567	5.6669	5.5088	5.3565
77	1.0719	1.0130	.9573	5.6864	5.5272	5.3739
78	1.0730	1.0139	.9580	5.7057	5.5455	5.3912
79	1.0742	1.0148	.9586	5.7250	5.5636	5.4083
80	1.0753	1.0156	.9593	5.7441	5.5817	5.4254
81	1.0764	1.0165	.9599	5.7632	5.5998	5.4424
82	1.0775	1.0174	.9606	5.7822	5.6177	5.4594
83	1.0786	1.0182	.9613	5.8012	5.6356	5.4763
84	1.0797	1.0191	.9619	5.8201	5.6535	5.4932
85	1.0808	1.0200	.9626	5.8390	5.6713	5.5100
86	1.0820	1.0209	.9632	5.8579	5.6892	5.5268
87	1.0831	1.0217	.9639	5.8768	5.7070	5.5436
88	1.0842	1.0226	.9645	5.8957	5.7248	5.5604
89	1.0853	1.0235	.9652	5.9146	5.7426	5.5773
90	1.0864	1.0244	.9659	5.9335	5.7605	5.5941

ϕ_c = 15° (continued)

Table 3

180°-ϕ°	ξ			η		
	f = 0.01	f = 0.02	f = 0.03	f = 0.01	f = 0.02	f = 0.03
45	-4.6660	-4.5564	-4.4503	1.2028	1.1611	1.1211
46	-4.6887	-4.5780	-4.4709	1.2259	1.1831	1.1420
47	-4.7104	-4.5987	-4.4905	1.2487	1.2049	1.1627
48	-4.7311	-4.6184	-4.5093	1.2713	1.2264	1.1832
49	-4.7509	-4.6372	-4.5272	1.2937	1.2476	1.2034
50	-4.7698	-4.6552	-4.5443	1.3159	1.2687	1.2234
51	-4.7879	-4.6724	-4.5606	1.3378	1.2895	1.2432
52	-4.8051	-4.6888	-4.5762	1.3595	1.3102	1.2628
53	-4.8217	-4.7045	-4.5911	1.3810	1.3306	1.2822
54	-4.8374	-4.7194	-4.6053	1.4023	1.3509	1.3015
55	-4.8525	-4.7338	-4.6189	1.4235	1.3709	1.3205
56	-4.8669	-4.7474	-4.6319	1.4444	1.3908	1.3394
57	-4.8807	-4.7605	-4.6443	1.4653	1.4106	1.3581
58	-4.8938	-4.7730	-4.6561	1.4859	1.4302	1.3767
59	-4.9064	-4.7849	-4.6674	1.5064	1.4496	1.3952
60	-4.9184	-4.7963	-4.6782	1.5267	1.4689	1.4135
61	-4.9298	-4.8071	-4.6885	1.5469	1.4880	1.4316
62	-4.9407	-4.8174	-4.6983	1.5670	1.5071	1.4497
63	-4.9511	-4.8273	-4.7076	1.5870	1.5260	1.4676
64	-4.9610	-4.8367	-4.7165	1.6068	1.5448	1.4854
65	-4.9704	-4.8456	-4.7249	1.6265	1.5635	1.5031
66	-4.9794	-4.8540	-4.7330	1.6462	1.5821	1.5207
67	-4.9879	-4.8621	-4.7406	1.6657	1.6006	1.5382
68	-4.9959	-4.8697	-4.7478	1.6851	1.6190	1.5557
69	-5.0035	-4.8769	-4.7546	1.7045	1.6373	1.5730
70	-5.0108	-4.8838	-4.7611	1.7238	1.6555	1.5903
71	-5.0176	-4.8902	-4.7672	1.7430	1.6737	1.6075
72	-5.0240	-4.8963	-4.7729	1.7621	1.6918	1.6246
73	-5.0300	-4.9019	-4.7783	1.7812	1.7099	1.6417
74	-5.0356	-4.9073	-4.7833	1.8003	1.7279	1.6587
75	-5.0409	-4.9123	-4.7880	1.8193	1.7458	1.6757
76	-5.0458	-4.9169	-4.7924	1.8382	1.7637	1.6926
77	-5.0503	-4.9212	-4.7965	1.8571	1.7816	1.7095
78	-5.0545	-4.9251	-4.8002	1.8760	1.7995	1.7264
79	-5.0583	-4.9288	-4.8036	1.8948	1.8173	1.7432
80	-5.0618	-4.9321	-4.8067	1.9137	1.8351	1.7600
81	-5.0650	-4.9350	-4.8095	1.9325	1.8528	1.7768
82	-5.0678	-4.9377	-4.8121	1.9513	1.8706	1.7935
83	-5.0703	-4.9400	-4.8143	1.9701	1.8883	1.8103
84	-5.0724	-4.9420	-4.8162	1.9889	1.9061	1.8271
85	-5.0742	-4.9437	-4.8178	2.0077	1.9239	1.8438
86	-5.0757	-4.9451	-4.8191	2.0265	1.9416	1.8606
87	-5.0769	-4.9462	-4.8201	2.0454	1.9594	1.8774
88	-5.0777	-4.9470	-4.8209	2.0643	1.9772	1.8941
89	-5.0782	-4.9475	-4.8213	2.0832	1.9950	1.9110
90	-5.0783	-4.9476	-4.8215	2.1021	2.0129	1.9278

ϕ_c = 30° (continued)

Table 3

180°-φ°	τ f = 0.01	τ f = 0.02	τ f = 0.03	σ f = 0.01	σ f = 0.02	σ f = 0.03
45	1.1698	1.1475	1.1256	2.0226	2.0027	1.9831
46	1.1755	1.1527	1.1304	2.0513	2.0309	2.0107
47	1.1811	1.1580	1.1353	2.0797	2.0587	2.0380
48	1.1868	1.1633	1.1402	2.1077	2.0862	2.0649
49	1.1925	1.1686	1.1452	2.1353	2.1132	2.0914
50	1.1982	1.1739	1.1501	2.1625	2.1399	2.1176
51	1.2039	1.1793	1.1551	2.1895	2.1663	2.1434
52	1.2097	1.1846	1.1601	2.2160	2.1923	2.1689
53	1.2154	1.1900	1.1651	2.2423	2.2180	2.1941
54	1.2212	1.1954	1.1702	2.2683	2.2435	2.2190
55	1.2270	1.2008	1.1752	2.2941	2.2687	2.2437
56	1.2328	1.2063	1.1803	2.3195	2.2936	2.2681
57	1.2386	1.2117	1.1854	2.3448	2.3183	2.2922
58	1.2445	1.2172	1.1905	2.3698	2.3428	2.3162
59	1.2503	1.2227	1.1956	2.3946	2.3670	2.3399
60	1.2562	1.2282	1.2008	2.4192	2.3911	2.3634
61	1.2621	1.2337	1.2059	2.4436	2.4150	2.3868
62	1.2680	1.2392	1.2111	2.4679	2.4387	2.4099
63	1.2739	1.2448	1.2163	2.4920	2.4622	2.4330
64	1.2799	1.2504	1.2215	2.5160	2.4857	2.4558
65	1.2859	1.2560	1.2268	2.5398	2.5089	2.4786
66	1.2919	1.2616	1.2321	2.5635	2.5321	2.5012
67	1.2979	1.2673	1.2373	2.5872	2.5552	2.5238
68	1.3039	1.2729	1.2427	2.6107	2.5782	2.5462
69	1.3100	1.2786	1.2480	2.6342	2.6011	2.5686
70	1.3161	1.2843	1.2534	2.6576	2.6240	2.5909
71	1.3222	1.2901	1.2588	2.6810	2.6468	2.6132
72	1.3284	1.2959	1.2642	2.7044	2.6696	2.6354
73	1.3346	1.3017	1.2696	2.7277	2.6923	2.6576
74	1.3408	1.3075	1.2751	2.7510	2.7151	2.6797
75	1.3470	1.3134	1.2806	2.7743	2.7378	2.7019
76	1.3533	1.3193	1.2861	2.7977	2.7606	2.7241
77	1.3597	1.3252	1.2917	2.8210	2.7834	2.7463
78	1.3660	1.3312	1.2973	2.8444	2.8062	2.7686
79	1.3724	1.3372	1.3029	2.8679	2.8291	2.7909
80	1.3789	1.3433	1.3086	2.8915	2.8520	2.8132
81	1.3854	1.3494	1.3143	2.9151	2.8750	2.8357
82	1.3919	1.3555	1.3201	2.9389	2.8982	2.8582
83	1.3985	1.3617	1.3259	2.9627	2.9214	2.8808
84	1.4052	1.3679	1.3317	2.9867	2.9447	2.9035
85	1.4118	1.3742	1.3376	3.0108	2.9682	2.9264
86	1.4186	1.3806	1.3435	3.0352	2.9919	2.9494
87	1.4254	1.3869	1.3495	3.0596	3.0157	2.9726
88	1.4323	1.3934	1.3556	3.0843	3.0397	2.9960
89	1.4392	1.3999	1.3616	3.1092	3.0639	3.0195
90	1.4462	1.4064	1.3678	3.1343	3.0884	3.0433

ϕ_c = 30° (continued)

Table 3

180°-ϕ°	ξ			η		
	f = 0.01	f = 0.02	f = 0.03	f = 0.01	f = 0.02	f = 0.03
45	-1.8604	-1.8430	-1.8258	.6584	.6497	.6411
46	-1.8806	-1.8628	-1.8452	.6790	.6698	.6608
47	-1.9001	-1.8819	-1.8640	.6995	.6900	.6806
48	-1.9190	-1.9005	-1.8821	.7202	.7102	.7004
49	-1.9373	-1.9184	-1.8997	.7408	.7305	.7203
50	-1.9550	-1.9357	-1.9167	.7616	.7508	.7402
51	-1.9721	-1.9525	-1.9331	.7823	.7711	.7601
52	-1.9887	-1.9687	-1.9490	.8031	.7915	.7801
53	-2.0047	-1.9844	-1.9643	.8240	.8119	.8001
54	-2.0202	-1.9995	-1.9792	.8449	.8324	.8201
55	-2.0351	-2.0141	-1.9935	.8658	.8529	.8401
56	-2.0495	-2.0283	-2.0073	.8868	.8734	.8603
57	-2.0635	-2.0419	-2.0206	.9079	.8940	.8804
58	-2.0769	-2.0550	-2.0335	.9290	.9147	.9006
59	-2.0899	-2.0677	-2.0459	.9501	.9353	.9208
60	-2.1023	-2.0799	-2.0578	.9713	.9561	.9411
61	-2.1144	-2.0917	-2.0693	.9926	.9769	.9614
62	-2.1259	-2.1030	-2.0804	1.0139	.9977	.9818
63	-2.1371	-2.1139	-2.0910	1.0353	1.0186	1.0022
64	-2.1478	-2.1243	-2.1012	1.0567	1.0395	1.0227
65	-2.1580	-2.1344	-2.1110	1.0783	1.0606	1.0432
66	-2.1679	-2.1440	-2.1204	1.0999	1.0817	1.0638
67	-2.1773	-2.1532	-2.1294	1.1215	1.1028	1.0845
68	-2.1863	-2.1620	-2.1380	1.1433	1.1241	1.1052
69	-2.1949	-2.1704	-2.1462	1.1651	1.1454	1.1260
70	-2.2031	-2.1784	-2.1540	1.1871	1.1668	1.1469
71	-2.2109	-2.1860	-2.1614	1.2091	1.1883	1.1679
72	-2.2183	-2.1932	-2.1685	1.2312	1.2099	1.1890
73	-2.2253	-2.2001	-2.1751	1.2535	1.2316	1.2101
74	-2.2320	-2.2065	-2.1814	1.2758	1.2534	1.2314
75	-2.2382	-2.2126	-2.1874	1.2983	1.2753	1.2528
76	-2.2441	-2.2183	-2.1929	1.3209	1.2974	1.2743
77	-2.2495	-2.2236	-2.1981	1.3436	1.3195	1.2958
78	-2.2546	-2.2285	-2.2029	1.3665	1.3418	1.3176
79	-2.2592	-2.2331	-2.2074	1.3895	1.3642	1.3394
80	-2.2636	-2.2373	-2.2114	1.4127	1.3868	1.3614
81	-2.2674	-2.2411	-2.2151	1.4360	1.4095	1.3835
82	-2.2710	-2.2445	-2.2185	1.4595	1.4324	1.4058
83	-2.2741	-2.2475	-2.2214	1.4831	1.4554	1.4282
84	-2.2768	-2.2502	-2.2240	1.5070	1.4786	1.4508
85	-2.2791	-2.2524	-2.2262	1.5310	1.5020	1.4736
86	-2.2810	-2.2543	-2.2280	1.5552	1.5256	1.4965
87	-2.2825	-2.2557	-2.2294	1.5797	1.5493	1.5197
88	-2.2836	-2.2568	-2.2304	1.6043	1.5733	1.5430
89	-2.2842	-2.2574	-2.2310	1.6292	1.5975	1.5665
90	-2.2845	-2.2576	-2.2313	1.6543	1.6220	1.5903

ϕ_c = 30° (continued)

Table 4

ϕ°	τ			σ		
	f = 0.01	f = 0.02	f = 0.03	f = 0.01	f = 0.02	f = 0.03
65	1.6722	1.6345	1.5975	3.9180	3.8525	3.7883
64	1.6837	1.6461	1.6093	3.9605	3.8940	3.8289
63	1.6955	1.6580	1.6214	4.0043	3.9369	3.8708
62	1.7076	1.6703	1.6339	4.0498	3.9813	3.9143
61	1.7200	1.6829	1.6467	4.0969	4.0274	3.9594
60	1.7328	1.6959	1.6598	4.1459	4.0754	4.0063
59	1.7460	1.7093	1.6735	4.1968	4.1252	4.0551
58	1.7596	1.7232	1.6875	4.2499	4.1772	4.1060
57	1.7736	1.7375	1.7021	4.3054	4.2315	4.1592
56	1.7882	1.7523	1.7171	4.3634	4.2884	4.2149
55	1.8032	1.7677	1.7328	4.4242	4.3479	4.2733
54	1.8189	1.7836	1.7490	4.4880	4.4105	4.3347
53	1.8352	1.8002	1.7660	4.5552	4.4765	4.3993
52	1.8521	1.8176	1.7837	4.6262	4.5461	4.4676
51	1.8698	1.8357	1.8022	4.7012	4.6197	4.5399
50	1.8884	1.8547	1.8216	4.7809	4.6980	4.6168
49	1.9078	1.8746	1.8420	4.8657	4.7813	4.6986
48	1.9283	1.8956	1.8635	4.9563	4.8703	4.7861
47	1.9500	1.9179	1.8863	5.0534	4.9658	4.8800
46	1.9729	1.9415	1.9106	5.1580	5.0687	4.9812
45	1.9973	1.9666	1.9364	5.2711	5.1801	5.0908
44	2.0234	1.9936	1.9642	5.3941	5.3012	5.2102
43	2.0515	2.0226	1.9940	5.5287	5.4338	5.3409
42	2.0819	2.0540	2.0265	5.6768	5.5799	5.4850
41	2.1150	2.0883	2.0619	5.8413	5.7423	5.6452
40	2.1513	2.1260	2.1010	6.0255	5.9242	5.8249
39	2.1917	2.1680	2.1445	6.2341	6.1305	6.0289
38	2.2371	2.2153	2.1937	6.4736	6.3675	6.2634
37	2.2890	2.2695	2.2502	6.7530	6.6444	6.5378
36	2.3495	2.3329	2.3163	7.0862	6.9749	6.8657
35	2.4221	2.4090	2.3960	7.4948	7.3809	7.2693
34	2.5126	2.5043	2.4960	8.0164	7.9003	7.7864
33	2.6323	2.6308	2.6294	8.7238	8.6064	8.4911
32	2.8078	2.8172	2.8267	9.7880	9.6721	9.5583
31	3.1296	3.1614	3.1936	11.7928	11.6905	11.5905

ϕ_c = 35°

ϕ°	τ			σ		
	f = 0.01	f = 0 02	f = 0.03	f = 0.01	f = 0.02	f = 0.03
90	1.6007	1.5639	1.5279	2.7330	2.6988	2.6652
89	1.6123	1.5755	1.5395	2.7611	2.7263	2.6921
88	1.6241	1.5872	1.5512	2.7897	2.7542	2.7194
87	1.6360	1.5992	1.5632	2.8187	2.7826	2.7471
86	1.6481	1.6113	1.5753	2.8482	2.8114	2.7753
85	1.6604	1.6237	1.5877	2.8782	2.8406	2.8040
84	1.6730	1.6362	1.6002	2.9087	2.8706	2.8332
83	1.6857	1.6489	1.6130	2.9398	2.9011	2.8630
82	1.6986	1.6619	1.6260	2.9716	2.9321	2.8933
81	1.7118	1.6752	1.6393	3.0040	2.9638	2.9243

$\phi_c = 30°$ (continued)

Table 4

$\phi°$	ξ			η		
	f = 0.01	f = 0.02	f = 0.03	f = 0.01	f = 0.02	f = 0.03
65	- 2.1023	- 2.0798	- 2.0579	2.4101	2.3589	2.3088
64	- 2.0841	- 2.0619	- 2.0404	2.4484	2.3963	2.3454
63	- 2.0645	- 2.0428	- 2.0217	2.4877	2.4347	2.3829
62	- 2.0435	- 2.0223	- 2.0016	2.5280	2.4741	2.4215
61	- 2.0210	- 2.0002	- 1.9801	2.5694	2.5147	2.4611
60	- 1.9969	- 1.9767	- 1.9570	2.6120	2.5564	2.5020
59	- 1.9710	- 1.9513	- 1.9322	2.6559	2.5993	2.5440
58	- 1.9433	- 1.9242	- 1.9056	2.7012	2.6437	2.5874
57	- 1.9135	- 1.8950	- 1.8770	2.7480	2.6895	2.6323
56	- 1.8815	- 1.8636	- 1.8463	2.7963	2.7369	2.6787
55	- 1.8470	- 1.8299	- 1.8132	2.8464	2.7860	2.7268
54	- 1.8100	- 1.7935	- 1.7776	2.8984	2.8369	2.7768
53	- 1.7700	- 1.7543	- 1.7391	2.9524	2.8899	2.8288
52	- 1.7268	- 1.7119	- 1.6975	3.0087	2.9451	2.8829
51	- 1.6801	- 1.6661	- 1.6525	3.0675	3.0028	2.9395
50	- 1.6294	- 1.6163	- 1.6036	3.1289	3.0631	2.9988
49	- 1.5743	- 1.5622	- 1.5505	3.1934	3.1265	3.0610
48	- 1.5143	- 1.5032	- 1.4925	3.2612	3.1931	3.1265
47	- 1.4486	- 1.4387	- 1.4290	3.3328	3.2636	3.1958
46	- 1.3766	- 1.3678	- 1.3594	3.4087	3.3382	3.2692
45	- 1.2973	- 1.2898	- 1.2825	3.4894	3.4176	3.3474
44	- 1.2096	- 1.2033	- 1.1974	3.5756	3.5025	3.4310
43	- 1.1120	- 1.1071	- 1.1026	3.6682	3.5938	3.5209
42	- 1.0027	- .9994	- .9963	3.7683	3.6925	3.6183
41	- .8796	- .8778	- .8763	3.8772	3.8000	3.7244
40	- .7395	- .7394	- .7396	3.9968	3.9182	3.8411
39	- .5784	- .5802	- .5822	4.1295	4.0493	3.9708
38	- .3910	- .3947	- .3986	4.2785	4.1968	4.1168
37	- .1693	- .1750	- .1809	4.4486	4.3653	4.2838
36	.0986	.0907	.0827	4.6467	4.5619	4.4788
35	.4313	.4213	.4113	4.8838	4.7975	4.7130
34	.8613	.8495	.8376	5.1791	5.0915	5.0057
33	1.4514	1.4384	1.4254	5.5693	5.4809	5.3944
32	2.3493	2.3377	2.3259	6.1404	6.0529	5.9671
31	4.0598	4.0598	4.0599	7.1859	7.1055	7.0269

$\phi_c = 35°$

$\phi°$	ξ			η		
	f = 0.01	f = 0.02	f = 0.03	f = 0.01	f = 0.02	f = 0.03
90	- 1.8838	- 1.8655	- 1.8475	1.5639	1.5384	1.5134
89	- 1.8835	- 1.8652	- 1.8473	1.5920	1.5659	1.5403
88	- 1.8828	- 1.8645	- 1.8466	1.6206	1.5938	1.5676
87	- 1.8815	- 1.8633	- 1.8454	1.6495	1.6221	1.5953
86	- 1.8797	- 1.8615	- 1.8437	1.6790	1.6509	1.6234
85	- 1.8774	- 1.8592	- 1.8414	1.7089	1.6802	1.6520
84	- 1.8744	- 1.8563	- 1.8386	1.7393	1.7099	1.6811
83	- 1.8709	- 1.8529	- 1.8352	1.7702	1.7402	1.7106
82	- 1.8668	- 1.8488	- 1.8313	1.8017	1.7709	1.7408
81	- 1.8620	- 1.8441	- 1.8267	1.8337	1.8023	1.7714

NAVAL SHIP RESEARCH AND DEVELOPMENT CENTER

GENERALIZED HYDRODYNAMIC LOADING FUNCTIONS FOR BARE AND FAIRED CABLES IN TWO-DIMENSIONAL STEADY-STATE CABLE CONFIGURATIONS

by

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Distribution of this document is unlimited

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NOTATION

$A_0, A_1, A_2 \ldots A_n$	Constants
$B_1, B_2 \ldots B_n$	Constants
B	Buoyancy
c	Chord length of faired cable
D_f	Friction drag
D_p	Pressure drag
F	Drag per unit length of cable when cable is parallel to the stream
f	Ratio F/R
H	Total hydrodynamic force
i	Initial point
n	An integer
P	Component of the external forces acting upon an element of cable in the direction of the element
Q	Component of the external forces, acting upon an element of cable, that is in the direction 90 degrees counterclockwise from the direction of the element
R	Drag per unit length of cable when cable is perpendicular to the stream
s	Distance along the cable measured positively in the sense of positive progression along the cable
T	Tension in the cable at an arbitrarily chosen point
t	Maximum thickness of cable fairing
W	Weight in air
w	Net weight, W-B
x, y	Rectangular coordinates of an arbitrarily chosen point on the cable
θ	Angle measured counterclockwise from the direction of motion to the direction of the total hydrodynamic force
ϕ	Angle measured counterclockwise from the direction of motion to the direction of the tangent to the cable at an arbitrarily chosen point on the cable, the direction of the tangent being taken in the sense of increasing s
ϕ_c	Critical angle of the cable, i.e., the value of the angle ϕ obtained when the cable is freely trailed in the stream

ABSTRACT

A generalized expression has been developed for representing the hydrodynamic loading functions for bare and faired towcables in two-dimensional steady flow. The expression is sufficiently general to represent the many loading functions previously published. In practical applications, it can also represent experimental data obtained from tests of both bare and faired cables by using the proper coefficients. Because of its versatility, the form is recommended for use in a digital computer program with the basic differential equations of a flexible cable to solve problems involving the towcable tension and geometry of cable-towed systems.

ADMINISTRATIVE INFORMATION

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INTRODUCTION

The Naval Ship Research and Development Center is engaged in a broad research program directed toward the development of improved experimental and analytical techniques for predicting the steady-state and dynamic characteristics of cable-towed systems. As part of this program, a development effort was initiated to establish a generalized form of the hydrodynamic loading functions for bare and faired towcables in two-dimensional steady flow. These functions are mathematical expressions representing the manner in which components of the hydrodynamic force act on an element of the cable. Once known, they can be used in a digital computer program with the basic differential equations for the equilibrium configuration of a flexible cable in a uniform stream to solve problems involving the towcable tension and geometry of cable-towed systems.

Within the limits of certain simplifying assumptions, the basic differential equations are well known and have been generally accepted and used by most agencies, but various investigators have developed different expressions for the hydrodynamic loading on an element of cable. Some expressions are based on theory, and others are based on limited data available from tests of bare and faired cables in wind tunnels and towing basins. This has led to the accumulation of a dozen or more expressions for the loading functions, since there has been an interest in cable theory for at least fifty years. Although these functions appear to be considerably different, it seemed possible to develop a general mathematical expression which would unify the many previous expressions.

To accomplish this objective, published literature[1-39] was searched for prior expressions of the loading functions, and a generalized mathematical expression was developed which applies to these previous expressions for both bare and faired cables. The application of the generalized form was also checked using existing experimental data for bare and faired cables. Based on these comparisons, the generalized mathematical expression was selected for future use because it has the desirable characteristic of unifying the previous mathematical expressions and also appears to have the ability to represent any future experimental data on loading functions that may be obtained for both bare and faired cables.

The purpose of this report is to summarize the prior expressions for loading functions, to present the generalized form, and to demonstrate the usefulness of the form when applied both to prior assumptions and to future data. For the sake of completeness, the assumptions and derivation of the basic differential equations are repeated in order to show clearly the relationship of the generalized loading functions to the equations. The report also presents conclusions concerning the use of the generalized functions in practical applications, and makes recommendations for the adoption of these functions as a standard method of describing the hydrodynamic loading on bare and faired cables.

CABLE CONFIGURATION EQUATIONS

The basic differential equations of a cable-towed system in two-dimensional steady flow are derived from the equilibrium of external forces acting on an element of cable, and from geometrical considerations. These equations appear frequently in the published literature, but most recently in References 10, 12, 13, and 16. For the sake of completeness, they are derived in detail in Appendix A of this report. The four basic equations are:

$$\frac{dT}{ds} + P(\phi) = 0, \qquad [1]$$

$$T\frac{d\phi}{ds} + Q(\phi) = 0, \qquad [2]$$

$$\frac{dx}{ds} - \cos\phi = 0, \qquad [3]$$

and

$$\frac{dy}{ds} - \sin\phi = 0. \qquad [4]$$

In their derivation, the following assumptions have been made:

1. The free-stream velocity of the flow is steady, uniform, and parallel,

[1] References are listed on page 19.

2. The hydrodynamic force on an element of cable is not affected by the curvature of the cable or the flow at adjacent elements,

3. The cable is inelastic, i.e., it does not change length or diameter with increasing tension,

4. The cable is perfectly flexible; i.e., it cannot support a bending moment, and

5. The entire cable lies in a plane and only constant forces in the same plane are applied to the cable.

It should be noted however that both the total hydrodynamic force (resolved into components normal and tangential to the element) and gravitational and hydrostatic forces (consisting of weight and buoyancy) are included in the formulation.

Once the required characteristics of the cable and the tension and angle at any arbitrary point in the configuration are known, the complete cable configuration and tension can be determined from the equations for any configuration which is physically possible. To help determine whether certain configurations are possible, Appendix B introduces and discusses a convenient artifice called the "cable circle".

For most towed systems, the hydrodynamic characteristics of the towed body can be either calculated or readily obtained using experimental equipment such as the Planar-Motion-Mechanism System or the Mark 1 Measurement System for Cable-Towed Bodies[40]. This information can be used to calculate the tension and angle at one point in the configuration since the towed body is usually an end point. However, only limited data are available concerning the exact magnitude of the hydrodynamic force acting on the element of cable. Consequently, the usual practice is to assume that the force is a function of the angle that the cable makes with respect to the stream. Several attempts have been made by investigators in the past to measure or resolve either the normal and tangential component or the lift and drag of an element of cable. They have been hampered by a variety of difficulties; inadequate instrumentation to measure the small tangential force; mounting techniques to avoid gap effects, end effects, etc.; and methods of resolving two-dimensional data from tests of three-dimensional models.

Because of the aforementioned uncertainties, it is not too surprising that a number of expressions representing the hydrodynamic loading on an element of cable have been developed by numerous investigators.

SUMMARY OF PRIOR LOADING FUNCTIONS

Because of the form of the basic differential equations as presented in this report, it is most convenient to consider the hydrodynamic loading functions based on the normal and tangential components of the total hydrodynamic force. Throughout this report, the following definitions of these hydrodynamic loading functions are used.

Normal loading function - the ratio of the steady-state, two-dimensional, component of hydrodynamic force perpendicular to the element of cable at angle ϕ to the drag when the element is at an angle ϕ of 90 degrees.

Tangential loading function - the ratio of the component of force parallel to the element at angle ϕ to the drag when the element is at an angle ϕ of 90 degrees.

For a given cable geometry and speed, it is assumed that these loading functions depend only on the cable angle.

Tables 1 and 2 itemize the loading functions assumed by various investigators for bare and faired cables, respectively. For purposes of comparison, the functions are presented in terms of the normal and tangential components although the original work may have used different components such as lift and drag. Inspection of the tables shows that most investigators have assumed that the normal loading function depends on the cable angle ϕ; whereas the tangential loading function variously has been assumed to be zero, independent of the cable angle, or dependent on the cable angle.

GENERALIZED LOADING FUNCTIONS

In spite of the apparent diversity in the prior functions, the expressions have a certain similar quality in that they are all related to a particular trigonometric expression. This similarity can be used to unify the past expressions with a generalized form.

The generalized mathematical form selected to describe the loading functions is the first few terms of the following trigonometric series:

$$S(\phi) = A_0 + \sum_{n=1}^{\infty} (A_n \cos n\phi + B_n \sin n\phi). \qquad [5]$$

With the proper choice of coefficients, Equation [5] can be used to represent both the normal and tangential hydrodynamic loading for bare and faired cables.

For example, consider the widely used functions of Reference 10 (Pode) where the normal loading function is given by $\sin^2\phi$ and the tangential loading function is given by f (a constant). From the trigonometric identity,

$$\cos 2x = 1-2 \sin^2 x,$$

TABLE 1

Hydrodynamic Loading Functions for Bare Cables

Item	Normal	Tangential	Investigator	Date	Reference
1	$\sin^2\phi$	"Small and nearly constant" (Data points given)	Relf and Powell	1917	1
2	$\sin^2\phi$	0	McLeod	1918	2
3	$\sin^2\phi$	0	Glauert	1934	3
4	$\sin^2\phi$	f (a constant)	Thews and Landweber	1936	4 & 5
5	$\sin^2\phi$	$\cos^2\phi$	Richtmyer	1941	6
6	$\sin^2\phi$	0	Reber	1942	7
7	$\sin^2\phi$	$\cos^2\phi$	Reber	1942	7
8	$\sin^2\phi + 0.022 \sin\phi$	$0.022 \cos\phi$	Reber	1944	8
9	$\sin^2\phi$	$f \cos\phi$	O'Hara	1945	9
10	$\sin^2\phi$	f (a constant)	Pode	1951	10
11	$\frac{D_p}{R} \sin^2\phi + \frac{D_f}{R} \sin\phi$	$\frac{D_f}{R} \cos\phi$	Hoerner	1951 1958	11
12	$0.98 \sin^2\phi + 0.02 \sin\phi$	$0.02 \cos\phi$	Eames	1956	12
13	$\sin^2\phi$	$0.083 \cos\phi - 0.035 \cos^2\phi$	Whicker, based on data by Relf and Powell	1957 1917	13 1

TABLE 2

Hydrodynamic Loading Functions for Faired Cables

Item	Normal	Tangential	Investigator	Date	Reference
14	$\frac{D_2}{R} \sin^2\phi + \frac{D_f}{R} \sin \phi$	$\frac{D_f}{R} \cos \phi$	Eames	1956	12
15	$\frac{t}{c} \sin^2\phi + (1 - \frac{t}{c}) \sin \phi$	$(0.386 - 0.303 \frac{t}{c}) \cos \phi$ $- (0.055 - 0.020 \frac{t}{c}) \cos^2\phi$	Whicker, based on data by Powell	1957 1919	13 14
16	$f_1 (\frac{t}{c}) \sin^2\phi + f_2 (\frac{t}{c}) \sin \phi$	$f_3 (\frac{t}{c}) \cos^2\phi + f_4 (\frac{t}{c}) \cos \phi$	Lofft	1958	15
17	Calculated from section series data	Calculated from section series data	Clark	1963	16

by rearranging terms,

$$\sin^2 x = 0.5 - 0.5 \cos 2x.$$

In terms of the coefficients of the generalized loading function, Equation [5], the coefficients for the normal loading function are

$$A_0 = 0.5,$$

$$A_2 = -0.5,$$

and all other coefficients are zero. For the tangential loading function,

$$A_0 = f$$

and all other coefficients are zero.

As another example, consider the functions of Reference 11 (Hoerner) which are listed in Table 1 for sub-critical Reynolds numbers. The normal loading function $\frac{D_p}{R} \sin^2\phi + \frac{D_f}{R} \sin\phi$ can be rearranged as above to get $\frac{D_p}{2R} + \frac{D_f}{R} \sin\phi - \frac{D_p}{2R} \cos 2\phi$. In this case, the coefficients of the generalized normal loading function are

$$A_0 = \frac{D_p}{2R},$$

$$B_1 = \frac{D_f}{R},$$

$$A_2 = -\frac{D_p}{2R},$$

and all other coefficients are zero. For the tangential loading function $\frac{D_f}{R} \cos\phi$,

$$A_1 = \frac{D_f}{R},$$

and all other coefficients are zero. It should be noted in this formulation that the exact values of the functions depend on Reynolds number because skin friction drag is involved. Nevertheless, the coefficients of the generalized loading functions can be computed for the desired Reynolds number and then used directly with the basic differential equations, Equations [1] through [4].

By similar analyses, all loading functions assumed by previous investigators listed in Tables 1 and 2 can be represented by the generalized form by the proper selection of coefficients. For future reference, the coefficients are itemized in Table 3. Items refer to Tables 1 and 2.

In cases where purely experimental data are available, the generalized form can also be used to represent the data and thus provide a mathematical functional relationship to be used in the basic differential equations.

TABLE 3

Coefficients for Generalized Loading Functions

Item	Normal					Tangential				
	A_0	A_1	B_1	A_2	B_2	A_0	A_1	B_1	A_2	B_2
1	0.5	0	0	-0.5	0	-	-	-	-	-
2, 3, 6	0.5	0	0	-0.5	0	0	0	0	0	0
4, 10	0.5	0	0	-0.5	0	f	0	0	0	0
5, 7	0.5	0	0	-0.5	0	0.5	0	0	0.5	0
8	0.5	0	0.022	-0.5	0	0	0.022	0	0	0
9	0.5	0	0	-0.5	0	0	f	0	0	0
11, 14	$\frac{D_p}{2R}$	0	$\frac{D_f}{R}$	$-\frac{D_p}{2R}$	0	0	$\frac{D_f}{R}$	0	0	0
12	0.49	0	0.02	-0.49	0	0	0.02	0	0	0
13	0.5	0	0	-0.5	0	-0.0175	0.083	0	-0.0175	0
15	$\frac{t}{2c}$	0	$1 - \frac{t}{c}$	$-\frac{t}{2c}$	0	$-0.0275 + 0.01\frac{t}{c}$	$0.386 - 0.303\frac{t}{c}$	0	$-0.0275 + 0.01\frac{t}{c}$	0
16	$0.5\, f_1(\frac{t}{c})$	0	$f_2(\frac{t}{c})$	$-0.5\, f_1(\frac{t}{c})$	0	$0.5\, f_3(\frac{t}{c})$	$f_4(\frac{t}{c})$	0	$0.5\, f_3(\frac{t}{c})$	0

The procedure at the Center is to use an IBM 7090 computer and a computer program developed by the Applied Mathematics Laboratory. This program, designated as BVPDE-3, determines a least-squares fit to the data for the first few terms of a prescribed trigonometric series with appropriate constraints for cable angles of 0 and 90 degrees.

Figure 1 shows experimental data taken from Reference 1 (Relf and Powell) in terms of the tangential loading on bare cables. The data have been fitted with the generalized loading function $0.0307 \cos \phi + 0.0003 \cos 2\phi + 0.0222 \sin 2\phi$. For comparison, the loading function proposed in Reference 13 (Whicker) for these data is also shown. It can be seen that, within the scatter of the data, the agreement between the generalized loading function and the data points is excellent, particularly at the lower angles where the curve of Reference 13 diverges from the data.

Figure 2 shows experimental data from Hydromechanics Laboratory Test Report 210-H-01 on the normal loading function for the DTMB B-5 trailing fairing and the generalized loading function $-1.0640 + 1.2633 \cos \phi + 1.8647 \sin \phi - 0.1993 \cos 2\phi - 0.6926 \sin 2\phi$. For comparison, the loading function proposed in Reference 12 (Eames) is also shown, based on coefficients obtained from the test results of the B-5 fairing and calculations of the friction drag associated with the fairing. The generalized function is seen to be in excellent agreement with the data points, where the function of Reference 12 is a maximum of 8 percent higher than the data points in the range of cable angles from 30 to 90 degrees.

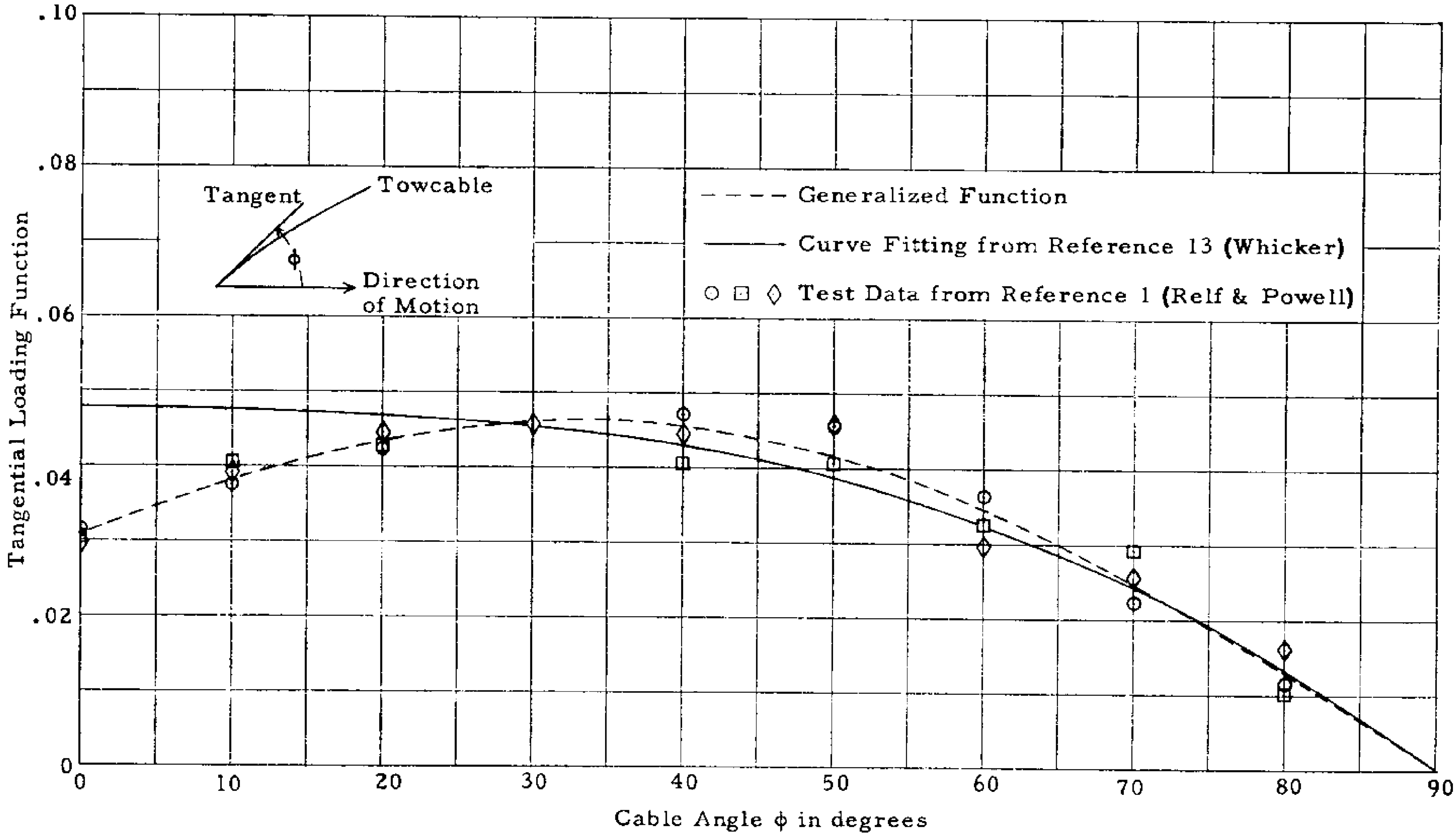

Figure 1 - Tangential Loading Function

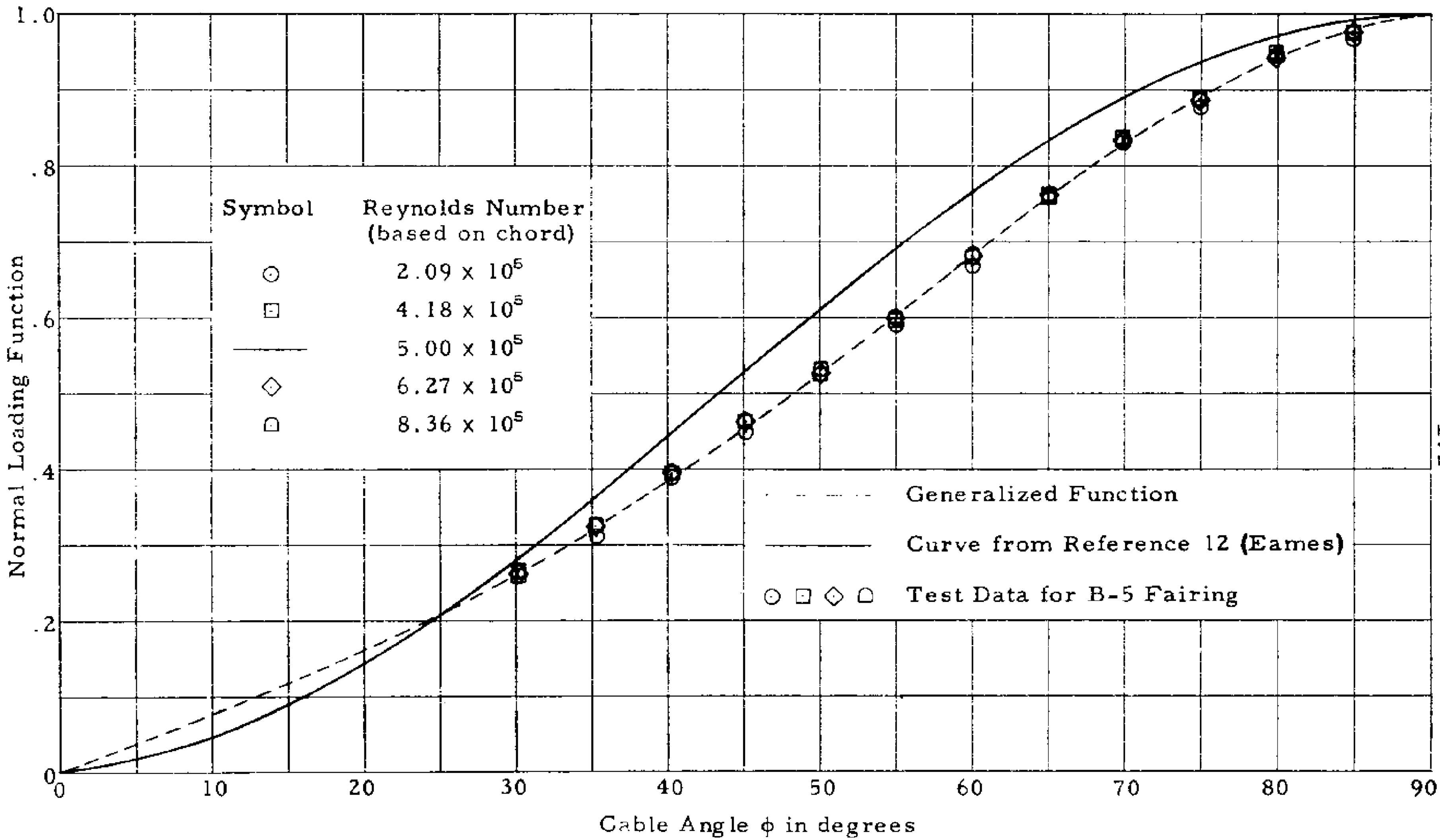

Figure 2 - Normal Loading Function

CONCLUSIONS AND RECOMMENDATIONS

Based on an analysis of existing expressions for loading functions and on studies of limited experimental data, a trigonometric expression has been selected for representing the hydrodynamic loading functions for an element of cable. It is concluded that the expression is sufficiently general to represent many previously assumed loading functions; and can also represent, with the proper coefficients, any theoretical or experimental data which may become available in the future. In practical applications, two terms ($n = 2$) or fewer of the trigonometric series have been adequate to give excellent agreement with experimental data obtained from tests of both bare and faired cables. Additional terms can be used if required for future data.

In view of the versatility of the generalized form, it is recommended that this form be adopted as a standard method for describing the hydrodynamic loading functions for bare and faired cables.

ACKNOWLEDGMENTS

The author wishes to express his sincere appreciation to Dr. Elizabeth H. Cuthill of the Applied Mathematics Laboratory for her contributions and assistance in selecting the trigonometric series as the generalized form for the loading functions and in generating the least-square curves from the experimental data points.

APPENDIX A

DERIVATION OF CABLE CONFIGURATION EQUATIONS

The basic differential equations of a cable-towed system in two-dimensional steady flow are derived from the equilibrium of external forces acting on an element of cable, and from geometrical considerations. Figure 3 shows a free-body diagram consisting of an element of cable of length ds being acted upon by hydrodynamic, hydrostatic, and gravitational forces and by a tension force applied in the plane of the cable.

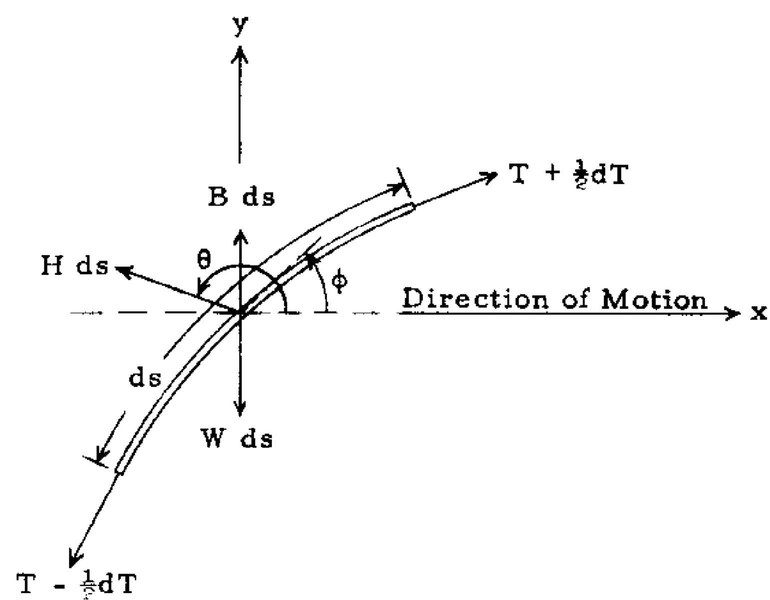

Figure 3 - Sketch of Forces Acting on an Element of Cable

The hydrodynamic force H ds can be resolved into components normal and tangential to the element; $H \sin(\theta - \phi)\, ds$ and $H \cos(\theta - \phi)\, ds$, respectively. These are the components on which the normal and tangential hydrodynamic loading functions are based. The weight W ds and buoyancy B ds can be added together and resolved into normal and tangential components; $(W-B) \cos \phi\, ds$ and $(W-B) \sin \phi\, ds$, respectively. Following the notation and sign convention of Reference 10, the sum of these components normal to the cable is

$$Q(\phi)\, ds = H \sin(\theta - \phi)\, ds - (W-B) \cos\, ds$$

and tangential to the cable is

$$P(\phi) = -H \cos(\theta - \phi)\, ds - (W-B) \sin \phi\, ds$$

as shown in Figure 4.

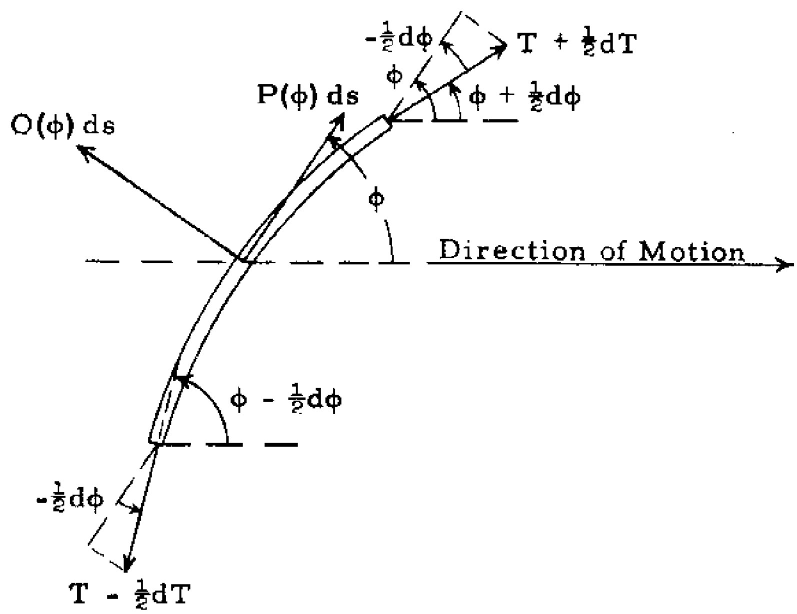

Figure 4 - Sketch of Components of Forces Acting on an Element of Cable

It should be noted that the positive direction of $d\phi$ is opposite to the positive direction of ϕ. For equilibrium, the sum of the normal and tangential components must be zero. In the P-Q plane, the sum of the tangential components is

$$P(\phi)\,ds + (T + \tfrac{1}{2}d\phi)\cos(\tfrac{1}{2}d\phi) + (T - \tfrac{1}{2}dT)\cos(\pi - \tfrac{1}{2}d\phi) = 0.$$

Since

$$\cos(\pi - \tfrac{1}{2}d\phi) = -\cos(\tfrac{1}{2}d\phi),$$

then

$$P(\phi)\,ds + dT\cos\tfrac{1}{2}d\phi = 0.$$

But $\cos x \simeq 1$ for small angles, so

$$\frac{dT}{ds} + P(\phi) = 0. \qquad [1]$$

The sum of the normal components is

$$Q(\phi)\,ds + (T + \tfrac{1}{2}dT)\sin(\tfrac{1}{2}d\phi) + (T - \tfrac{1}{2}dT)\sin(\pi - \tfrac{1}{2}d\phi) = 0.$$

Since

$$\sin(\pi - \tfrac{1}{2}d\phi) = \sin(\tfrac{1}{2}d\phi),$$

then

$$Q(\phi)\, ds + 2T \sin(\tfrac{1}{2}d\phi) = 0.$$

But $\sin x \simeq x$ in radians for small angles, so

$$Q(\phi)\, ds + T d\phi = 0,$$

or

$$T \frac{d\phi}{ds} + Q(\phi) = 0. \qquad [2]$$

In the x-y plane,

$$\cos\phi = \frac{dx}{ds}$$

or

$$\frac{dx}{ds} - \cos\phi = 0. \qquad [3]$$

Also,

$$\sin\phi = \frac{dy}{ds},$$

or

$$\frac{dy}{ds} - \sin\phi = 0. \qquad [4]$$

APPENDIX B

THE CABLE CIRCLE AND SIGN CONVENTION

In Reference 10, it was stated "that any section of a known cable configuration is also the solution of a cable problem". By a sort of reverse reasoning, one could imagine that if all the solutions to all cable problems were added together, the result might be the most general cable configuration, of which all cable problems are a part. Since cable angles between 0 and 360 degrees are all physically possible in one cable configuration or another, a convenient artifice is to think of the most general cable configuration as a circle.

Figure 5 shows the "cable circle" and sign convention. The cable is considered to lie along the circumference of the circle, although it should be realized that the radius of curvature is not constant in a real cable configuration. The cable angles ϕ are listed around the circumference of the circle starting with $\phi = 0$ at the top. The positive direction of the cable scope s is clockwise, and the cable angle ϕ is measured counterclockwise from the direction of motion to the positive tangent to the cable. The origin of the x-y coordinate system is located at the initial point i on the circumference of the circle. The notation near the center of the circle indicates that, at angles between 0 and 180 degrees (Quadrant 1 and Quadrant 2), all values of w = W-B including zero are possible. Between 180 and 270 degrees (Quadrant 3) only positive w's are permitted; and between 270 and 360 degrees (Quadrant 4), only negative w's. To determine in which quadrant a cable configuration occurs, the cable angle at some point in the configuration must be compared with the critical angle of the cable as explained in Reference 10.

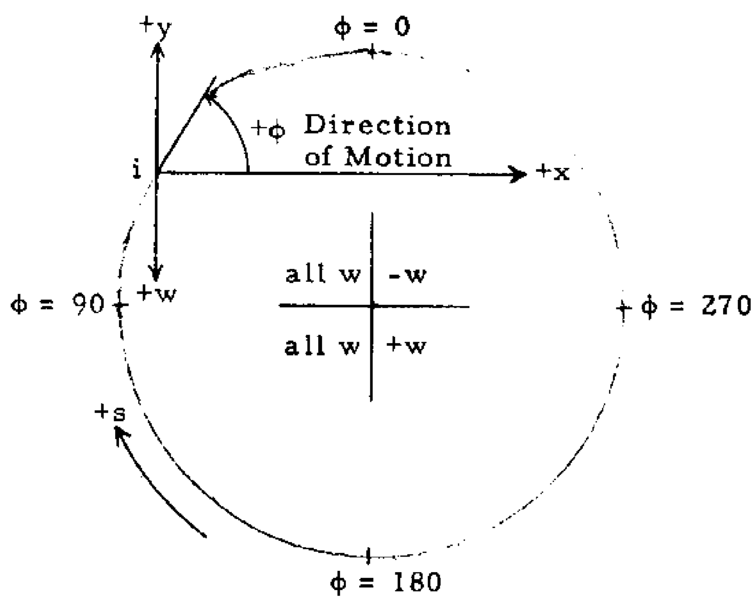

Figure 5 - Sketch Showing the Cable Circle and Sign Convention

To illustrate the relationship between the cable circle and some actual cable configurations, Figure 6 shows a number of sketches of cable configurations, some of which frequently occur in towed systems, each with a cable circle marked to indicate the section of the configuration. The direction of motion of the system is to the right in the cases of ship towing, and the direction of flow is to the left in the cases of moored cables. Figure 6a shows a surface ship towing a body in Quadrant 1 as is typical of present VDS systems. Figure 6b shows the Quadrant 2 configuration of a buoy towed by a submarine. Figure 6c combines Quadrant 2 and Quadrant 3 in the case of a surface ship towing a barge. Figure 6d shows a surface ship towing a submerged fuel tank in Quadrant 3. Figure 6e is the case of a surface buoy which is moored with a buoyant cable. Figure 6f combines Quadrant 4 and Quadrant 1 in the case of a buoyant cable with both ends attached to the bottom. Whether a particular configuration falls in Quadrant 1 (Figure 6a) or Quadrant 3 (Figure 6d), for example, is usually determined by comparing the cable angle at the body with the critical angle ϕ_0 of the cable. If it is known, however, that w has a negative sign (positively buoyant cable), then the configuration can only be in Quadrant 1 since negative w's are not permitted in Quadrant 3. Similarly, in the cases of Quadrant 2 (Figure 6b) and Quadrant 4 (Figure 6e), if the cable weight w has a positive sign (negatively buoyant cable), then the configuration can only be in Quadrant 2 since positive w's are not permitted in Quadrant 4.

Thus, the cable circle is a valuable aid in visualizing various towing configurations and in determining in some cases whether the configurations are physically possible.

It should be remembered that the cable tension and angle at some point in the configuration must be known before the basic differential equations can be solved for the complete configuration and tensions. But given the tension and angle at some initial point i, the integration can proceed either in the direction of positive scope s or in the direction of negative scope. For example, in the case of the surface ship towing the barge (Figure 6c), the integration could start at the towing ship and proceed toward the barge (positive s direction), or it could start at the barge and proceed toward the towing ship (negative s direction), depending on the location of the initial point at which the conditions are known. The calculated configuration and tension are the same, of course, regardless of the direction of integration.

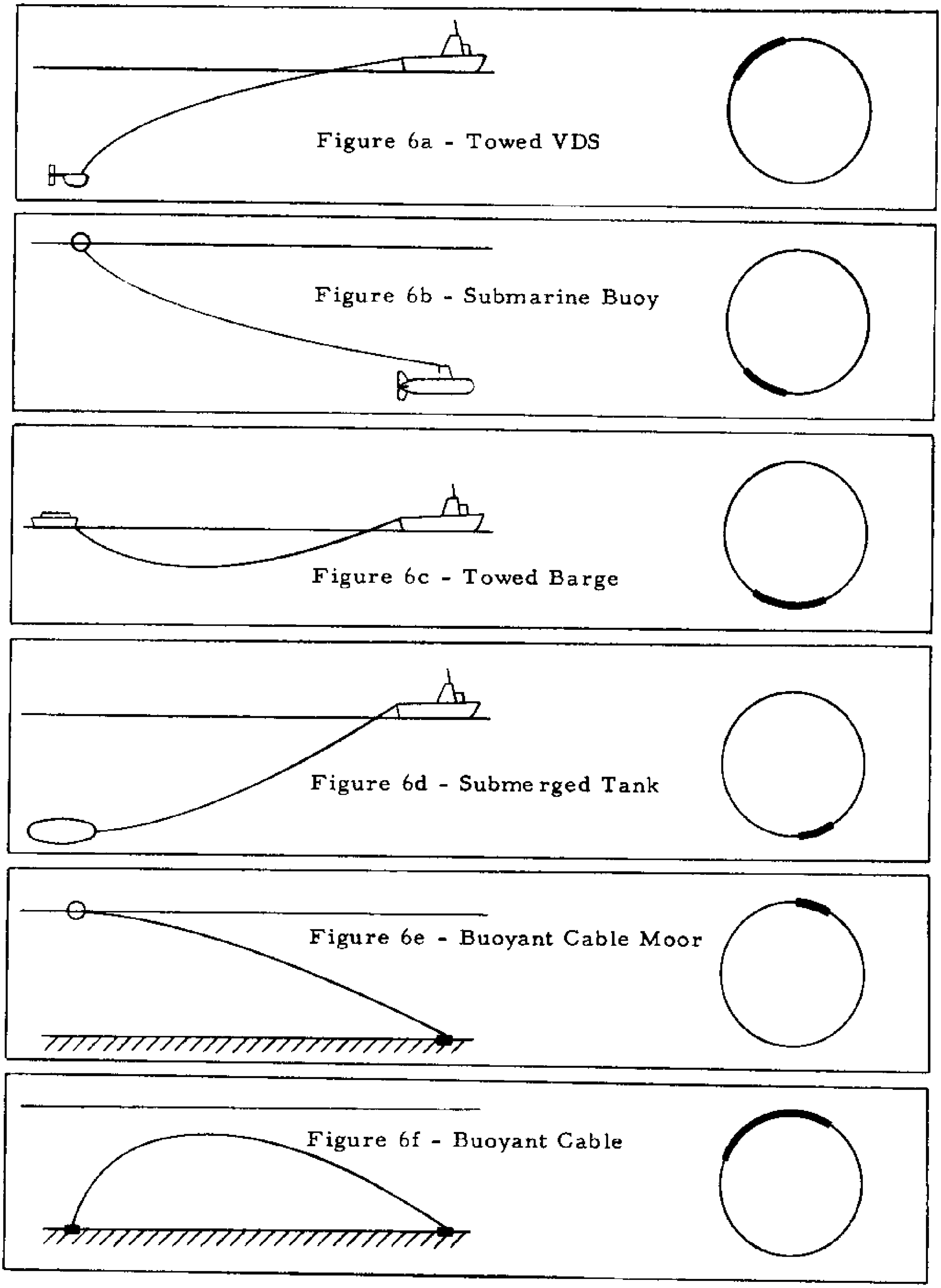

Figure 6 - Typical Cable Configurations

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CASE STUDIES OF TYPICAL OCEAN VEHICLES

Throughout the text certain analysis techniques, mathematical modelling, and solution procedures for vehicle motion response to the sea environment has been presented for the ultimate purpose of evaluating the capability of a vehicle or system to accomplish certain missions at sea. This pertains to both the determination of the level of "operability" of a vehicle or system already built and operating, and the evaluating of the "operability" of proposed designs and/or alternate systems for determining the best design, (also whether a given design can be expected to operate in an acceptable manner at sea).

The "case study" approach essentially demonstrates how the methods of analysis have been or can be applied to evaluate the motion responses of a broad variety of existing ocean vehicle types. For each vehicle type, there are certain mission requirements which predominate. Once these missions are defined, those motion responses which are critical to the vehicle operations can be established fairly well. This in turn sets the stage for determining what properties of the sea environment (the environment in a geographical and seasonal sense as determined by the mission) are important in exciting the critical motion response. As a result, the excitation can be formulated, the critical responses can be solved for, and the "operability" of the vehicle can be established based on "levels of acceptability" of these responses or by correlating the magnitude of the responses with the degradation in the probability of accomplishing the mission. With this experience from these case studies, it is expected that the student will be in a position to intelligently go about the analysis and evaluation of existing vehicle types in different operational and/or environmental conditions and of new vehicle types which are yet to be designed for accomplishing newly devised missions.

First, a vehicle is typed such as fishing trawler, aircraft carrier, etc., followed by some discussion of the major mission requirements. These steps are then followed in proper sequence by a determination of which responses are critical to the operation, what particular property of the ocean environment is involved in the excitation. This is followed by discussion of the form of the particular linearized equations, the physical nature of and the methods for determining the hydrodynamic coefficients involved, the sensitivity of the coefficients to environmental factors, etc.

An "agenda" for class discussion for the "case study" of a particular vehicle type is presented here. Where vehicle type appears, one can substitute the following particular vehicles chosen for class discussion, as can be seen from the proposed lecture schedule given in Appendix IV.

Torpedo
Submarine and Submersibles
Commerical Displacement Ship
Military Displacement Ships, Carrier and Destroyer
Flip Ship
Free Falling Body (instrument package)
Hydrofoil Boat
Air Cushion Vehicles
Oil Drilling Platform
Catamarans
Fishing Trawlers
Deep Submergence Rescue Vehicle
Towed Oceanographic Instrument Package

Tethered Sonar System
Free Floating and Moored Surface Buoys
Bottom Moored Buoys

No attempt will be made in the text to cover the details of the discussion of each of the above vehicles. The job would require very extensive text with the devotion of excessive time on the part of the author. To fill in the discussion "agenda", in a significantly meaningful way with particular information for each vehicle type, is also beyond the capacity (time and funds) of the intended initial incomplete form of the text. The "agenda" followed in the "case study" discussion is given below:

1. Vehicle or system type
2. General aspect of missions
   a. Primary
   b. Secondary
3. Establishing of operational conditions and requirements.
   a. Vehicle requirements such as speed, maneuverability, seakeeping, etc. for missions and "adjacent" conditions.
   b. Determination of the ocean environment involved in mission and "adjacent" conditions.
4. Evaluation of motion and associated responses which may affect the operations in mission and "adjacent" conditions.
5. Evaluation of the environmental conditions to determine what properties of the environment may excite the pertinent motions and associated responses, or may produce restraints.
   a. Surface gravity waves with its associated subsurface orbital velocities and accelerations.
   b. Water currents and turbulence, either arising from general ocean environment or resulting from proximity to other vehicles in the system.

   c. Wind
   d. Shallow water and restricted waterways
   e. Etc.

6. Determination whether the problem can be reduced from the general six degrees of freedom to some more restricted situation such as horizontal plane and vertical plane.
7. Determination of the proper equations of motion.
   a. Linearized equations involving the pertinent hydrodynamic coefficients and linearized excitation terms.
   b. If prior analysis indicates linearized equations are not valid, then formulation of a proper non-linear model should be attempted.
8. Evaluation of the hydrodynamic coefficients and excitation involved.
   a. Are the coefficients affected by environmental conditions such as:
      i. On or near the free surface where gravity wave, cavitation and aeration, surface tension, etc. can affect the value of the coefficients.
      ii. In proximity to a boundary - such as in shallow water, canal, in replenishment formation, which can affect the hydrodynamic coefficients.
   b. Analyze the hydrodynamic situation and determine what forces arise from hydrostatics, free surface effects, vicosity (friction and separation), and circulation phenomena. Are the coefficients frequency dependent?

c. Application of the various theoretical means for calculating the hydrodynamic coefficients and excitation.
   i. How valid are the theoretical methods for this vehicle in the environment?
   ii. Consider recourse to published data on tests of model series or specific model tests on the vehicle model.

9. Solution of the linearized equations for the motion and associated critical responses.
   a. Discuss the level of stability in the various motion responses.
   b. Determine approximate acceptable range of linearity for the particular situation.
   c. Obtain the curves of response operators for each of the motion and associated responses which possibly affect operations.
   d. Evaluate the environment involved and obtain "excitation" spectra corresponding to the various environmental situations encountered in operations.
   e. Obtain response spectra.
   f. Determine the various "statistical" quantities and probabilities of response occurrence.

10. Evaluation of the affect of these responses on operations.
   a. What levels of acceptability have been established?
   b. Determine level of degradation in operations and in what sea environments operations can proceed, be of reduced effectiveness, or must cease.

11. Evaluation of whether additional motion capabilities are necessary.

12. Determination of whether improvements are best made through:
   a. Change in the inherent characteristics of the vehicle

such as addition of stabilizing surfaces, increase in rudder size, addition of fixed appendages such as bilge keels, skegs, etc., or a major design change.

b. Adoption of automatic controls to introduce necessary new sensitivities and/or to improve inherent sensitivities.
    i. Determination of the number and type.
    ii. Evaluation of the effect of control constants and time lags.
c. Both inherent changes and automatic controls.

13. Evaluation of critical motion responses with automatic controls.
    a. Determination of response operators with the various controls.
    b. Obtain "statistical" quantitative information on the various responses.
14. Evaluate "operational performance",
    a. Is the predicted performance in the environment acceptable?
    b. Under what sea conditions can successful (acceptable) operations take place?
15. Consideration of survivability of the vehicle in severe ocean situations.
16. Summary of the vehicle "characteristics" and operational aspects.
    a. Why the basic shape -
    b. Why the various appendages -
    c. Why the particular selection of control effectors -
    d. Why the level of sophistication in the control system -
    e. Why the "levels of acceptability" on critical responses -
    f. What are the realistic implications of serious degradation in "operability" -

g. What tradeoffs exist in establishing the levels of acceptability. For example, accepting slamming damage (involving repair costs) in order to maintain higher sea speed (more sailings per year and more income.)

APPENDIX I

LECTURES ON SHIP HYDRODYNAMICS
STEERING AND MANEUVERABILITY

by

Martin A. Abkowitz

Report No. Hy-5 May 1964

Hydrodynamics Department
Hydro-og Aerodynamisk Laboratorium
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CHAPTER I

Equations of Motion for a Body Moving With Six Degrees of Freedom

In the study of ship motions, on which subject we are about to embark, parameters associated with the body motion become important - such as components of the linear velocity in addition to the forward velocity, components of angular velocity, and various accelerations both linear and angular. The general field of ship motions is usually divided into the areas of a) steering and maneuverability and b) seakeeping, both areas being concerned with the concepts of motion stability and control.

a) Steering and maneuverability usually deal with the motion of a ship in the absence of excitation from the sea (calm water). The motion results from the excitation forces applied through the deflecting of control surfaces.

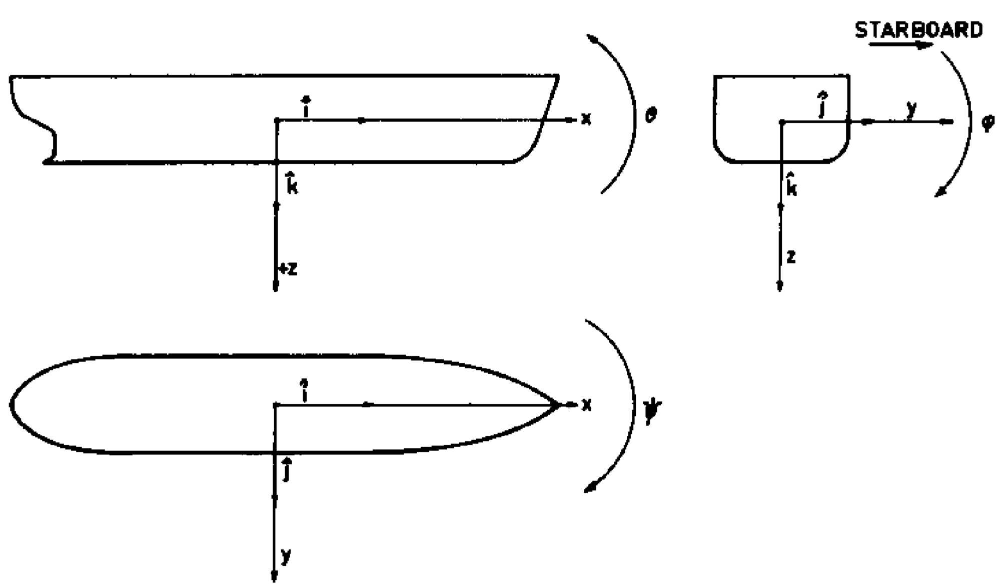
b) Seakeeping deals with the motion of a ship resulting from the excitation forces of the sea (such as waves). When control surfaces are used either to counter the sea excitation or to effect a maneuver in the presence of the sea excitation, then there is a combination of the two areas referred to as maneuvering in a seaway.

c) Motion stability deals with the aspects of ship motion in the absence of any excitation either from control surface deflection or from the seaway.

d) Motion control deals with the effects of the forces excited on the ship through manual or automatic application of control surfaces or other devices.

A ship at sea, or a body moving in a fluid, is allowed to move, and many times does move, in all the six degrees of freedom of motion -

i.e. translation along three orthogonal axes and rotating about each of the three axes. It is therefore necessary to choose an axis system to describe these motion freedoms and the choice should be one which is most convenient for the development of the motion analysis. Practically all vehicles and fluid dynamic bodies have a plane of symmetry - i.e. the centerline plane - since the port and starboard have the same geometry and represent reflections of each other in the centerline plane. This symmetry in body shape can be observed in ships, submarines, rockets, boats, torpedoes, hydrofoil boats, airplanes, dirigibles, fish, birds, etc. (Some asymmetry may be caused by a preferred direction of propeller rotation on a single screw ship but this slight diversion can be readily handled). An axis system which takes advantage of this plane symmetry is chosen. Hence, two of the three axes are in the plane of symmetry (and define the plane) and the third is perpendicular to the plane. Some bodies, such as rockets, and torpedoes, have a second plane of symmetry, where the upper and lower halves (keel and deck) are symmetrical, and this plane of symmetry is perpendicular to the other plane of symmetry. Axes, at least two of them, in the plane of symmetry are chosen, because the expressions for the hydrodynamic forces are simplified through symmetry and the equations of motion are simplified through the fact that axes oriented by symmetry are usually parallel to principal axes of inertia. The sketch below defines the axis system chosen.



x-axis = longitudinal axis in the plane of symmetry positive forward. Usually parallel to the keel or calm water line. If upper and lower half are symmetrical then the axis is the intersection of the two planes of symmetry. A unit vector along the x-axis is designated by $\hat{i}$.

y-axis = transverse axis, perpendicular to the plane of symmetry, positive to starboard. A unit vector along the y-axis is designated by $\hat{j}$.

z-axis = 'vertical axis', (perpendicular to water line planes), in the plane of symmetry, positive downward towards the keel. A unit vector along the z-axis is designated by $\hat{k}$.

These axes form a consistent right-handed coordinate system. A clockwise rotation, looking in the direction of the positive axis, would advance a right hand thread along the positive axis. Positive rotation about the x-axis tends to rotate the y-axis in the direction of the z-axis, positive rotation about the y-axis tends to rotate the z-axis towards the x-axis, and positive rotation about the z-axis tends to rotate the x-axis towards the y-axis. If φ is the roll angle, θ is the pitch angle, and ψ the yaw angle, then positive rotations are indicated in the sketch above. A consistent set of axes furnishes the convenience of being able to derive the remaining two components of a vector quantity from a general expression of the component along one of the axes, as will be demonstrated later.

In dealing with ship motion one needs to exploit the fundamentals of rigid body dynamics in order to develop the analysis. Hence, one begins with Newton's laws of motion, expressed as follows:

$$\vec{F} = \frac{d}{dt}\,(\overrightarrow{\text{Momentum}})$$

$$\vec{\mathcal{M}} = \frac{d}{dt}\,(\overrightarrow{\text{Angular momentum}})$$

$\vec{F}$ is the vector force acting on a body. The components of this force along the x, y, and z axes are X, Y, and Z respectively.

$$\vec{F} = \hat{i}X + \hat{j}Y + \hat{k}Z$$

$\vec{\mathcal{M}}$ is the vector moment acting on the body. The components along the x, y, and z axes are K, M, and N respectively.
$\vec{\mathcal{M}} = \hat{i}K + \hat{j}M + \hat{k}N.$

The origin for the axis system is taken at the center of gravity, G; this is necessary in order to write Newton's law in the form of separate force and moment equations. In addition, the axes are assumed to be the principal axes of inertia through the origin at G, thereby simplifying the momentum expressions. The force expression is written as

$$\vec{F} = \frac{d}{dt}(\overrightarrow{mU}) = m\frac{d\vec{U}}{dt} + \vec{U}\frac{dm}{dt} = m\dot{\vec{U}} + \vec{U}\frac{dm}{dt}$$

where $\vec{U}$ is the linear velocity vector, having components of u, v, w along the x, y, z axes respectively.
$\vec{U} = \hat{i}u + \hat{j}v + \hat{k}w$

m is the mass of the body.

$\frac{d}{dt}$ is the derivative with respect to time. The usual convention of denoting this derivative by a dot over the quantity is used, i.e. $\dot{\vec{U}} = \frac{d}{dt}(\vec{U})$

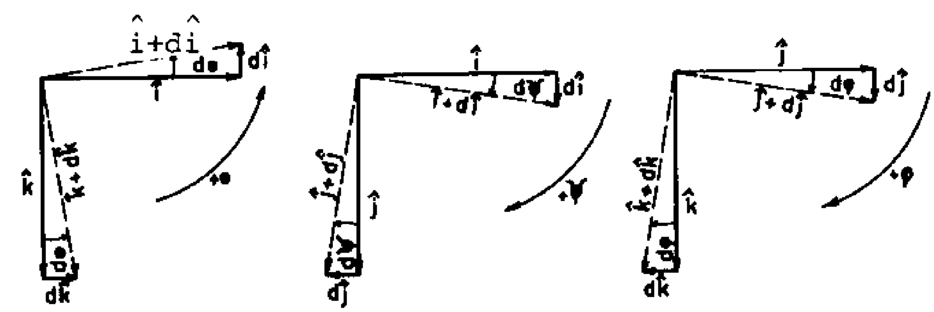
Since in most marine vehicles the time rate of change in mass due to fuel consumption is negligible, the mass of the body will be considered constant in time, hence $\frac{dm}{dt} = 0$. (This is not so for rockets).

The axis system chosen is fixed in the ship in order to use the symmetry of the ship to more easily calculate the hydrodynamic and hydrostatic forces in the vector quantity, $\vec{F}$. Since the ship moves in space, the axes are moving axes which somewhat complicate the expressions for momentum change on the right hand side of the equation. This complication is minor compared to the gain effected in the ability to express $\vec{F}$ through use of symmetry considerations. Before considering the nature of $\vec{F}$ and $\vec{\mathcal{M}}$, let us develop the momentum change (right hand side of equations) for the moving axis system chosen.

On substituting the component expression for $\vec{U}$, the force equation becomes (under the constant mass assumption)

$$\vec{F} = m\frac{d}{dt}(\vec{U}) = m\frac{d}{dt}(\hat{i}u + \hat{j}v + \hat{k}w)$$
$$= m\left[\hat{i}\frac{du}{dt} + u\frac{d\hat{i}}{dt} + \hat{j}\frac{dv}{dt} + v\frac{d\hat{j}}{dt} + \hat{k}\frac{dw}{dt} + w\frac{d\hat{k}}{dt}\right]$$

A change in a vector quantity can occur only by a change in length and/or a change in direction. Since $\hat{i}$, $\hat{j}$, and $\hat{k}$ are unit vectors they do not change their length. However, their directions are along axes fixed in a moving ship and their direction change as the ship moves in space. Hence, the quantities $\frac{d\hat{i}}{dt}$, $\frac{d\hat{j}}{dt}$, and $\frac{d\hat{k}}{dt}$ are not zero for the moving axes system. The change in a unit vector is a change in direction brought about by the rotation of the body and does not depend on the translation of the body. The change in direction of the unit vectors for rotation about each of the body axes are demonstrated by the sketch below. The length of the vectors $d\hat{i}$, $d\hat{j}$, and $d\hat{k}$ are given by the unit radius multiplied by the radian measure of the small (differential) angles of rotation, The directions of $d\hat{i}$, $d\hat{j}$, $d\hat{k}$ are perpendicular to $\hat{i}$, $\hat{j}$, and $\hat{k}$ respectively.



Rotation in θ, (pitch) (about y-axis)	Rotation in ψ, (yaw) (about z-axis)	Rotation in φ, (roll) (about x-axis)
$d\hat{i} = -\hat{k}d\theta$	$d\hat{i} = \hat{j}d\psi$	$d\hat{i} = 0$
$d\hat{j} = 0$	$d\hat{j} = -\hat{i}d\psi$	$d\hat{j} = \hat{k}d\varphi$
$d\hat{k} = \hat{i}d\theta$	$d\hat{k} = 0$	$d\hat{k} = -\hat{j}d\varphi$

For a general small rotation about the three axes, the three contributions are added to give

$$d\hat{i} = \hat{i}0 + \hat{j}d\psi - \hat{k}d\theta \quad \text{or} \quad \frac{d\hat{i}}{dt} = \hat{i}0 + \hat{j}\frac{d\psi}{dt} - \hat{k}\frac{d\theta}{dt}$$

$$d\hat{j} = -\hat{i}d\psi + \hat{j}0 + \hat{k}d\varphi \quad \text{or} \quad \frac{d\hat{j}}{dt} = -\hat{i}\frac{d\psi}{dt} + \hat{j}0 + \hat{k}\frac{d\varphi}{dt}$$

$$d\hat{k} = \hat{i}d\theta - \hat{j}d\varphi + \hat{k}0 \quad \text{or} \quad \frac{d\hat{k}}{dt} = \hat{i}\frac{d\theta}{dt} - \hat{j}\frac{d\varphi}{dt} + \hat{k}0$$

The vector angular velocity is designated by $\vec{\Omega}$, and has the components of p, q, and r about the x, y, and z axes respectively, i.e.

$\vec{\Omega} = \hat{i}p + \hat{j}q + \hat{k}r$. Since $\frac{d\varphi}{dt} = p$, $\frac{d\theta}{dt} = q$, and $\frac{d\psi}{dt} = r$, (within linear theory)*

$$\frac{d\hat{i}}{dt} = \hat{i}0 + \hat{j}r - \hat{k}q \quad \text{or} \quad \frac{d\hat{i}}{dt} = \vec{\Omega} \times \hat{i} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ p & q & r \\ 1 & 0 & 0 \end{vmatrix}$$

$$\frac{d\hat{j}}{dt} = -\hat{i}r + \hat{j}0 + \hat{k}p \quad \text{or} \quad \frac{d\hat{j}}{dt} = \vec{\Omega} \times \hat{j} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ p & q & r \\ 0 & 1 & 0 \end{vmatrix}$$

$$\frac{d\hat{k}}{dt} = \hat{i}q - \hat{j}p + \hat{k}0 \quad \text{or} \quad \frac{d\hat{k}}{dt} = \vec{\Omega} \times \hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ p & q & r \\ 0 & 0 & 1 \end{vmatrix}$$

The expressions for $\frac{d\hat{j}}{dt}$ and $\frac{d\hat{k}}{dt}$ can be derived from the expression for $\frac{d\hat{i}}{dt}$ by a process of permutation of the components, a property resulting from the use of a consistent set of axes. The permutation procedure is as follows:

$x \rightarrow y \rightarrow z \rightarrow x$

$\hat{i} \rightarrow \hat{j} \rightarrow \hat{k} \rightarrow \hat{i}$

$p \rightarrow q \rightarrow r \rightarrow p$

$u \rightarrow v \rightarrow w \rightarrow u$

$X \rightarrow Y \rightarrow Z \rightarrow X$

$K \rightarrow M \rightarrow N \rightarrow K$

If one takes a general expression involving a component or a vector, the expressions for the other components, or similar vectors can be obtained by moving every item one down the index scale. For deriving $\frac{d\hat{j}}{dt}$ and $\frac{d\hat{k}}{dt}$ from $\frac{d\hat{i}}{dt}$, permute as follows:

$$\begin{array}{ccccccc} \frac{d\hat{i}}{dt} & = & \hat{i}0 & + & \hat{j}r & - & \hat{k}q \\ \downarrow & & \downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow \\ \frac{d\hat{j}}{dt} & = & \hat{j}0 & + & \hat{k}p & - & \hat{i}r \\ \downarrow & & \downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow \\ \frac{d\hat{k}}{dt} & = & \hat{k}0 & + & \hat{i}q & - & \hat{j}p \end{array}$$

Let us return to the force equation and substitute into the equation the expressions for $\frac{d\hat{i}}{dt}$, $\frac{d\hat{j}}{dt}$, and $\frac{d\hat{k}}{dt}$, using the dot over a quantity to indicate the derivative of the quantity with respect to time.

$$\vec{F} = m\left[\hat{i}\dot{u} + u(\hat{j}r-\hat{k}q) + \hat{j}\dot{v} + v(\hat{k}p-\hat{i}r) + \hat{k}\dot{w} + w(\hat{i}q-\hat{k}p)\right]$$

*See Appendix II

The quantities are grouped under the respective directional components, together with the defined components of $\vec{F}$, to give

$$\hat{i}X + \hat{j}Y + \hat{k}Z = m\left[\hat{i}(\dot{u}+qw-rv) + \hat{j}(\dot{v}+ru-pw) + \hat{k}(\dot{w}+pv-qu)\right]$$

or

$$X = m(\dot{u}+qw-rv)$$
$$Y = m(\dot{v}+ru-pw)$$
$$Z = m(\dot{w}+pv-qu)$$

The expressions for Y and Z could have been obtained from the expression for X by the process of permutation,

$$\begin{array}{l} X = m(\dot{u}+qw-rv) \\ \downarrow \qquad\quad \downarrow\ \downarrow\downarrow\ \downarrow\downarrow \\ Y = m(\dot{v}+ru-pw) \end{array}$$

Since the terms $\dot{u}$, $\dot{v}$, and $\dot{w}$ represent apparent acceleration components within the moving axis system, the terms (qw-rv), (ru-pw), and (pv-qu) must represent the components of centripetal accelerations on the body arising from the moving coordinate system.

The angular momentum about a set of axes fixed in the vehicle can be expressed as:

$$\overrightarrow{(\text{angular momentum})} = \begin{bmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{yx} & I_y & -I_{yz} \\ -I_{zx} & -I_{zy} & I_z \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

where the terms I_{ab}, $(a \neq b)$ are the products of inertia. If the axes chosen are principal axes of inertia (of necessity with the origin at C.G.) then the products of inertia are zero and we have:

$$\overrightarrow{(\text{Angular momentum})} = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

or

$$\overrightarrow{(\text{Angular momentum})} = \hat{i}\, I_x\, p + \hat{j}\, I_y\, q + \hat{k}\, I_z\, r$$

(Continue to Page I-7b)

(Continue below)

$$\vec{\mathcal{M}} = \frac{d}{dt}\,(\overrightarrow{\text{Ang.mom.}}) = \frac{d}{dt}\,(\hat{i}I_x p + \hat{j}I_y q + \hat{k}I_z r)$$

$$= \hat{i}\,\frac{d}{dt}(I_x p) + I_x p\,\frac{d\hat{i}}{dt} + \hat{j}\,\frac{d}{dt}(I_y q) + I_y q\,\frac{d\hat{j}}{dt} + \hat{k}\,\frac{d}{dt}(I_z r) + I_z r\,\frac{d\hat{k}}{dt}$$

Since the mass of the ship is assumed constant in time, then also the inertia of the ship (mass distribution) is assumed constant in time. Hence, $\frac{d}{dt}(I_x p) = I_x \dot{p}$, with analogous results for the similar terms. On incorporation of the expressions for $\frac{d\hat{i}}{dt}$, $\frac{d\hat{j}}{dt}$, and $\frac{d\hat{k}}{dt}$ into the moment equation, there results

$$\vec{\mathcal{M}} = \hat{i}I_x\dot{p} + I_x p(\hat{j}r-\hat{k}q) + \hat{j}I_y\dot{q} + I_y q(\hat{k}p-\hat{i}r) + \hat{k}I_z\dot{r} + I_z r(\hat{i}q-\hat{j}p)$$

With the vector components of moment defined as

$$\vec{\mathcal{M}} = \hat{i}K + \hat{j}M + \hat{k}N \ ,$$

the grouping of the vector quantities into components in the $\hat{i}$, $\hat{j}$, $\hat{k}$ directions gives

$$K = I_x\dot{p} + (I_z - I_y)qr$$

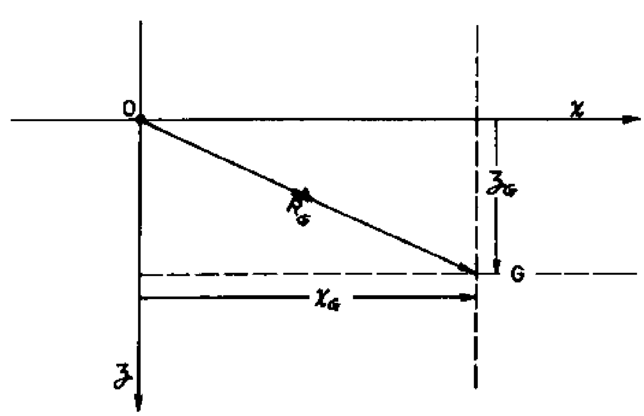
$$M = I_y\dot{q} + (I_x - I_z)rp$$

$$N = I_z\dot{r} + (I_y - I_x)pq$$

The expressions for M and N could have been deduced from the expression for K by the permutation process. Since $\dot{p}$, $\dot{q}$, $\dot{r}$ are "apparent" angular accelerations in the moving system, the terms $(I_z - I_y)qr$, $(I_x - I_z)rp$, and $(I_y - I_x)pq$ represent gyroscopic moments arising from the moving axis system.

The equations have been developed for the case of the origin located at the center of gravity of the body, but the center of gravity is not necessarily located at the center of geometry or buoyancy of the body. Since hydrostatic and hydrodynamic forces depend greatly on the geometry of the body, it would be very convenient to develop the equations for an arbitrary origin so as to provide the flexibility to choose an origin which takes advantage of body geometrical symmetries to more easily express the hydrostatic and hydrodynamic forces acting on the body. The equations of motion will be developed for an axis system parallel to the principal axes of inertia through the center of gravity, G, but for a location of the origin, O, not necessarily at the center of gravity.

In order to use the separate force and moment equations, the forces and moment acting at the center of gravity, G, will be used, but they will be expressed in terms of components measured relative to an origin O in the body. The vector distance that the center of gravity is from the origin is designated by $\vec{R}_G = \hat{i}x_G + \hat{j}y_G + \hat{k}z_G$; x_G, y_G, and z_G are the distances of the center of gravity, G, from O, along the x, y, and z axes respectively, as can be observed from the following sketch.



The equation $\vec{F} = \frac{d}{dt}(m\vec{U}_G)$ is the proper Newtonian expression, since $\vec{U}_G$ refers to the velocity at the center of gravity, but it is desired to develop this expression for a velocity $\vec{U}$ as measured at the origin O. The velocity $\vec{U}_G$ at G must equal the velocity $\vec{U}$ at O plus the velocity of G relative to O, or

$$\vec{U}_G = \vec{U} + \frac{d}{dt}(\vec{R}_G) = \vec{U} + \dot{\vec{R}}_G$$

Since $\vec{R}_G$ is a vector fixed in the body, it cannot change its length but only its direction as the body moves about. Hence, the velocity of G relative to O can result only from rotation - hence from the product of angular velocity and the radius. $\dot{\vec{R}}_G$ can then be expressed as

$$\dot{\vec{R}}_G = \vec{\Omega} \times \vec{R}_G$$

This expression can be obtained by carrying through the time derivative of the vector expression of $\vec{R}_G$.

$$\dot{\vec{R}}_G = \frac{d}{dt}(\hat{i}x_G+\hat{j}y_G+\hat{k}z_G) = \hat{i}\dot{x}_G+x_G\frac{d\hat{i}}{dt}+\hat{j}\dot{y}_G+y_G\frac{d\hat{j}}{dt}+\hat{k}\dot{z}_G+z_G\frac{d\hat{k}}{dt}$$

$\dot{x}_G = \dot{y}_G = \dot{z}_G = 0$, since $\vec{R}_G$ is fixed in the body. Recall that

$$\frac{d\hat{i}}{dt} = \hat{j}r-\hat{k}q \qquad \frac{d\hat{j}}{dt} = \hat{k}p-\hat{i}r \qquad \frac{d\hat{k}}{dt} = \hat{i}q-\hat{j}p$$

and when these are substituted into the expression for $\dot{\vec{R}}_G$ there results

$$\dot{\vec{R}}_G = x_G(\hat{j}r-\hat{k}q) + y_G(\hat{k}p-\hat{i}r) + z_G(\hat{i}q-\hat{j}p)$$

This reduces to

$$\dot{\vec{R}}_G = \hat{i}(qz_G-ry_G)+\hat{j}(rx_G-pz_G)+\hat{k}(py_G-qx_G) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ p & q & r \\ x_G & y_G & z_G \end{vmatrix} = \vec{\Omega} \times \vec{R}_G$$

The force equation is therefore written as

$$\vec{F} = m\frac{d}{dt}(\vec{U} + \vec{\Omega}\times\vec{R}_G)$$

$$= m[\dot{\vec{U}} + \dot{\vec{\Omega}}\times\vec{R}_G + \vec{\Omega}\times\dot{\vec{R}}_G]$$

The term $\dot{\vec{U}} = \frac{d\vec{U}}{dt}$ has been developed above and can be expressed as

$$\dot{\vec{U}} = \frac{d}{dt}(\hat{i}u+\hat{j}v+\hat{k}w) = \hat{i}(\dot{u}+qw-rv)+\hat{j}(\dot{v}+ru-pw)+\hat{k}(\dot{w}+pv-qu)$$

The components of $\dot{\vec{\Omega}}$ are developed as follows

$$\dot{\vec{\Omega}} = \frac{d}{dt}(\hat{i}p+\hat{j}q+\hat{k}r) = \hat{i}\dot{p} + p\frac{d\hat{i}}{dt} + \hat{j}\dot{q} + q\frac{d\hat{j}}{dt} + \hat{k}\dot{r} + r\frac{d\hat{k}}{dt}$$

$$p\frac{d\hat{i}}{dt} + q\frac{d\hat{j}}{dt} + r\frac{d\hat{k}}{dt} = p(\hat{j}r-\hat{k}q)+q(\hat{k}p-\hat{i}r)+r(\hat{i}q-\hat{j}p) = \hat{i}0+\hat{j}0+\hat{k}0$$

Therefore, $\dot{\vec{\Omega}} = \hat{i}\dot{p}+\hat{j}\dot{q}+\hat{k}\dot{r}$, and the expression for $\dot{\vec{\Omega}} \times \vec{R_G}$ becomes

$$\dot{\vec{\Omega}} \times \vec{R_G} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dot{p} & \dot{q} & \dot{r} \\ x_G & y_G & z_G \end{vmatrix} = \hat{i}(z_G\dot{q}-y_G\dot{r})+\hat{j}(x_G\dot{r}-z_G\dot{p})+\hat{k}(y_G\dot{p}-x_G\dot{q})$$

and the expression for $\vec{\Omega} \times \dot{\vec{R}}_G$ becomes (on substituting the components of $\dot{\vec{R}}_G$)

$$\vec{\Omega} \times \dot{\vec{R}}_G = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ p & q & r \\ qz_G-ry_G & rx_G-pz_G & py_G-qx_G \end{vmatrix} = \hat{i}(qpy_G-q^2x_G-r^2x_G+prz_G)+\hat{j}(\ldots\ldots\ldots) +\hat{k}(\ldots\ldots\ldots).$$

To obtain the expression for the X component of force, the $\hat{i}$ component of $m[\dot{\vec{U}}+\dot{\vec{\Omega}}\times\vec{R_G}+\vec{\Omega}\times\dot{\vec{R}}_G]$ is formulated as follows:

$$\dot{\vec{U}} \qquad\qquad \dot{\vec{\Omega}}\times\vec{R_G} \qquad\qquad\qquad \vec{\Omega}\times\dot{\vec{R}}_G$$

$$X = m\left[\dot{u}+qw-rv + z_G\dot{q}-y_G\dot{r} + qpy_G-x_G(q^2+r^2)+prz_G\right]$$

$$X = m\left[\dot{u}+qw-rv-x_G(q^2+r^2)+y_G(pq-\dot{r})+z_G(pr+\dot{q})\right]$$

The Y and Z components can be formulated by permuting the terms.

$$Y = m\left[\dot{v}+ru-pw-y_G(r^2+p^2)+z_G(qr-\dot{p})+x_G(qp+\dot{r})\right]$$

$$Z = m\left[\dot{w}+pv-qu-z_G(p^2+q^2)+x_G(rp-\dot{q})+y_G(rq+\dot{p})\right]$$

The equations for the X, Y, and Z components for an origin not at the center of gravity, differ from the equations for the origin at the center of gravity in the additional terms involving x_G, y_G, and z_G. On studying the X equation, for example, the physical significance of these additional terms become apparent. The terms resulting from $\vec{\Omega} \times \dot{\vec{R}_G} = \vec{\Omega} \times (\vec{\Omega} \times \vec{R_G})$, represent centrifugal forces acting at the origin because of the center of gravity not being at the origin, and the terms resulting from $\dot{\vec{\Omega}} \times \vec{R_G}$ represent the inertial reaction forces felt at the origin by the acceleration of the C.G. relative to the origin.

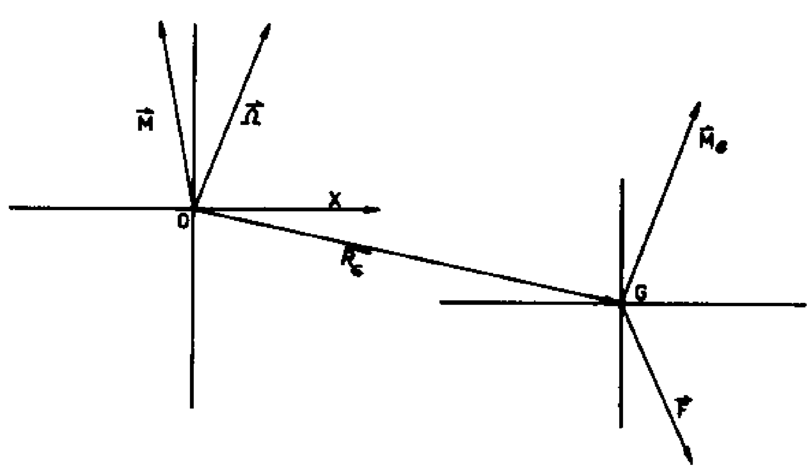
The expressions for the components of $\vec{\mathcal{M}}$ for an origin off the C.G. will now be developed, using the equation

$$\vec{\mathcal{M}}_G = \frac{d}{dt}\,(\overrightarrow{\text{Ang.Mom.}})_{C.G.}$$

which refers to the moment at the center of gravity. The sketch shown below serves as an aid in visualizing the relationships between the two origins.

The moment $\vec{\mathcal{M}}$ experienced at O equals the moment experienced at G plus the moment caused by the force $\vec{F}$ acting over the radius $\vec{R_G}$, i.e.

$$\vec{\mathcal{M}} = \vec{\mathcal{M}}_G + \vec{R_G} \times \vec{F}$$



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The angular momentum for an origin at G with axes being the principal axes of inertia are:

$$(\overrightarrow{\text{ang. mom.}})_G = \hat{i}' I'_x p' + \hat{j}' I'_y q' + \hat{k}' I'_z r'$$

where the prime indicates a reference to an origin at the center of gravity. For an axis system parallel to the principal axis system, with origin at O, the unit vectors are identical i.e. $\hat{i} = \hat{i}'$, $\hat{j}=\hat{j}'$, and $\hat{k}=\hat{k}'$ since the vectors have the same direction and the same magnitude (unity). Since the axes are parallel to the principal axes of inertia, then by the parallel axis theorem:

$$I_x' = I_x - m(y_G^2 + z_G^2)$$

$$I_y' = I_y - m(z_G^2 + x_G^2)$$

$$I_z' = I_z - m(x_G^2 + y_G^2)$$

where I_x, I_y, and I_z refer to moments of inertia about the x,y,and z axes with origin at O.

Therefore:

$$(\overrightarrow{\text{ang.mom.}})_G = \hat{i} I_x p + \hat{j} I_y q + \hat{k} I_z r$$

$$-m[\hat{i}(y_G^2 + z_G^2)p + \hat{j}(z_G^2+x_G^2)q+\hat{k}(x_G^2+y_G^2)r]$$

$$= \hat{i} I_x p + \hat{j} I_y q + \hat{k} I_z r - m\vec{R}_G x(\vec{\Omega} \, x\vec{R}_G)$$

since $$\vec{\Omega} x \, \vec{R}_G = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ p & q & r \\ x_G & y_G & z_G \end{vmatrix}$$

and

$$\vec{R}_G \times (\vec{\Omega} \times \vec{R}_G) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_G & y_G & z_G \\ (qz_G - ry_G) & (rx_G - pz_G) & (py_G - qx_G) \end{vmatrix}$$

We now have

$$\vec{\mathcal{M}}_G = \vec{\mathcal{M}} - \vec{R}_G \times \vec{F} = \frac{d}{dt}[\hat{i}I_x p + \hat{j}I_y q + \hat{k}I_z r - m\vec{R}_G \times (\vec{\Omega} \times \vec{R}_G)]$$

Since $\vec{F} = m\dot{\vec{U}}_G$ and since $\vec{U}_G = \vec{U} + \vec{\Omega} \times \vec{R}_G$ the above formulation becomes:

$$\vec{\mathcal{M}} = \frac{d}{dt}(\hat{i}I_x p + \hat{j}I_y q + \hat{k}I_z r) - m\frac{d}{dt}[\vec{R}_G \times (\vec{\Omega} \times \vec{R}_G)]$$

$$+ m\vec{R}_G \times \frac{d}{dt}[\vec{U} + \vec{\Omega} \times \vec{R}_G]$$

$$= \frac{d}{dt}(\hat{i}I_x p + \hat{j}I_y q + \hat{k}I_z r) - m\dot{\vec{R}}_G \times (\vec{\Omega} \times \vec{R}_G) - m\vec{R}_G \times \frac{d}{dt}(\vec{\Omega} \times \vec{R}_G)$$

$$+ m\vec{R}_G \times \frac{d\vec{U}}{dt} + m\vec{R}_G \times \frac{d}{dt}(\vec{\Omega} \times \vec{R}_G)$$

Since $\dot{\vec{R}}_G = \vec{\Omega} \times \vec{R}_G$

$$\vec{\mathcal{M}} = \frac{d}{dt}(\hat{i}I_x p + \hat{j}I_y q + \hat{k}I_z r) - m(\vec{\Omega} \times \vec{R}_G) \times (\vec{\Omega} \times \vec{R}_G) + m\vec{R}_G \times \frac{d\vec{U}}{dt}$$

In previous derivations, it has been shown that

$$\dot{\vec{U}} = \frac{d}{dt}(\hat{i}u + \hat{j}v + \hat{k}w) = \hat{i}(\dot{u} + qw - rv) + \hat{j}(\dot{v} + ru - pw) + \hat{k}(\dot{w} + pv - qu)$$

and

$$\vec{R}_G \times \dot{\vec{U}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_G & y_G & z_G \\ (\dot{u} + qw - rv) & (\dot{v} + ru - pw) & (\dot{w} + pv - qu) \end{vmatrix} = \hat{i}\left[y_G(\dot{w} + pv - qu) - z_G(\dot{v} + ru - pw)\right]$$

$$+ \hat{j}\left[\dots\dots\right] + \hat{k}\left[\dots\dots\right].$$

From previous development of the derivative of angular momentum, it has been shown that

$$\frac{d}{dt}(\hat{i}I_x p + \hat{j}I_y q + \hat{k}I_z r) = \hat{i}\left[I_x\dot{p} + (I_z - I_y)qr\right] + \hat{j}\left[\dots\right] + \hat{k}\left[\dots\right]$$

The various terms associated with unit vector $\hat{i}$ in the expression

$$\vec{\mathcal{M}} = \frac{d}{dt}(\hat{i}I_xp + \hat{j}I_yq + \hat{k}I_zr) + m\vec{R}_G \times \dot{\vec{U}} = \hat{i}K + \hat{j}M + \hat{k}N$$

are grouped together to form an expression for K, the $\hat{i}$ component of $\vec{\mathcal{M}}$. Hence,

$$K = I_x\dot{p} + (I_z - I_y)qr + m\left[y_G(\dot{w} + pv - qu) - z_G(\dot{v} + ru - pw)\right]$$

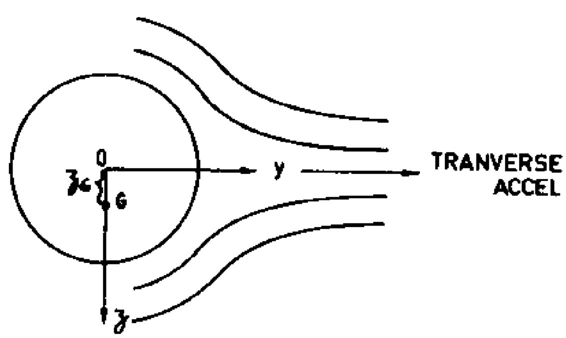
and by permutation

$$M = I_y\dot{q} + (I_x - I_z)rp + m\left[z_G(\dot{u} + qw - rv) - x_G(\dot{w} + pv - qu)\right]$$

$$N = I_z\dot{r} + (I_y - I_x)pq + m\left[x_G(\dot{v} + ru - pw) - y_G(\dot{u} + qw - rv)\right].$$

The apparent physical significance of the additional terms involving x_G, y_G, and z_G is the introduction into the moment equation of those moments resulting from inertial reaction forces caused by acceleration of the center of gravity.

The equations of motion for a body have now been expressed in a flexible form, allowing the choice of origin for the coordinate system. A simple example of the advantage of choosing an origin off the center of gravity is given by the transverse acceleration of a body like a torpedo. If the origin is chosen at O, a position of symmetry, as shown below,



then a transverse acceleration produces no hydrodynamic roll moment, K, because of the symmetry of flow relative to O. The formula gives the roll moment about O caused by G not being at the origin, - i.e. $mz_G \times$ (transverse acceleration). If the origin were at G, it would have been necessary to calculate a hydrodynamic moment.

CHAPTER II

Forces and Moments Acting on a Body

The forces and moments acting on a body, which in turn cause the ship to move, need now be studied in order to analyze the motion of a body.

Through the dependence of various phenomena on the properties of the body, properties of the motion, and properties of the fluid, the relationship for the forces and moments (in unrestricted water) become

$$\begin{matrix}\text{Forces}\\ \text{Moments}\end{matrix} = f\Big\{\underbrace{L, \text{geom}, m, \vec{R}_G, I;}_{\text{Properties of body}}\ \underbrace{\vec{R}_O, \varphi, \theta, \psi; \vec{U}, \dot{\vec{U}}, \vec{\Omega}, \dot{\vec{\Omega}}, n, \dot{n}, \delta, \dot{\delta}, \ddot{\delta};}_{\text{Properties of motion}}\ \underbrace{\rho, \mu, g, \tau, p, p_v, E, \ldots}_{\text{Properties of fluid}}\Big\}$$

On reduction to non-dimensional form, the properties of the fluid were analyzed which resulted in the terms of Reynolds' number, Froude number, etc. and their significance in modelling was demonstrated. Since the fluid forces acting on the body depend on the orientation and motion of the body relative to the fluid, the parameters in the above function can be expressed in terms of the orientation and the motion of the body relative to fixed axes in space plus the orientation and motion of the fluid relative to fixed axes in space. Hence, if one prefers to call the motion properties listed in the function as referring to space axes, then additional parameters involving the orientation and motion of the fluid must be included in the parameters of the function. Such items as wave-shape, size, and particle orbital velocity would then appear in the function. One can characterize these fluid motion properties as an excitation parameter.

In order to concentrate on the effects of the dynamic parameters in the function, the dimensional form will be used and a given fluid and a given ship size will be considered. The results from the analysis of the function in dimensional form can be readily reduced to non-dimensional form for considerations of model work in maneuvering and seakeeping.

For a given ship in a given fluid, in the absence of excitation forces, one can express the general function as

$$\left.\begin{matrix}\vec{F}\\ \vec{\mathcal{M}}\end{matrix}\right\} = f(x_0, y_0, z_0, \varphi, \theta, \psi, u, v, w, p, q, r, \dot{u}, \dot{v}, \dot{w}, \dot{p}, \dot{q}, \dot{r}, \delta, \dot{\delta}, \ddot{\delta})$$

since if a function depends on $\vec{U}, \vec{\Omega}, \dot{\vec{U}}, \dot{\vec{\Omega}}$, then the function is also dependent on the components of these vectors. If the function were dependent on only one variable it may be possible to calculate the force and moment as the variable changed its value. For instance, in calm water with the ship not moving, and at even keel, the force exerted on the ship if it is moved vertically to the water surface is expressed as a simple function of z_0

$$\vec{F} = f_1(z_0)$$

The force is readily calculable as a vertical force equal to the change in displacement caused by increasing the draft of the ship. However, if this vertical motion is varied at the same time other variables are not zero, additional forces, not readily calculable are introduced. On the other hand, if the only variable were the forward speed, u, then even the simple function

$$\vec{F} = f_2(u)$$

is not calculable and resistance tests on models need to be run to estimate the force. It is necessary to develop the function of these many variables into a useful form for analysis purposes.

The function describing the forces and moments acting on a given ship in a given fluid involves the many motion and orientation parameters. The function can be reduced to useful mathematical form by the use of the Taylor expansion of a function of several variables. To use the expansion, the function and its derivatives need to be continuous and not go to infinity (blow up) in the region of the values of the variables under consideration. This assumption holds very well with respect to hydrodynamic bodies in the region of their operating conditions, especially ships.

Let us observe how the Taylor expansion works with one variable, say x as an example. If the value of the function f(x) is desired for a certain value of x, it can be described in terms of the value of the function and its derivatives at some other value of x, say at $x = x_0$, as follows:

$$f(x) = f(x_0)+(x-x_0)\frac{df(x_0)}{dx} + \frac{(x-x_0)^2}{2!}\frac{d^2f(x_0)}{dx^2} + \frac{(x-x_0)^3}{3!}\frac{d^3f(x_0)}{dx^3} + ---$$

where $f(x_0)$ indicates the value of the function at $x = x_0$

$\frac{d^nf(x_0)}{dx^n}$ indicates the $n^{\underline{th}}$ derivative of the function evaluated at $x = x_0$.

On introducing the differential operator $\mathcal{D}_x = \frac{d}{dx}$, $(\mathcal{D}_x^{\ n} = \frac{d^n}{dx^n})$, and $(x - x_0) = \Delta x$, then the form of the expansion becomes

$$f(x) = f(x_0) + \Delta x\,\mathcal{D}_x f(x_0) + \frac{(\Delta x\,\mathcal{D}_x)^2}{2!} f(x_0) + \frac{(\Delta x\,\mathcal{D}_x)^3}{3!} f(x_0) + -----$$

or

$$f(x) = \left[1 + (\Delta x\,\mathcal{D}_x) + \frac{(\Delta x\,\mathcal{D}_x)^2}{2!} + \frac{(\Delta x\,\mathcal{D}_x)^3}{3!} + -----\right] f(x_0).$$

This form is exactly the form for a series expansion of the exponential

$$e^a = 1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + -----$$

so that the Taylor expansion can be expressed as

$$f(x) = e^{\Delta x\,\mathcal{D}_x} \cdot f(x_0).$$

Similarly, for more than one variable, say the two variables x and y, the Taylor expansion takes the form

$$f(x,y) = e^{\Delta x\,\mathcal{D}_x + \Delta y\,\mathcal{D}_y} \cdot f(x_0,y_0)$$

where $\mathcal{D}_x = \frac{\partial}{\partial x}$ and $\mathcal{D}_y = \frac{\partial}{\partial y}$ since partial derivatives are required for more than one variable. On expanding one obtains

$$f(x,y) = f(x_0,y_0)+(\Delta x\mathcal{D}_x+\Delta y\mathcal{D}_y)f(x_0,y_0)+\frac{(\Delta x\mathcal{D}_x+\Delta y\mathcal{D}_y)^2}{2!}f(x_0,y_0)+ ---$$

$$= f(x_0,y_0)+\frac{\Delta x\,\partial f(x_0,y_0)}{\partial x}+\Delta y\frac{\partial f(x_0,y_0)}{\partial y}+\frac{1}{2!}\left[\frac{(\Delta x)^2\partial^2 f(x_0,y_0)}{\partial x^2}+\frac{(\Delta y)^2\partial^2 f(x_0,y_0)}{\partial y^2}+2\Delta x\,\Delta y\frac{\partial^2 f(x_0,y_0)}{\partial x\,\partial y}\right] + ----$$

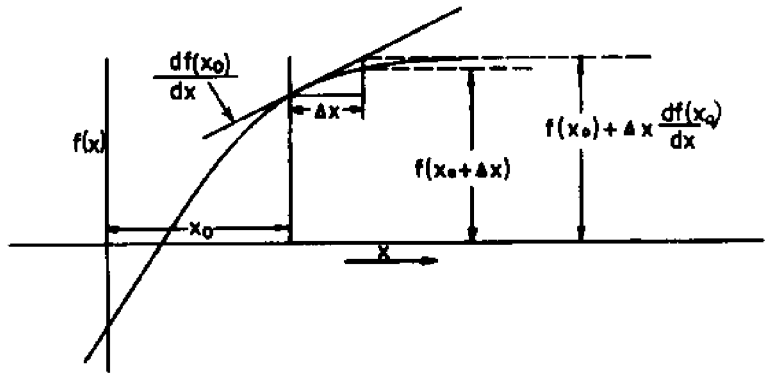
(Remember that x and y above are sample variables and bear no relationship to the variables x_0, y_0 in our function for $\vec{F}$ and $\vec{\mathcal{M}}$).

The Taylor expansion for the forces and moments acting on the ship would then be expressed as

$$\text{or}\ \left.\begin{matrix}\vec{F}\\ \vec{\mathcal{M}}\end{matrix}\right\}=\left[e^{\Delta x_0\mathcal{D}_{x_0}+\Delta y_0\mathcal{D}_{y_0}+\Delta z_0\mathcal{D}_{z_0}+\quad + --- +\Delta v\mathcal{D}_v+\Delta\dot{r}\mathcal{D}_{\dot{r}}}\right] f\left[(x_0)_0,(y_0)_0, --- v_0, ---, \dot{r}_0\right].$$

and the expansion of the power series into the actual functional form would indeed give an extremely long, cumbersome, and almost impossible to handle expression for $\vec{F}$ and $\vec{\mathcal{M}}$.

Hence, for simplicity and for the sake of reducing the equation to solvable form, the function is "linearized" about an initial equilibrium condition of motion. On linearization, only the linear terms in the change of the value of the variable from the equilibrium (initial condition) are maintained, i.e. terms of the order Δx and Δy in the example. Terms of higher order (i.e. $(\Delta x)^2$, $(\Delta y)^2$, $(\Delta x)(\Delta y)$, $(\Delta x)^3$, etc. in the example) are considered small compared to the first order terms and are neglected. This limits the validity of the analysis to relatively small changes in the variables (i.e. small Δx, Δy, etc.). Linearization, in effect, estimates the value of the function by multiplying the slope of the function by the change in the variable as shown below.



A familiar example of linearization of a function is the use of the product of metacentric height and the angle of heel to estimate the function of righting arm vs. angle of heel, which is the curve of statical stability.

In the domain of ship motions, we are mostly interested in those ship motions which depart from the condition of straight ahead motion such as various maneuvers from straight ahead motion or the pitching and heaving of the surface ship about a mean straight path. Hence, the initial condition of motion equilibrium is chosen as straight ahead motion at constant speed. This is indeed a condition of equilibrium since no forces and moments are acting on the body because there are no accelerations either angular or linear in this condition. The propeller forces are cancelling the resistance forces (through thrust deduction) with no net force acting on the body. The equilibrium condition of the function (straight ahead motion and designated by the subscript 0 on the variables) becomes

$$\left.\begin{matrix}\vec{F}_0\\ \vec{m}_0\end{matrix}\right\} = f\left\{(x_0)_0,(y_0)_0,(z_0)_0,\varphi_0,\theta_0,\psi_0,u_0,v_0,w_0,p_0,q_0,r_0,\dot{u}_0,\ldots\ldots\dot{r}_0,\delta_0,\ldots\ldots\right\}$$

For straight ahead motion at constant speed (using a chosed orientation for reference) all the initial values (equilibrium values) of the variables are zero except for u_0 which is the value of the forward speed. Hence,

$$(x_0)_0 = (y_0)_0 = \text{-----} = \psi_0 = v_0 = w_0 = \text{-----} = \delta_0 = 0$$

$$u_0 \neq 0$$

The changes in the value of the variables from the value at the equilibrium condition already has been designated by a preceeding Δ , i.e. $\Delta u = u-u_0$, $\Delta v = v-v_0$, etc. Since all the variables have equilibrium values of 0, except for u, a change in value for all the variables, excluding u, can be written if the form

$$\Delta\text{variable} = \text{variable} - (\text{variable})_0$$

$$(\text{variable})_0 = 0$$

$$\Delta\text{variable} = \text{variable}$$

For example $\Delta v = v$, $\Delta \dot{u} = \dot{u}$, $\Delta \dot{q} = q$, etc., but $u = u_0 + \Delta u$. If the force and moment are functions of a set of variables so also are the components of the force and moment. Hence, X, Y, Z, K, M, and N can be expressed as functions of these many variables. Let us take for example

$$X = X(x_0, y_0, ----. u, v, ---, \dot{u}, \dot{v}, --- \dot{r} ---)$$

indicating the X component is some function X of the variables. The linear terms of the Taylor expansion of this function would appear as follows:

$$X = X_0 + \left(\frac{\partial X}{\partial x_0}\right)_0 \Delta x_0 + \left(\frac{\partial X}{\partial y_0}\right)_0 \Delta y_0 + \left(\frac{\partial X}{\partial z_0}\right)_0 \Delta z_0 + --- + \left(\frac{\partial X}{\partial u}\right)_0 \Delta u$$

$$+ \left(\frac{\partial X}{\partial v}\right)_0 \Delta v + --- + \left(\frac{\partial X}{\partial \dot{r}}\right)_0 \Delta \dot{r} + ---.$$

A convenient notation for writing the derivative of a function taken at the equilibrium value of the function (or variable) uses a subscript to denote the variable involved in the differential, as demonstrated in the following examples.

$$\left(\frac{\partial X}{\partial u}\right)_0 = \left(\frac{\partial X}{\partial u}\right)_{\substack{u=u_0 \\ v=v_0=0 \\ w=w_0=0 \\ \vdots}} = X_u$$

$$\left(\frac{\partial X}{\partial \dot{r}}\right)_0 = \left(\frac{\partial X}{\partial \dot{r}}\right)_{\substack{\dot{r}=\dot{r}_0=0 \\ u=u_0 \\ v=v_0=0 \\ \vdots}} = X_{\dot{r}}$$

The linear expansion for the X function using this notation, together with the substitution of $\Delta v = v$, $\Delta \dot{u} = \dot{u}$, etc., as previously developed, gives

$$X = X_0 + X_{x_0} x_0 + X_{y_0} y_0 + X_{z_0} z_0 + X_\phi \phi + \text{-----} + X_u \Delta u + X_v v + \text{------} + X_r r + \text{---}$$

with similar expressions for the Y, Z, K, M, and N components. In the equilibrium condition of straight ahead motion at constant speed there are no forces acting on the body, hence $X_0=Y_0=Z_0=K_0=M_0=N_0=0$. In order to keep the development of the solution of the equations of motion somewhat less complicated for the purposes of understanding the phenomenon, let us devote our efforts at the present time to the analysis of motion in the horizontal plane (maneuvering) without rolling. This involves the three degrees of motion freedom of translation along the x and y axes and rotation about the z axis, (forward, transverse and yaw motions). Under this limitation, only the following variables will appear in the function (allowing no deflection of the rudder for the present).

$$x_0,\ y_0,\ \psi,\ u,\ v,\ r,\ \dot{u},\ \dot{v},\ \dot{r}$$

and the force and moment components of interest are X, Y, and N. A comparable restriction to motion in the vertical plane (seakeeping or submarine maneuvering) would involve only X, Z, and M and the variables x_0, z_0, θ, u, w, q, $\dot{u}$, $\dot{w}$, and $\dot{q}$. The equation for roll involving K, φ, p, and $\dot{p}$ is usually taken together with the equations for motion in the horizontal plane, since this motion excites roll due to asymmetry of the hull[x)], or is treated separately as a one degree of freedom system.

The linearized force and moment functions have now been developed and it is now necessary to equate these forces and moments to the dynamic response terms - i.e. the right hand side of the equations of motion. However, since the force expression has been linearized, only the linear terms of the right side of the equation need be retained. Let us assume that the center of gravity lies in the centerline plane (since any good naval architect would design it so) and therefore $y_G = 0$. For motion in the horizontal plane (no rolling) the right side of the equations reduce to

----

x) Motion in the vertical plane (at least within the linear theory), does not excite roll because of the symmetry of port and starboard.

$$X = m(\dot{u}-rv-x_G r^2)$$

$$Y = m(\dot{v}+ru+x_G \dot{r})$$

$$N = I_z \dot{r}+mx_G(\dot{v}+ru)$$

A linearization of the right hand side of the Y equation proceeds as follows:

$$\dot{v}+ru+x_G\dot{r} = (\dot{v}_0+\Delta\dot{v})+(r_0+\Delta r)(u_0+\Delta u)+x_G(\dot{r}_0+\Delta\dot{r})$$

$$= \Delta\dot{v}+\Delta r(u_0+\Delta u)+x_G\Delta\dot{r} = \Delta\dot{v}+\Delta r u_0+\Delta r\,\Delta u+x_G\Delta\dot{r}$$

since

$$\dot{v}_0 = r_0 = \dot{r}_0 = 0$$

The term $\Delta r\,\Delta u$ is second order and must be dropped since similar second order terms have been neglected in the force and moment function on the left side of the equation. Since $\Delta v = v-v_0 = v$, etc., the linearized right side of the Y equation becomes

$$m(\dot{v}+ru_0+x_G\dot{r}).$$

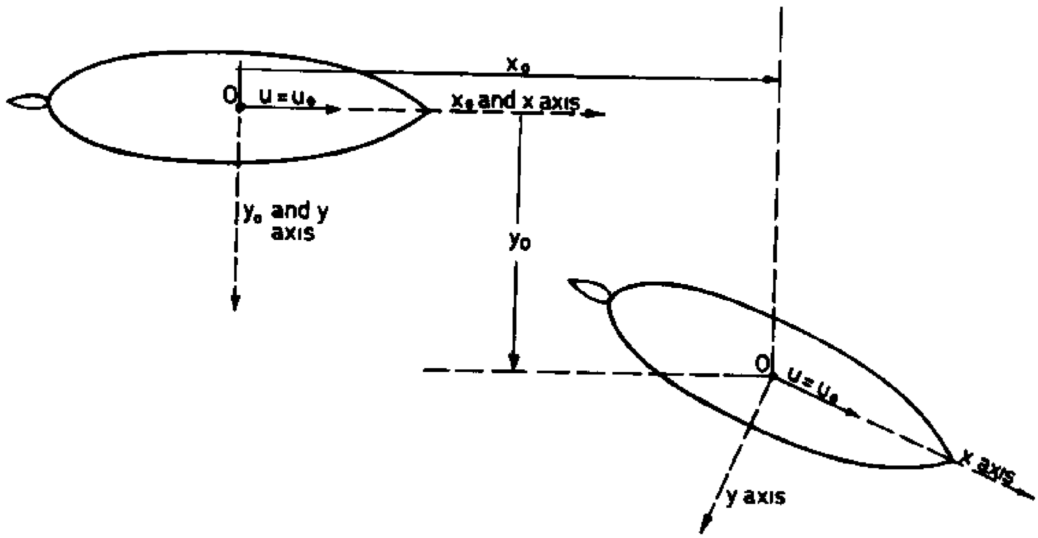
Similar linearization of the right side of the X equation gives mu and the N equation gives $I_z\dot{r}+mx_G(\dot{v}+ru_0)$. The linearized equations for motion in the horizontal plane can now be written as

$$X_{x_0}x_0+X_{y_0}y_0+X_\psi\psi+X_{\dot{u}}\dot{u}+X_u\Delta u+X_{\dot{v}}\dot{v}+X_v v+X_{\dot{r}}\dot{r}+X_r r = m\dot{u}$$

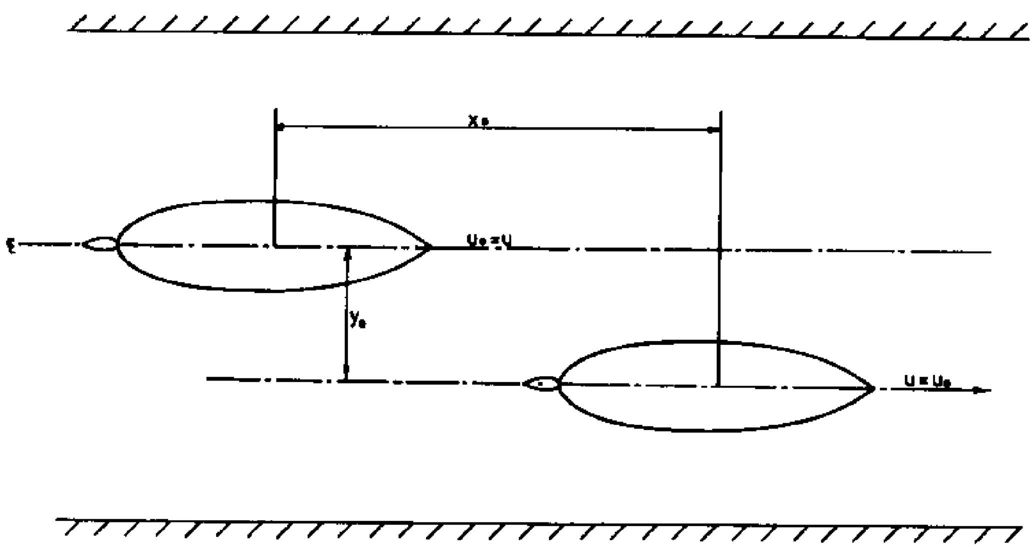
$$Y_{x_0}x_0+Y_{y_0}y_0+Y_\psi\psi+Y_{\dot{u}}\dot{u}+Y_u\Delta u+Y_{\dot{v}}\dot{v}+Y_v v+Y_{\dot{r}}\dot{r}+Y_r r = m\ (\dot{v}+ru_0+x_G\dot{r})$$

$$N_{x_0}x_0+N_{y_0}y_0+N_\psi\psi+N_{\dot{u}}\dot{u}+N_u\Delta u+N_{\dot{v}}\dot{v}+N_v v+N_{\dot{r}}\dot{r}+N_r r = I_z\dot{r}+mx_G(\dot{v}+ru_0)$$

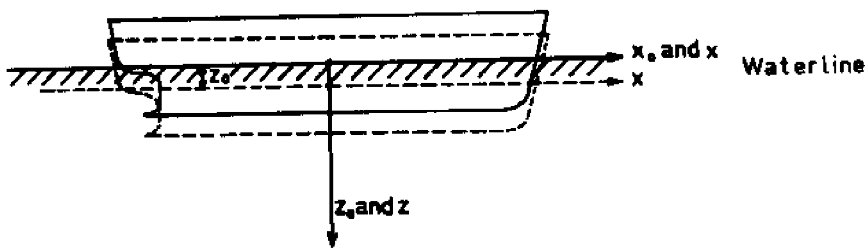
It will now be shown that the derivatives X_{x_0}, X_{y_0}, X_ψ, Y_{x_0}, Y_{y_0}, Y_ψ, N_{x_0}, N_{y_0}, N_ψ are all zero. These derivatives indicate the change brought about in the function when a given variable is changed slightly from the equilibrium value, with all other variables remaining at their equilibrium values. Hence, if the equilibrium condition is straight ahead motion at constant speed, the fact that the ship is oriented differently on the surface of the fluid, but still going straight ahead at constant speed, does not cause any forces to be exerted on the ship. For example, in the sketch below,



the orientation parameters of the ship are changed while all the remaining variables are at the equilibrium condition and it is clear that no forces or moments are exerted on the ship due to this change in orientation. This condition does not apply when a ship is sailing in a narrow canal, since, if the ship becomes oriented closer to the wall of the canal, hydrodynamic forces are created tending to draw the ship toward the near wall.



In unrestricted water, the forces on a ship due to orientation are essentially hydrostatic and hence in the vertical direction, therefore no hydrostatic forces are expected for orientation changes in the horizontal plane. For motion in the vertical plane, significant forces due to a change in vertical orientation are produced as can be seen from the example shown below for a change in vertical orientation, z_0.



A change in vertical orientation, caused by depressing the ship an amount of z_0 into the water, causes a hydrostatic force upward equal to the change in displacement resulting from the amount of orientation change, z_0. With the hydrodynamic derivatives involving the orientation parameters set equal to zero, and with the terms on the right side of the equation brought over to the left side of the equation and combined with similar terms on the left side, the equations of motion become

$$(X_{\dot{u}}-m)\dot{u}+X_u\,\Delta u+X_{\dot{v}}\dot{v}+X_v v+X_{\dot{r}}\dot{r}+X_r r = 0$$

$$Y_{\dot{u}}\dot{u}+Y_u\,\Delta u+(Y_{\dot{v}}-m)\dot{v}+Y_v v+(Y_{\dot{r}}-mx_G)\dot{r}+(Y_r-mu_0)r = 0$$

$$N_{\dot{u}}\dot{u}+N_u\,\Delta u+(N_{\dot{v}}-mx_G)\dot{v}+N_v v+(N_{\dot{r}}-I_z)\dot{r}+(N_r-mx_G u_0)r = 0$$

It is interesting to note that the coefficients of the "acceleration" terms $\dot{u}$ and $\dot{v}$ essentially have the mass increased by $X_{\dot{u}}$ and $Y_{\dot{v}}$ respectively and the coefficient of angular acceleration $\dot{r}$ has the inertia increased by $N_{\dot{r}}$. These acceleration derivatives are a result of hydrodynamic forces and represent the linear term of the Taylor expansion of the force and moment due to acceleration. $X_{\dot{u}}$, $Y_{\dot{v}}$, and $N_{\dot{r}}$ are all negative in value (as will be shown later) and therefore add in absolute magnitude to the mass or inertia in the coefficients of the accelerations. Hence, the labels of "added mass" are sometimes given to $X_{\dot{u}}$ and $Y_{\dot{v}}$ and "added inertia" to $N_{\dot{r}}$, and the combination of the mass and inertia respectively with these terms are sometimes called "virtual mass" or "virtual inertia" since the ship behaves in water with respect to acceleration as if the mass and inertia had these increased values. Some like to consider these added quantities as the amount of water the ship drags along with it as it accelerates, but this concept is physically wrong.

CHAPTER III

Solution of the Linearized Equation of Motion

We now wish to solve the three equations of motion in order to determine what the motion of the ship will be when disturbed from its original equilibrium condition of straight ahead motion. From this solution we shall develop an analysis of the motion to determine and test under what condition the motion will be stable, i.e. whether the ship can indeed maintain straight line motion with its rudder undeflected.

The solution will give as to how each of the variables, Δu, $\dot{u}$, v, $\dot{v}$, r, and $\dot{r}$ vary with time after the disturbance from the equilibrium condition. On first appearance it looks like there are six unknowns and only three equations. However, if solutions are obtained for Δu, v, and r as functions of time, then $\dot{u}$, $\dot{v}$, and $\dot{r}$ as functions of time can be obtained by differentation with respect to time of the functions Δu, v, and r. Hence, $\dot{u}$, $\dot{v}$, and $\dot{r}$ are dependent variables and there are only three independent parameters with the three equations. We are in a position now to solve the equations for the unknown Δu, v, and r as functions of time.

The equations will be solved using the operational calculus technique since only the elementary aspects of this technique need be explained to carry through the solution. Regular integral calculus or Laplace transforms could be used as well. When the differential operator, $\mathcal{D} = \frac{d}{dt}$, is introduced and used in the equations replacing the time derivatives in the manner indicated below:

$$\dot{u} = \frac{du}{dt} = \frac{d}{dt}(u_0 + \Delta u) = \frac{d}{dt}(\Delta u) = \mathcal{D}\,(\Delta u)$$

$$\dot{v} = \mathcal{D} v\,, \quad \dot{r} = \mathcal{D} r$$

the equations take the form

$$\overset{a_{11}}{\left[(X_{\dot{u}}-m)\mathcal{D}+X_u\right]}\Delta u+\overset{a_{12}}{\left[X_{\dot{v}}\mathcal{D}+X_v\right]}v+\overset{a_{13}}{\left[X_{\dot{r}}\mathcal{D}+X_r\right]}r=0$$

$$\overset{a_{21}}{\left[Y_{\dot{u}}\mathcal{D}+Y_u\right]}\Delta u+\overset{a_{22}}{\left[(Y_{\dot{v}}-m)\mathcal{D}+Y_v\right]}v+\overset{a_{23}}{\left[(Y_{\dot{r}}-mx_G)\mathcal{D}+(Y_r-mu_0)\right]}r=0$$

$$\overset{a_{31}}{\left[N_{\dot{u}}\mathcal{D}+N_u\right]}\Delta u+\overset{a_{32}}{\left[(N_{\dot{v}}-mx_G)\mathcal{D}+N_v\right]}v+\overset{a_{33}}{\left[(N_{\dot{r}}-I_z)\mathcal{D}+(N_r-mx_G\ u_0)\right]}r=0.$$

The letter "a" with the various subscripts are used to denote the nine coefficients of the variables u, v, and r involved in the three equations. If one could use a straightforward algebraic technique to solve these equations - i.e. if a_{11}, a_{12}, etc. were regular numerical coefficients, the solution for the variable r, for example, would be:

$$r=\frac{\begin{vmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}=\frac{0}{|\mathcal{D}et.|}$$

where $|\mathcal{D}et.|$ is used to designate the determinant in the denominator, with similar expressions for the solutions for Δu, and v. In operational calculus, it is shown that the operator $\mathcal{D}$ can be treated as an algebraic quantity provided the other terms in the coefficients are constants with respect to time. Since the hydrodynamic derivatives, defined as the slope of the force or moment versus a dynamic variable <u>taken at the equilibrium condition</u>, the terms in the coefficients other than $\mathcal{D}$ are constants in time. Hence, $\mathcal{D}$ can be treated as an algebraic quantity but certain interpretations with respect to the algebraic solution must be made in order to give the same result as would be obtained through formal integral calculus. To develop these interpretations, two simple examples are given below, where in each example the solution is demonstrated as calculated by the algebraic process and by formal calculus. Let us consider the differential equation

$$\frac{dz}{dt} = f(t) \quad \text{or} \quad \mathcal{D}z = f(t)$$

where z is some function of time. (z is some arbitrary variable not to be associated with the z used in the ship axis system). The algebraic solution for z is

$$z = \frac{1}{\mathcal{D}} f(t)$$

and the formal solution is

$$z = \int f(t)dt$$

If the algebraic solution is to be made equal to the formal solution, then one must interpret the operation $\frac{1}{\mathcal{D}}$ as follows,

$$\frac{1}{\mathcal{D}} = \int (\quad)dt$$

or that the inverse of differentiation is integration, as one well knows. The other example is the differential equation

$$\frac{dz}{dt} - az = f(t) \quad \text{or} \quad (\mathcal{D} - a)z = f(t)$$

The algebraic solution is given by

$$z = \frac{1}{(\mathcal{D} - a)} f(t)$$

and the formal calculus solution (as you may recall from previous mathematics) is given by

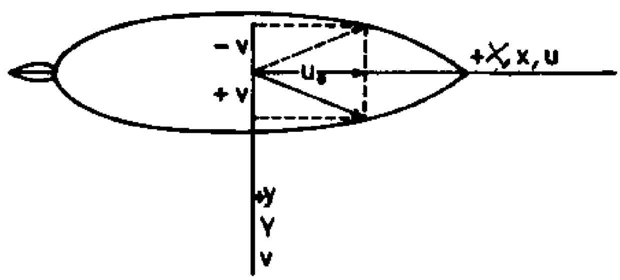
$$z = e^{at} \int e^{-at} f(t)dt$$

If the algebraic solution is to be made equal to the calculus solution, then the operation $\frac{1}{\mathcal{D} - a}$ must be interpreted as

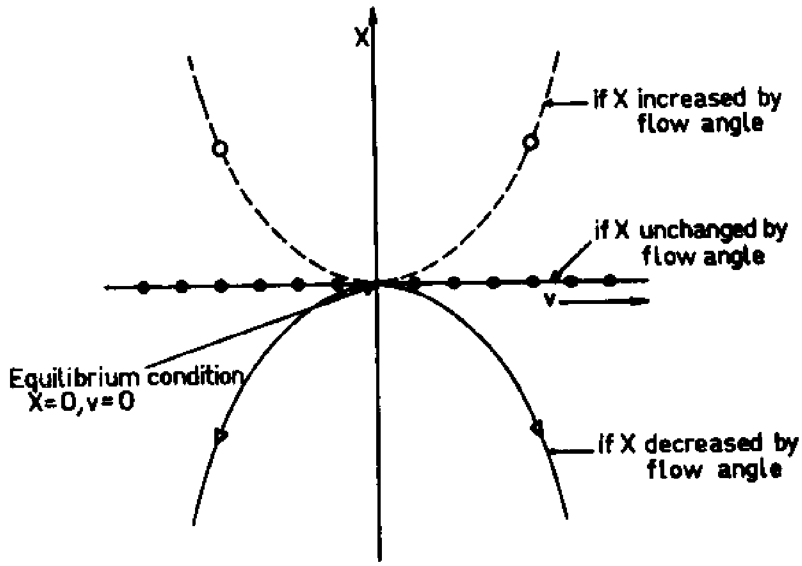
$$(\frac{1}{\mathcal{D} - a}) = e^{at} \int e^{-at}(\quad)dt$$

where whatever is operated on is inserted into the parenthesis.

Before returning to the solution of the equations of motion, we shall show that the derivatives X_v, $X_{\dot{v}}$, X_r, and $X_{\dot{r}}$ are all zero for any ship or body with symmetrical shape port and starboard. This is one of the advantages, previously mentioned, of choosing axis systems in the plane of symmetry of the body. The derivative X_v represents the slope of the X force vs. v curve taken at the equilibrium condition of $u = u_0$, $v = 0$, $\dot{u} = \dot{v} = r = \dot{r} = 0$. The sketch below indicates a ship slightly disturbed from the equilibrium condition by a small disturbance + v, and then by a disturbance - v.

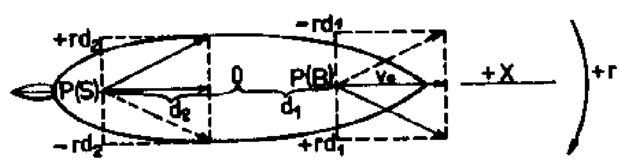


In considering how the X force varies with v, one notices that for a positive v, the approach angle of the flow to the ship is $\tan^{-1}\frac{v}{u_0}$ from starboard. Similarly, for a negative value of v, the approach angle is $\tan^{-1}\frac{v}{u_0}$ from port. Since the port and starboard side are symmetrical in shape, if an angle of flow from starboard (+v) decreases X (i.e. increases drag), then the same angle of flow from the port side (-v) must also decrease X. Similarly, if a flow from starboard increased X, then a similar flow from port will also increase X, and if flow from one side did not change X, then equal flow from the other side would not alter X either. All these deductions result from the symmetry of port and starboard. Hence for any shape, provided the port and starboard are symmetrical, the curve of X vs. v can take only one of the following shapes:

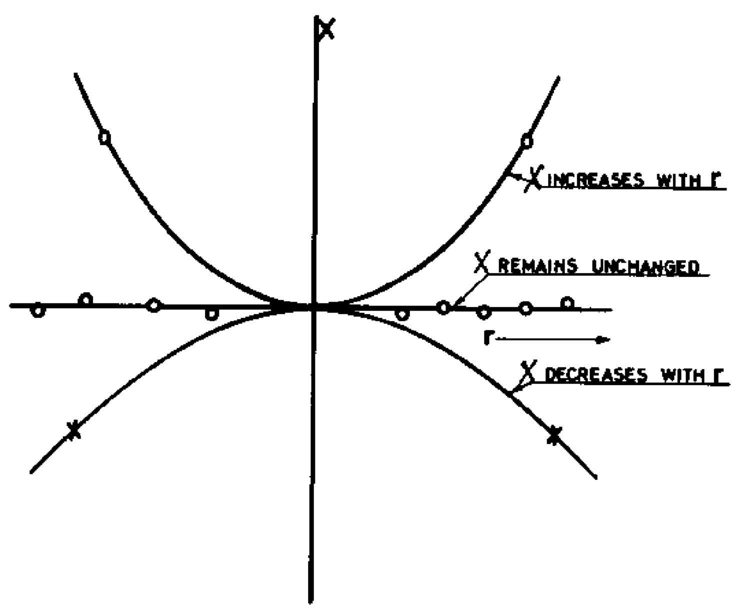


The sketch above indicates that the curve of X versus v must be symmetrical about the X axis for symmetry of port and starboard, hence the slope of the X versus v curve taken at v = 0, i.e. X_v, must be zero.

A similar situation results when considering a small disturbance in angular velocity r from the equilibrium condition, as can be seen from the following sketch.



A point B located a distance d, forward of the origin would have a transverse velocity to starboard of rd resulting from an angular velocity +r. This transverse velocity coupled with the forward velocity u_0, creates at inflow angle, at various bow positions, of $\tan^{-1}\frac{rd_1}{u_0}$, from starboard. Similarly, it can be seen that at different stern locations, the inflow angle for a +r is $\tan^{-1}\frac{rd_2}{u_0}$, from port. This type of flow, depending on the geometry of the body may increase X, decrease X, or leave it unchanged. However, for a -r, one observes that the bow sections experience an inflow angle of $\tan^{-1}\frac{rd_1}{u_0}$ from port and the stern sections an inflow angle of $\tan^{-1}\frac{rd_2}{u_0}$ from starboard. Since port and starboard have the same geometry (reflected in the x axis) and since the flow angles (or geometry of flow) are reflected in the x axis in going from +r to -r, if +r increased the X force then also -r must increase the force, with similar results for a decrease or no change in the X forces. Hence, from the symmetry properties of port and starboard, the function of X versus r must take one of the following three shapes.



Again since the function X versus r is symmetrical about the X axis, (even function), the derivative of X versus r taken at r = 0, must be zero. Hence, $X_r = 0$.

With respect to the derivatives $X_{\dot{v}}$ and $X_{\dot{r}}$, similar arguments can be presented to show that the functions X vs. $\dot{v}$ and X vs. $\dot{r}$ must be even functions because of symmetry of port and starboard. Therefore, $X_v = X_r = X_{\dot{v}} = X_{\dot{r}} = 0$ and the coefficients a_{12} and a_{13} in the equations of motion are thereby also zero.

The solution for r can now be written as

$$r = \frac{0}{\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} = \frac{0}{a_{11}(a_{22}a_{33}-a_{23}a_{32})}$$

If the denominator in this expression is other than zero, the solution for r (and subsequently v, and Δu) would be identically zero for all time for a small disturbance from the equilibrium condition. This is physically impossible, hence a solution exists only if the denominator is equal to zero. Setting the denominator equal to zero gives

$$a_{11}(a_{22}a_{33}-a_{23}a_{32}) = 0$$

and this condition is satisfied only if

$$a_{11} = 0 \text{ or } a_{22}a_{33}-a_{23}a_{32} = 0$$

$$a_{11} = (X_{\dot{u}}-m)\mathcal{D} + X_u = 0 \text{ or } (X_{\dot{u}}-m)(\mathcal{D} + \frac{X_u}{X_{\dot{u}}-m}) = (X_{\dot{u}}-m)(\mathcal{D}-\sigma_3) = 0$$

where we define

$$\sigma_3 = -\frac{X_u}{X_{\dot{u}}-m} .$$

If one expands the product $a_{22}a_{33}-a_{23}a_{32}$, the product contains terms in $\mathcal{D}^2$, $\mathcal{D}$, and independent of $\mathcal{D}$.

The product as expanded and set equal to zero becomes

$$a_{22}a_{33}-a_{23}a_{32} = A\mathcal{D}^2 + B\mathcal{D} + C = 0$$

where

$$A = (Y_{\dot{v}}-m)(N_{\dot{r}}-I_z)-(Y_{\dot{r}}-mx_G)(N_{\dot{v}}-mx_G)$$

$$B = (Y_{\dot{v}}-m)(N_r-mx_Gu_0)+(N_{\dot{r}}-I_z)Y_v-(Y_{\dot{r}}-mx_G)N_v-(N_{\dot{v}}-mx_G)(Y_r-mu_0)$$

$$C = Y_v(N_r-mx_Gu_0)-N_v(Y_r-mu_0).$$

The quadratic equation in $\mathcal{D}$ can be written in the form

$$A\mathcal{D}^2+B\mathcal{D}+C = A(\mathcal{D}-\sigma_1)(\mathcal{D}-\sigma_2) = A\mathcal{D}^2-A(\sigma_1+\sigma_2)\mathcal{D}+A\sigma_1\sigma_2$$

where σ_1 and σ_2 are the roots of the equation as given by the well known quadratic solution

$$\left.\begin{matrix}\sigma_1\\ \sigma_2\end{matrix}\right\} = \frac{-\frac{B}{A} \pm \sqrt{(\frac{B}{A})^2 - \frac{4C}{A}}}{2}$$

where it is clear that $A\sigma_1\sigma_2 = C$, and $-A(\sigma_1+\sigma_2) = B$. The solution for r now becomes

$$r = \frac{0}{A(X_u-m)(\mathcal{D}-\sigma_3)(\mathcal{D}-\sigma_2)(\mathcal{D}-\sigma_1)} = (\frac{1}{\mathcal{D}-\sigma_3})(\frac{1}{\mathcal{D}-\sigma_2})(\frac{1}{\mathcal{D}-\sigma_1})(\ 0\ )$$

The solution will result from a sequence of operations of the form $(\frac{1}{\mathcal{D}-\sigma})$ on the value 0. The first operation gives (using the definition of the operator previously developed)

$$(\frac{1}{\mathcal{D}-\sigma_1})\ 0 = e^{\sigma_1 t}\int e^{-\sigma_1 t}\,0\ dt = e^{\sigma_1 t}\int 0\ dt = e^{\sigma_1 t}\left[C_1\right] = C_1 e^{\sigma_1 t}$$

where C_1 is a constant of integration.

Continuing the operation, one obtains

$$(\frac{1}{\mathcal{D}-\sigma_2})(C_1 e^{\sigma_1 t}) = e^{\sigma_2 t}\int e^{-\sigma_2 t} C_1 e^{\sigma_1 t} dt = e^{\sigma_2 t}\int C_1 e^{(\sigma_1-\sigma_2)t} dt$$

$$= e^{\sigma_2 t}\left[(\frac{C_1}{\sigma_1-\sigma_2})\ e^{(\sigma_1-\sigma_2)t} + C_2\right] = C_1^{\,1} e^{\sigma_1 t} + C_2 e^{\sigma_2 t}$$

where $C_1^{\,1}$ is an arbitrary constant of integration since C_1, being arbitrary, divided by a fixed quantity $\sigma_1 - \sigma_2$, is also arbitrary. (In the exceptional case where $\sigma_1 = \sigma_2$, the integration

$$\int e^{(\sigma_1-\sigma_2)t} dt = \int e^0 dt = \int dt = t+C_2$$

and this results in the form $C_1 t e^{\sigma_2 t} + C_2 e^{\sigma_2 t}$).

The final operation gives the solution for r.

$$r = \left(\frac{1}{\mathscr{D}-\sigma_3}\right)(C_1{}^1 e^{\sigma_1 t} + C_2 e^{\sigma_2 t}) = e^{\sigma_3 t}\int e^{-\sigma_3 t}\left[C_1{}^1 e^{\sigma_1 t} + C_2 e^{\sigma_2 t}\right] dt$$

$$= e^{\sigma_3 t}\int\left[C_1{}^1 e^{(\sigma_1-\sigma_3)t} + C_2 e^{(\sigma_2-\sigma_3)t}\right] dt$$

$$= \left(\frac{C_1{}^1}{\sigma_1-\sigma_3}\right) e^{\sigma_1 t} + \left(\frac{C_2}{\sigma_2-\sigma_3}\right) e^{\sigma_2 t} + C_3 e^{\sigma_3 t}$$

$$r = r_1 e^{\sigma_1 t} + r_2 e^{\sigma_2 t} + r_3 e^{\sigma_3 t}$$

where r_1, r_2, and r_3 are constants of integration. Since the algebraic solutions for Δu and v are the same form as the solution for r, namely

$$\left.\begin{matrix}\Delta u\\ v\\ r\end{matrix}\right\} = \frac{0}{(\mathscr{D}-\sigma_3)(\mathscr{D}-\sigma_2)(\mathscr{D}-\sigma_1)}$$

then the actual solution for Δu and v is

$$\Delta u = u_1 e^{\sigma_1 t} + u_2 e^{\sigma_2 t} + u_3 e^{\sigma_3 t}$$

$$v = v_1 e^{\sigma_1 t} + v_2 e^{\sigma_2 t} + v_3 e^{\sigma_3 t}$$

where u_1, u_2, u_3, v_1, v_2, and v_3 are constants of integration.

The solutions obtained describe how the motion of the ship will vary with time after an initial disturbance from straight line motion. Analysis of this solution leads us to determine under what conditions the ship will be stable in straight line motion and will furnish us with a criteria for this stability.

CHAPTER IV

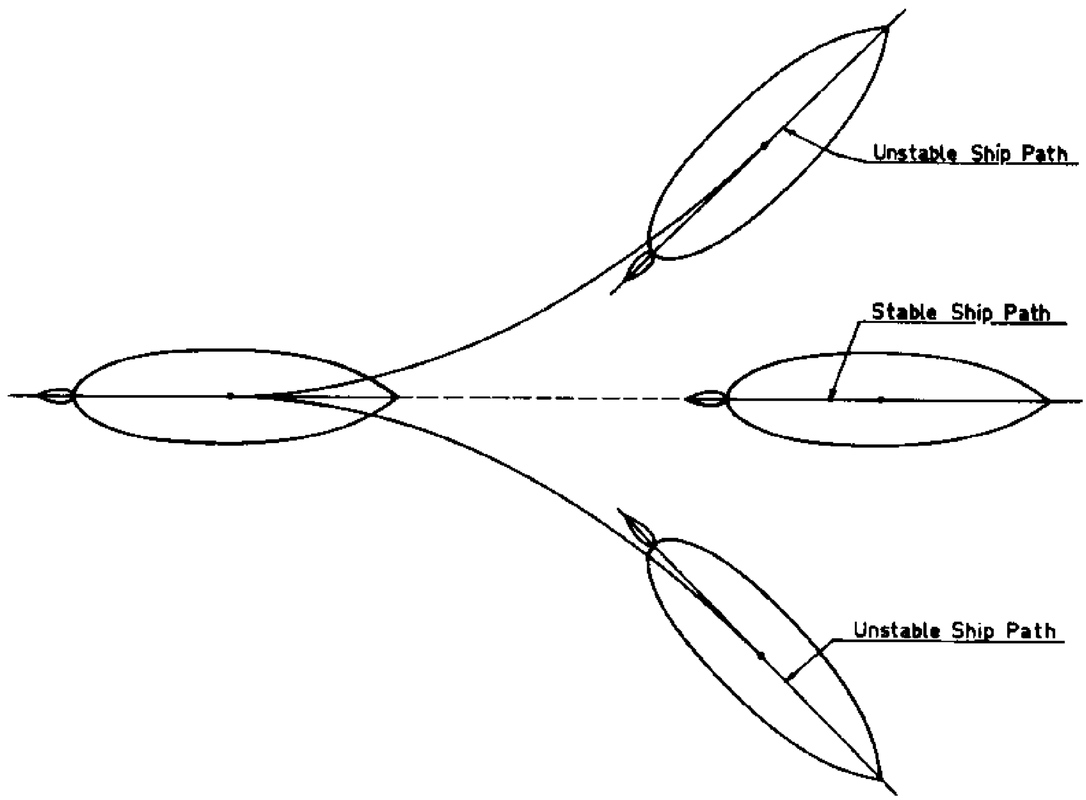
Stability of a Ship in Straight Ahead Motion

The test for any type of stability is to establish an equilibrium situation, and determine whether the system returns to the original condition of equilibrium after a disturbance of the smallest amount (infinitesimal disturbance). If it returns, or tends to return, to the original equilibrium condition when the disturbance is removed, it is stable. If it departs, or has the tendency to depart, from the original equilibrium condition, the original equilibrium condition is unstable. The usual case for a body which is unstable in a given condition of equilibrium, is to depart from that condition until it reaches another equilibrium condition (not the original one) which is a stable one. This is the way one goes about testing a ship for stability in roll. The ship is disturbed slightly from its upright equilibrium position and when the disturbance is removed the tendency to return to the original upright position is observed. If it returns, it is stable, if it departs, it is unstable. An unstable ship in heel, cannot remain in the upright equilibrium condition, but, in the absence of disturbance, heels (flops) either to starboard or port until a new angle of heel is reached which is a stable one (new position of equilibrium).

In this manner, one tests the equilibrium condition of straight ahead motion at constant speed for stability*). Just as in the case of stability in roll where an unstable ship cannot remain upright when there is no heeling moment, a ship which is dynamically unstable in straight

*) Straight ahead motion at constant speed is a condition of equilibrium, since there are no linear or angular accelerations and therefore no net forces or moments acting on the body.

line motion cannot maintain straight line motion when there is no rudder deflection. The unstable ship will go into a starboard or port turn without any rudder deflection as indicated by the sketch below.



The ship which is dynamically unstable in straight line motion, can maintain a straight course (on the average) only by continuous use of the rudder.

As mentioned earlier, the linearized equations of motion were solved, furnishing certain parameters of the ship motion as functions of time. These solutions will now be used to analyse whether a ship is capable of maintaining straight ahead motion (without rudder application) and thereby determine whether it is dynamically stable in this motion. The solution for the angular velocity, r, as a function of time was

$$r = r_1 e^{\sigma_1 t} + r_2 e^{\sigma_2 t} + r_3 e^{\sigma_3 t}$$

where r_1, r_2, and r_3 were arbitrary constants of integration depending on initial conditions, and the roots σ_1, σ_2, and σ_3 were expressed in terms of certain combinations of the various hydrodynamic derivatives.

(In the case of equal roots, the solution was $r = r_1 e^{\sigma_1 t} + r_2 t e^{\sigma_1 t} + r_3 e^{\sigma_3 t}$). The equations represented the ship motions in the absence of any disturbance and therefore represent the behavior of the ship when a (slight) disturbance is removed. Since r is the angular velocity, straight ahead motion is only satisfied when r = 0. Therefore, the test for stability in straight line motion is for r to go to zero as time increases (time being counted from when the disturbance is removed). Since r_1, r_2, and r_3 are arbitrary constants, and in addition, since in general σ_1, σ_2, and σ_3 are different in value, the terms $r_1 e^{\sigma_1 t}$, $r_2 e^{\sigma_2 t}$, and $r_3 e^{\sigma_3 t}$ cannot negate one another. Hence, the only condition under which r will go to zero in time is for each term to go to zero as time increases. The only way that each term can go to zero with increasing time is for each of the exponents to be negative, i.e. that σ_1, σ_2, and σ_3 all be negative quantities if they are real numbers, since as t increases to infinity, $e^{kt} \rightarrow 0$, if k is negative. If σ is a complex number in the form $\sigma = a+ib$, the following relationships hold

$$e^{\sigma t} = e^{(a+ib)t} = e^{at} e^{ibt} = e^{at}(\cos bt + i \sin bt)$$

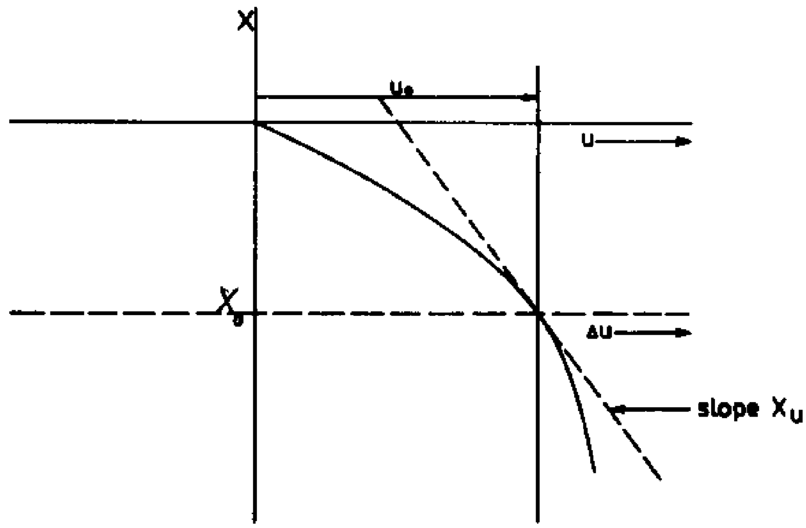
and the condition for stability requires that the real parts of σ_1, σ_2, and σ_3 be negative as they are complex numbers. (The imaginary part of the number indicates the angular frequency of oscillation, in which the motion dies down or is amplified). (In the case of equal roots, $\sigma_1 = \sigma_2$, the term $r_2 t e^{\sigma_2 t}$ goes to zero as $t \rightarrow \infty$ for σ_2 negative, since $t e^{-|\sigma_2| t} \rightarrow 0$ as $t \rightarrow \infty$).

We shall now analyse under what conditions, all three roots σ_1, σ_2, and σ_3 will be negative, if real, or have real parts which are negative, if complex. This will furnish us with a criteria for determining whether a given ship is stable or not in straight line motion.

The value of σ_3 was previously determined as

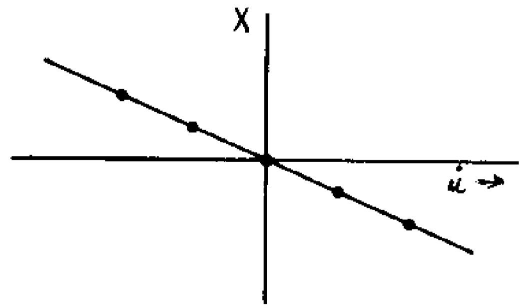
$$\sigma_3 = - \frac{X_u}{X_{\dot{u}} - m}$$

and it will be shown that σ_3 is always negative for the regular displacement type ship because of the basic nature of hydrodynamic drag. Since the direction of the positive X force is in the opposite direction to the direction of the drag (or resistance) force, the plot of X vs. speed, in the absence of any propeller thrust, would appear as follows,



with the drag increasing (X force decreasing) with speed u in some power function of u (approximately u^2 if the resistance coefficient does not change significantly). At the equilibrium speed of u_o, the propeller thrust, through the thrust deduction factor, will overcome the resistance so that at a speed u_o, the X force is zero when propeller effect is included. Hence, the plot of X vs. Δu appears to the right and below the axis for the minus drag vs. u curve. The derivative X_u is the slope of this curve taken at $\Delta u = 0$, and X_u will be negative as long as the net drag increases with speed (propeller effect included in net drag - say at constant r.p.m.). Since the net drag for displacement ships increases markedly with speed, X_u will be a relatively large negative number. For the case of a planing boat just about to plane, the value of X_u may be positive if a decrease in drag results from an increase in speed.

The derivative $X_{\dot{u}}$ is a negative quantity because of the following hydrodynamic reasons. The term $X_{\dot{u}}\dot{u}$ represents the force that a body experiences in the x direction as the result of an acceleration in the x direction. The body must accelerate the water and there is an inertial reaction force of the water (because of its density) on the body. This reaction force is in the opposite direction to the acceleration. Hence, the plot of X vs. $\dot{u}$ will look as follows



and the value of $X_{\dot{u}}$ will be negative - i.e. $\left(\frac{\partial X}{\partial \dot{u}}\right)_{\dot{u}=0} = \ominus$. For elongated bodies, such as normal ship types $X_{\dot{u}}$ is the order of about 5-10% of the mass of the ship. The value of σ_3 is then

$$\sigma_3 = \frac{-X_u}{X_{\dot{u}}-m} = -\frac{-|X_u|}{-|X_{\dot{u}}|-m} = -\frac{|X_u|}{|X_{\dot{u}}|+m} = \ominus$$

$\ominus$ indicates a minus quantity. Since X_u and $X_{\dot{u}}$ have been shown to be negative, these quantities can be designated by a minus sign times their absolute values. (Absolute value denoted by $|\ |$). Hence, σ_3 is real and negative and therefore stable. It indicates that a disturbance in speed (Δu) will tend to zero after the disturbing forces are removed - i.e. ship will return to the original equilibrium condition.

With σ_3 always a stable root for normal ship types, the stability then depends on the value of σ_1 and σ_2. Therefore, one must consider under what conditions of the coefficients A, B, and C (defined earlier) will the roots σ_1 and σ_2 have real parts which are negative. The solution for these roots were

$$\left.\begin{matrix}\sigma_1\\ \sigma_2\end{matrix}\right\} = \frac{\frac{-B}{A} \pm \sqrt{\left(\frac{B}{A}\right)^2 - \frac{4C}{A}}}{2}$$

a) For any value of $\frac{B}{A}$, whether a positive or a negative value, if $\frac{C}{A}$ is negative (i.e. $\frac{C}{A} < 0$), then

$$\left.\begin{matrix}\sigma_1\\ \sigma_2\end{matrix}\right\} = \frac{1}{2}\left(-\frac{B}{A} \pm Q\right) \text{ when } Q > \left|\frac{B}{A}\right|$$ since, if $\frac{C}{A} < 0$, $-\frac{4C}{A}$ is a positive quantity and this is added to a positive quantity $\left(\frac{B}{A}\right)^2$. The

quantity under the radical sign is positive and equals Q^2.
($Q^2 = (\frac{B}{A})^2 + \frac{4C}{A}$). The roots σ_1 and σ_2 are real quantities. However, since $Q > \frac{B}{A}$, whether $\frac{B}{A}$ is positive or negative, one of the roots, σ_1 or σ_2, must be a positive quantity and therefore unstable. Therefore, one of the conditions for stability is that $\frac{C}{A}$ not be negative - i.e. $\frac{C}{A} > 0$ for stability.

b) If $\frac{C}{A} > 0$, and $\frac{4C}{A} < (\frac{B}{A})^2$, then

$$\left.\begin{matrix}\sigma_1\\ \sigma_2\end{matrix}\right\} = \frac{1}{2}(-\frac{B}{A} \pm Q) \text{ where } Q < \left|\frac{B}{A}\right|$$

If $\frac{B}{A}$ is negative, then both roots will be real and positive. If $\frac{B}{A}$ is positive then both roots will be real and negative. Hence, an additional condition for stability (over and above $\frac{C}{A} > 0$) is that $\frac{B}{A} > 0$.

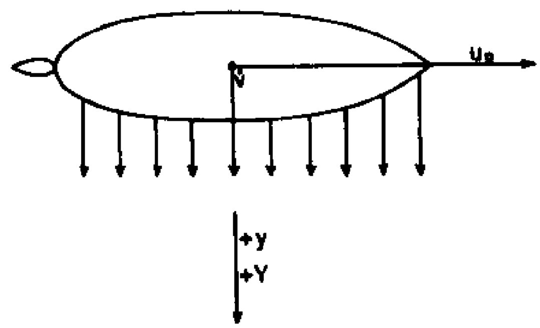
c) If $\frac{C}{A} > 0$ and $\frac{4C}{A} > (\frac{B}{A})^2$, then the roots are complex of the form

$$\left.\begin{matrix}\sigma_1\\ \sigma_2\end{matrix}\right\} = \frac{1}{2}(-\frac{B}{A} \pm iQ) \text{ where } Q = \sqrt{\frac{4C}{A} - (\frac{B}{A})^2} .$$

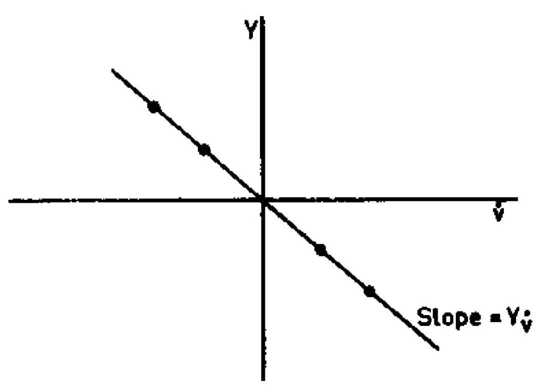
The real part of the roots is $-\frac{B}{A}$. Hence, $\frac{B}{A}$ has to be positive for stability (i.e. $-\frac{B}{A}$ has to be negative).

The conditions for stability have now been reduced to the requirements that $\frac{B}{A}$ and $\frac{C}{A}$ must both be positive quantities. The hydrodynamic derivatives appearing in the definitions of A, B, and C will now be analysed to see under what conditions $\frac{B}{A}$ and $\frac{C}{A}$ are positive and thereby develop a criterion for stability. It is necessary to establish the order of magnitude and the sign (whether positive or negative) of the various derivatives. The analysis is intended to show that for ships, A and B are always positive quantities and that the condition of stability rests on $\frac{C}{A}$ (or C) being positive.

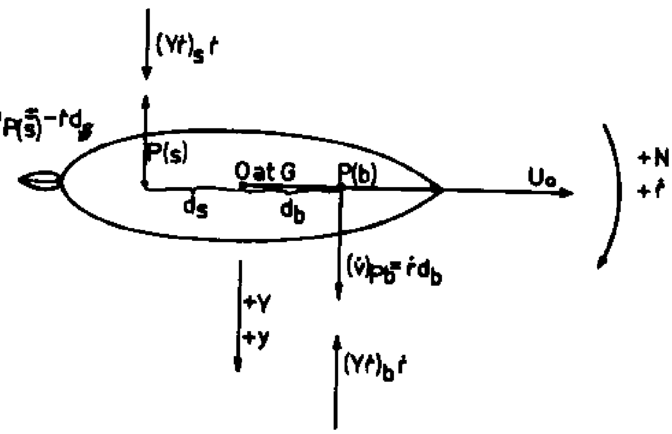
The term $Y_{\dot{v}}\dot{v}$ represents the linear approximation of the Y force resulting from an acceleration in the y direction. The sketch shown below represents the ship with an acceleration $+\dot{v}$.



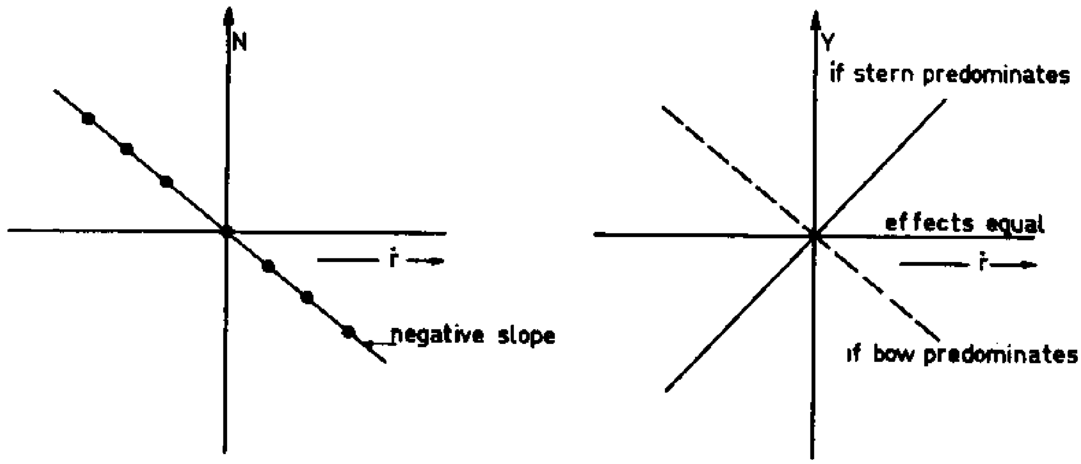
Both bow and stern experience a $\dot{v}$ acceleration in the y direction. Inertial reaction pressures of the water being accelerated by the hull produce forces in the negative y direction on both the bow and stern. Hence bow and stern effects add to give a relatively large negative Y force resulting from a positive $\dot{v}$. If a disturbance of a negative $\dot{v}$ is placed on the ship, the inertial pressures on bow and stern add together to give a relatively large Y force in the positive y direction. Hence the plot of Y versus $\dot{v}$ would appear as follows.



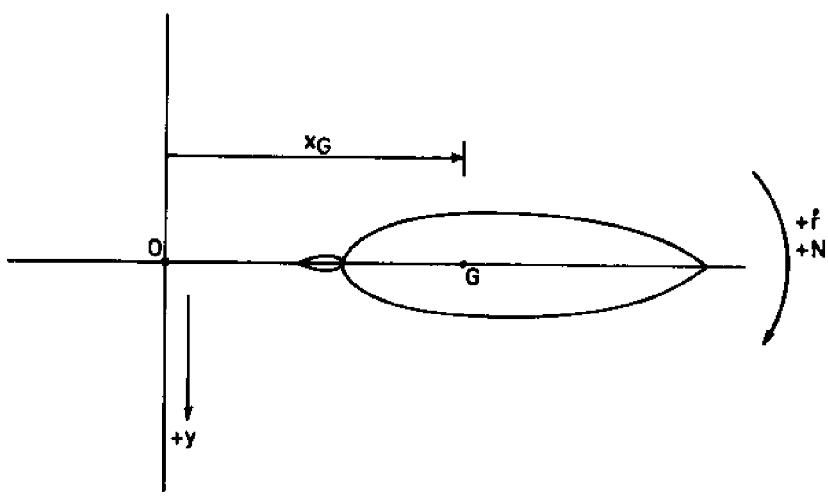
The slope taken at $\dot{v} = 0$, would be a negative value of relatively large magnitude. For elongated bodies, like ships with usual length to beam ratios, the magnitude of $Y_{\dot{v}}$ is approximately that of the ship's mass, m. (Ship in neutral buoyancy). For example, theoretically calculated (potential theory) values of $Y_{\dot{v}}$ for ellipsoids give values of -0.5 m for $\frac{L}{B} = 1$, -0.9 m for $\frac{L}{B} = 5$, -0.95 m for $\frac{L}{B} = 8.5$, and -1.0 m for $\frac{L}{B} = \infty$. Since $Y_{\dot{v}}$ for most ships are of the order of -m, then the term $(Y_{\dot{v}}-m)$ is about -2 m and represents a relatively large negative number. Similarly the derivative $N_{\dot{r}}$ is negative and relatively large as demonstrated in the following sketch.



A positive $\dot{r}$ produces a local $\dot{v}$ of $+\dot{r}d_b$ at a point P(b) at the bow and a $-\dot{r}d_s$ at the stern. The hydrodynamic force is an inertial reaction force of the water on the hull, in the opposite direction to the local acceleration. Hence, for a positive $\dot{r}$, the bow experiences a force in the negative y direction, producing a negative N moment and the stern experiences a positive Y force, but also producing a negative N moment. Hence, the moments produced at the bow and stern add to give a significant negative value for a positive $\dot{r}$. However, bow and stern produce Y forces in the opposite direction to one another - i.e. bow fights stern. A similar situation arises for a negative $\dot{r}$ - a large positive moment (bow adds to stern) and a small net Y force. Hence, $N_{\dot{r}}$ is a relatively good-size negative quantity. If bow and stern have equal effect, then $Y_{\dot{r}}$ is 0, if the bow predominates (greater pressure distribution) over the stern $Y_{\dot{r}}$ is negative. If the stern predominates, then $Y_{\dot{r}}$ is positive. Since, bow fights stern, whether positive or negative, $Y_{\dot{r}}$ will be a relatively small quantity. A sample plot of N vs. $\dot{r}$ and Y vs. $\dot{r}$ are given below.



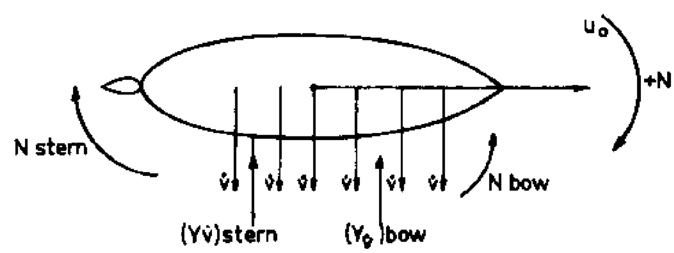
In the example discussed, the origin of the axis system was taken at the center of gravity, G, so that the value of $Y_{\dot{r}}$ represented the magnitude of the term $(Y_{\dot{r}}-mx_G)$ since $x_G = 0$. One can choose the origin arbitrarily, the resulting expressions, and the relative magnitudes of the terms in A, B, C, should not change. For example, the term $Y_{\dot{r}}-mx_G$ can be made a very large quantity by choosing an origin so that x_G is large, as indicated in the following sketch.



Under the above situation $Y_{\dot{r}}$ is made a very large negative quantity, something of the order of $(Y_{\dot{v}})x_G \approx -mx_G$, since a positive $\dot{r}$ produces a local $\dot{v}$ at G of $x_G\dot{r}$, and the hydrodynamic reaction is a large force in the negative Y direction. Hence, with a large x_G, the term $Y_{\dot{r}}$ can be made very large merely by choice of origin. However, at the same time $N_{\dot{r}}$

is increased in negative value by about $Y_{\dot{v}} x_G^{\;2}$ and I_z is increased by mx_G^2 so that the term $(N_{\dot{r}} - I_z)$ is increased by a much larger amount than $(Y_{\dot{r}} - mx_G)$. Hence, the same relationship is maintained - $(N_{\dot{r}} - I_z)$ is a relatively large quantity (negative) and $(Y_{\dot{r}} - mx_G)$ is a relatively small quantity. A similar situation obtains with the other derivatives and terms when the choice of origin is changed. Therefore, an analysis of the magnitude of the terms, with the c.g. at the origin or x_G small (the case of practically all displacement type ships) will give a proper indication of the relative magnitude of the various derivatives.

As was indicated in the analysis of the derivative $Y_{\dot{v}}$, both bow and stern add to contribute to a negative $Y_{\dot{v}}$. In the case of $N_{\dot{v}}$, the bow fights the stern as can be seen from the sketch below.



For a positive $\dot{v}$ the bow contributes to a negative N value whereas the stern contributes to a positive, and for a negative $\dot{v}$ the bow contributes to a positive value of N and the stern to a negative value. Hence, bow and stern fight each other and $N_{\dot{v}}$ is expected to be a relatively small quantity, positive if the stern predominates and negative if the bow predominates.

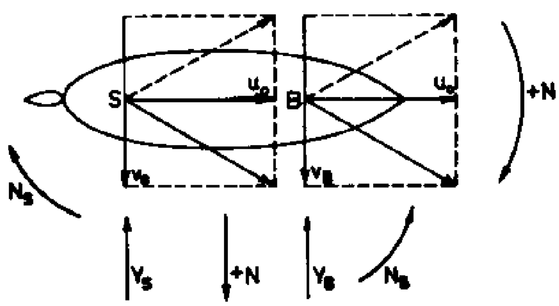
Since

$$A = (Y_{\dot{v}} - m)\,(N_{\dot{r}} - I_z) - (Y_{\dot{r}} - mx_G)\,(N_{\dot{v}} - mx_G)$$

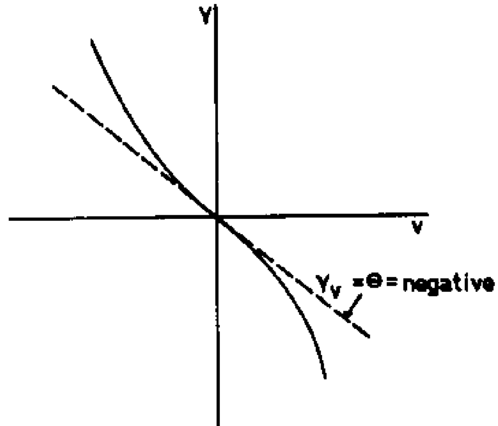
- $(Y_{\dot{v}} - m)$: large negative about $-2\ m$
- $(N_{\dot{r}} - I_z)$: large negative about $-1.8\ I_z$
- $(Y_{\dot{r}} - mx_G)$: small positive or negative
- $(N_{\dot{v}} - mx_G)$: small positive or negative

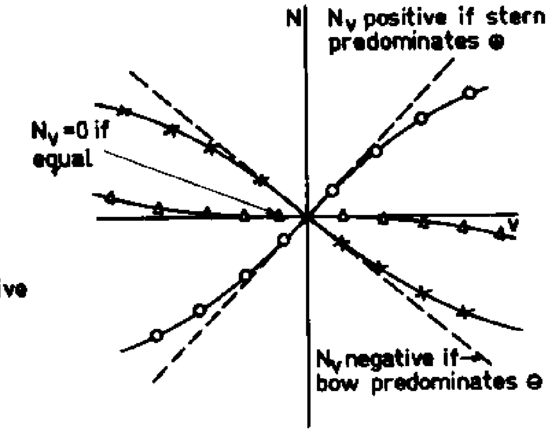
it is clear from the nature of the derivatives that the product of the first two terms is a much larger quantity than the product of the second two terms. Since the product of the first two terms is very large and positive, A must be a substantial positive quantity. Therefore, the conditions for stability of $\frac{B}{A} > 0$ and $\frac{C}{A} > 0$ become $B > 0$ and $C > 0$.

To evaluate the relative magnitudes of B and C it is necessary to look at the nature of the derivatives Y_v, N_v, Y_r, and N_r. In the following sketch the nature of the forces acting on a body with a velocity v added to a forward velocity u_o, is shown.

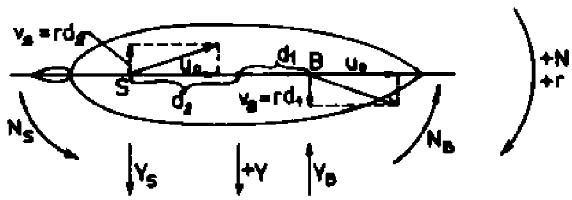


At a point B on the bow, a positive v together with a u_o give an angle of attack on the bow section, the resulting lift of which produces a force in the negative Y direction (Y_B) and a moment in the negative N direction (N_B). At point S on the stern, a positive v produces an angle of attack resulting in negative Y forces at the stern (Y_S) and a positive N moment at the stern (N_S). A negative v will reverse the direction of all the above forces and moments (angle of attack is changed to negative). Hence, in the case of Y_v, bow and stern add to give a substantial negative value for Y_v whereas bow and stern fight each other in the case of N_v. Therefore N_v is relatively small (for a moment) and is positive if the stern predominates, is negative if the bow predominates, and is zero if the contributions of bow and stern are equal. The plots of Y versus v and N versus v will have the form indicated below.

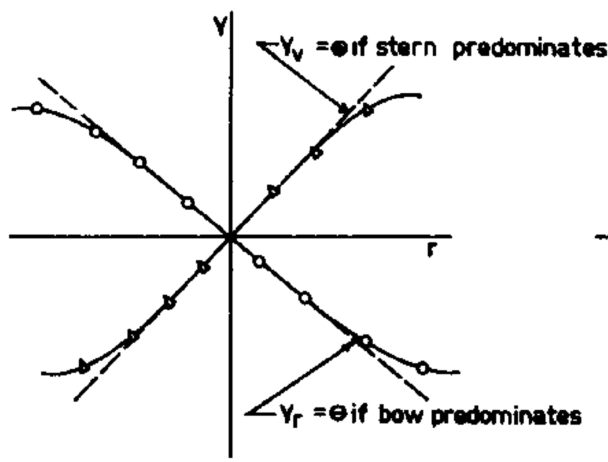


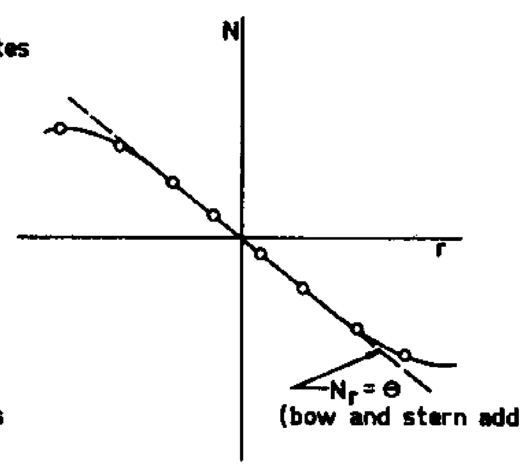


In analyzing the effect of an angular velocity r on Y and N, again a location B on the bow and S on the stern are followed.



When the ship is moving ahead with a velocity u_o and an angular velocity +r is added, point B at the bow has an angle of attack from starboard ($\approx \frac{rd_1}{u_o}$ for small r) producing a negative Y force and a negative N moment on the bow. For a +r, point S at the stern experiences an angle of attack from the port side producing a positive Y force at the stern and a negative N moment. Hence, bow and stern add to give a large negative N for a positive r, whereas bow and stern fight each other to give either a positive or negative Y force for a positive r, negative if the bow predominates. For a negative r, angles of attack change to opposite sides and hence the force and moment contributions change sign. Sample curves are indicated in the plots below.





Hence, N_r is always a substantial negative quantity (for ships) since bow and stern effects add whereas Y_r is a relatively small quantity since bow fights stern and is positive if the stern predominates and negative if the bow predominates. Also, the quantity Y_r is always in combination with mu_o and is very small compared to mu_o.**

The term, B, has been defined as

$$B = (Y_{\dot{v}}-m)(N_r-mx_Gu_o)+(N_{\dot{r}}-I_z)Y_v-(Y_{\dot{r}}-mx_G)N_v-(N_{\dot{v}}-mx_G)(Y_r-mu_o)$$

It has been indicated that $Y_{\dot{v}}-m$ is a large negative term and this term is multiplied by $(N_r-mx_Gu_o)$ which is negative and relatively large since in N_r both bow and stern add to give a negative value.*) Hence the product $(Y_{\dot{v}}-m)(N_r-mx_Gu_o)$ is a large positive number. To this is added another large positive number, that is the product $(N_{\dot{r}}-I_z)Y_v$, since it was already shown that $(N_{\dot{r}}-I_z)$ is a large negative term of the order of $-1.8\ I_z$ and that Y_v is substantially negative since bow and stern effects add to give a negative value. On the other hand the products $(Y_{\dot{r}}-mx_G)N_v$ and $(N_{\dot{v}}-mx_G)(Y_r-mu_o)$ are small and either positive or negative by virtue of the fact that the terms $(Y_{\dot{r}}-mx_G)$ and $(N_{\dot{v}}-mx_G)$ are relatively small, since bow fights stern effects in these derivatives. Hence, the sum of the first two large positive products in B greatly outweigh the possible magnitude of the last two products; therefore, B is always a positive quantity for ships. For ships, therefore, one of the conditions for stability, i.e. $B > 0$, is satisfied.

With both $A > 0$ and $B > 0$ for ships, the condition for dynamic stability in straight line motion essentially reduces to the condition that $C > 0$. The term C was defined as

$$C = Y_v(N_r-mx_Gu_o)-N_v(Y_r-mu_o).$$

The product $Y_v(N_r-mx_Gu_o)$ is a good size positive quantity in that both Y_v and N_r are negative in value with bow and stern adding to give negative values for both. On the other hand N_v can be positive or negative and since Y_r is small (positive or negative since bow fights stern), the term (Y_r-mu_o) is a relative large negative quantity. If N_v is positive, the product $N_v(Y_r-mu_o)$ is negative which when subtracted from the large positive product $Y_v(N_r-mx_Gu_o)$ further increases the positive value of C. Hence, the condition that $C > 0$ can always be satisfied (and hence insure stability) provided N_v is positive. However, it is not necessary

*) x_G is usually small. Also in a previous discussion it was shown that arbitrary choices of axes well removed from the center of gravity or midship section do not change relative values.

**) this is not so for a high lift body such as a hydrofoil where a Kutta condition has to be satisfied.

for N_v to be positive for C to be positive and in the usual case for ships N_v is not positive. With A and B positive for ships, the criteria for dynamic stability in straight line motion becomes

$$Y_v(N_r-mx_Gu_o)-N_v(Y_r-mu_o) > 0.$$

Whereas in the single degree of freedom for roll motion the criterion rested on the value of one term i.e. metacentric height being positive or negative, in the case of the two degrees of motion freedom for motion in the horizontal plane (Y and N), the four terms Y_v, N_r, N_v, and Y_r are involved in the criterion for stability. The hydrodynamic analysis of these four derivatives, carried out previously indicated the following.

- Y_v bow and stern effects add to give negative values.
- N_r bow and stern effects add to give negative values.
- N_v bow and stern effects fight each other. If bow predominates N_v is negative, if the stern predominates, N_v is positive.
- Y_r bow and stern effects fight each other. If bow predominates Y_r is negative, if stern predominates Y_r is positive. Y_r is not a sensitive parameter since mu_o is the predominant term in (Y_r-mu_o).

Since making C more positive improves the stability, design changes which in effect add positive amounts to C will improve the stability. If the bow lifting forces are increased relative to those at the stern, by a lines change or by the addition of lifting surfaces or fins at the bow, then N_r and Y_v are made somewhat more negative and their product more positive. However, additional forces at the bow make N_v less positive or more negative. Since in N_v, bow fights stern, small magnitude changes in N_v are large percentage changes so that the product $N_v(Y_r-mu_o)$ can be changed markedly in relative magnitude as well as changing sign. Y_v and N_r changes by adding lifting surfaces are smaller percentages than occur with N_v.

Hence, lifting surfaces at the bow tend to make C more negative (or less positive) than if the surfaces were more aft. If additional lifting forces are produced at the stern, Y_v and N_r are made more negative, as in the case of forces at the bow, but N_v is made more positive (or less negative) thereby improving the stability. Therefore, adding lifting surfaces towards the bow are relatively ineffective and perhaps have a negative effect on stability , but adding lifting surfaces at the stern has a strong tendency to stabilize. Hence, a fine stern with neat flow lines (no separation), and deadwood or stabilizing fins aft will improve stability. Since small length/beam ratios prevent fine sterns, a tendency to instability may exist on ships with small length to beam ratios.

One may be tempted to design a ship so that N_v is positive and hence guaranteeing a very stable ship. It must be remembered that stability indicates the tendency to go in straight line motion when subjected to small transient disturbances. Since steering of a ship is effected by producing disturbing force and moment by a rudder deflection, a too stable ship will not turn as tight as a somewhat less stable ship. Hence, a too stable ship may compromise the maneuverability of the ship. On the other hand, an unstable ship will not be able to go straight but will require constant use of the rudder. A ship should be designed for a moderate amount of stability so as to be able to go straight but not so much stability as to compromise maneuverability. This situation is similar to having a positive metacentric height to ensure stability, but not too large a metacentric height so as to have too rapid a rolling motion and rolling accelerations.

It can be seen that the further forward the center of gravity is from the center of geometry (midship section) the more stable the situation by observing the effect of increasing the positive value of x_G in the term $(N_r - m x_G u_o)$. In fact, qualitatively speaking, if the center of dynamic pressure is aft of the center of gravity, the ship will be stable in straight line motion. This is analogous to the condition in roll stability that the center of hydrostatic pressure (metacenter) has to be above the center of gravity for the ship to be stable in roll (i.e. positive metacentric height).

Before indicating how the actual values of the various derivatives are obtained for a given ship design, a rather simple stability analysis of the very familiar case of a ship in roll will be carried out by means of the more formal approach taken in the analysis of the more complicated case of ship motion in the horizontal plane. This is done in order to show, in a formal way, the criterion of positive metacentric height for stability. For the single degree of motion freedom in roll, the linearized equation of motion becomes

$$K = K_{\dot{p}}\dot{p} + K_p p + K_\varphi \varphi = I_x \dot{p}$$

where it may be recalled that

- K is the roll moment
- φ is the angle of roll (or heel)
- p is the angular velocity of roll, i.e. $p = \dot{\varphi} = \mathcal{D}\varphi$ *)
- $\dot{p}$ is the angular acceleration, i.e. $\dot{p} = \ddot{\varphi} = \mathcal{D}^2\varphi$
- I_x is the moment of inertia about the x axis

The roll moment experienced by the ship is a function of the orientation and motion variables φ, p, and $\dot{p}$ (single degree of freedom system) and the linearization of the function of these variables gives the derivatives $K_{\dot{p}}$, K_p, and K_φ .

The roll equation can be written as

$$\Big[\underbrace{(K_{\dot{p}}-I_x)}_{A}\mathcal{D}^2+\underbrace{K_p}_{B}\mathcal{D}+\underbrace{K_\varphi}_{C}\Big]\varphi = 0$$

or

$$\left[A\mathcal{D}^2+B\mathcal{D}+C\right]\varphi = 0.$$

Following the system of solving linear differential equations with constant coefficients by use of the operator $\mathcal{D}$ as was done previously, the solution for φ as a function of time is given by

$$\varphi = \frac{0}{A\mathcal{D}^2+B\mathcal{D}+C} = \frac{0}{A(\mathcal{D}-\sigma_1)(\mathcal{D}-\sigma_2)} = \varphi_1 e^{\sigma_1 t} + \varphi_2 e^{\sigma_2 t}$$

where σ_1 and σ_2 are the roots of the quadratic in the denominator and φ_1 and φ_2 are constants of integration.

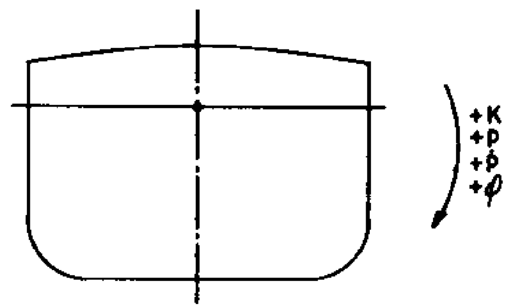
$$\left.\begin{matrix}\sigma_1\\ \sigma_2\end{matrix}\right\} = \frac{1}{2}\left[-\frac{B}{A} \pm \sqrt{\left(\frac{B}{A}\right)^2 - \frac{4C}{A}}\right]$$

Again, as in the previous case, the roots σ_1 and σ_2 will be stable roots if $\frac{B}{A} > 0$ and $\frac{C}{A} > 0$. These conditions are

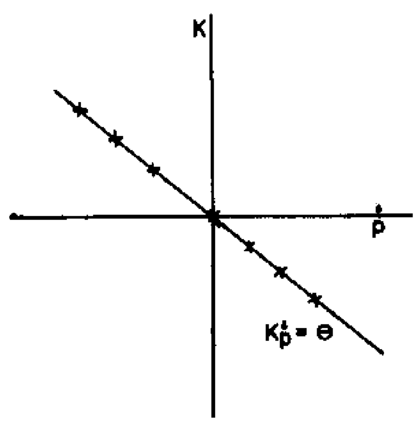
$$\frac{K_p}{K_{\dot{p}}-I_x} > 0 \quad \text{and} \quad \frac{K_\varphi}{K_{\dot{p}}-I_x} > 0.$$

*) $\mathcal{D}$ is the differential operator, $\mathcal{D} = \frac{d}{dt}$.

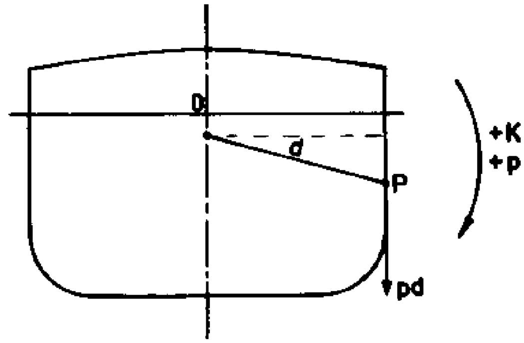
To evaluate the nature of $K_{\dot{p}}$, one observes the hydrodynamic effect of an acceleration in roll using the sketch below.



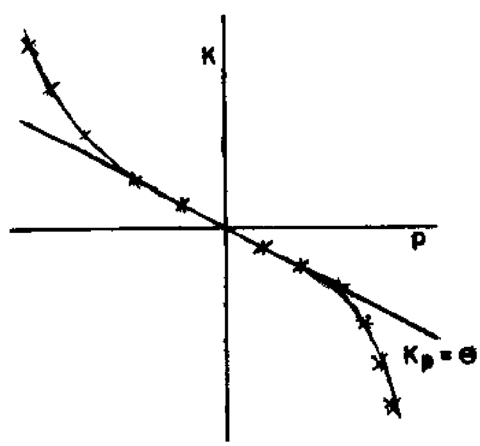
For a positive $\dot{p}$, the hydrodynamic pressure on the hull results from the inertial reaction of the hull accelerating fluid along with it. Hence, the reaction moment is opposed to the direction of acceleration. The plot of K versus $\dot{p}$ would appear as follows (positive $\dot{p}$ producing a negative K and a negative $\dot{p}$ producing a positive K).



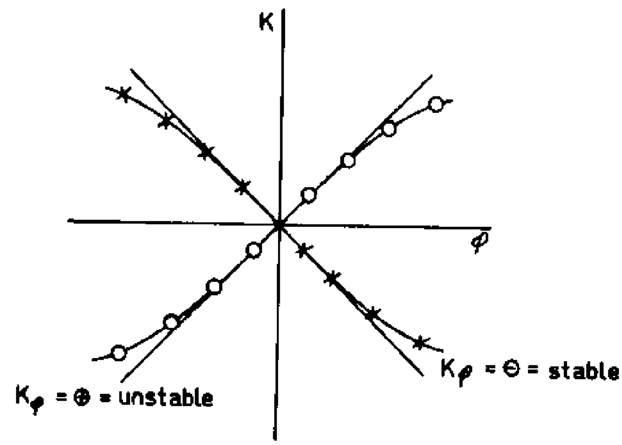
The derivative $K_{\dot{p}}$ is therefore negative and since I_x is a positive quantity, $(K_{\dot{p}}-I_x)$ is always negative in value. The condition for stability then becomes that K_p and K_ϕ are both negative in value. The derivative K_p can be shown to be negative for the following reasons.



A positive angular velocity p produces a linear velocity pd at point P on the ship surface. This results in a skin friction force opposite to pd which produces a rolling moment opposite in direction to p. Also eddy resistance around the bilge will produce a moment opposite to p. In addition, these pressures can cause surface waves to be generated and radiated from the ship (this is energy dissipation and indicates damping). Hence, the curve of K versus p will have a negative slope (i.e. K_p is negative in value) and will appear as



With K_p always negative, the criterion for stability now becomes the condition that K_φ be negative. This means that in the plot of K versus φ, the slope K_φ must be negative for stability.



Since for a positive angle of roll φ , the righting moment is opposite to the direction of roll, and since a positive metacentric height produces a positive righting moment, the condition that K_{φ} be negative is identical to the condition that the metacentric height be positive. Hence, the criteria for roll stability of positive metacentric height (or negative K_{φ}) has been demonstrated in the more formal manner.

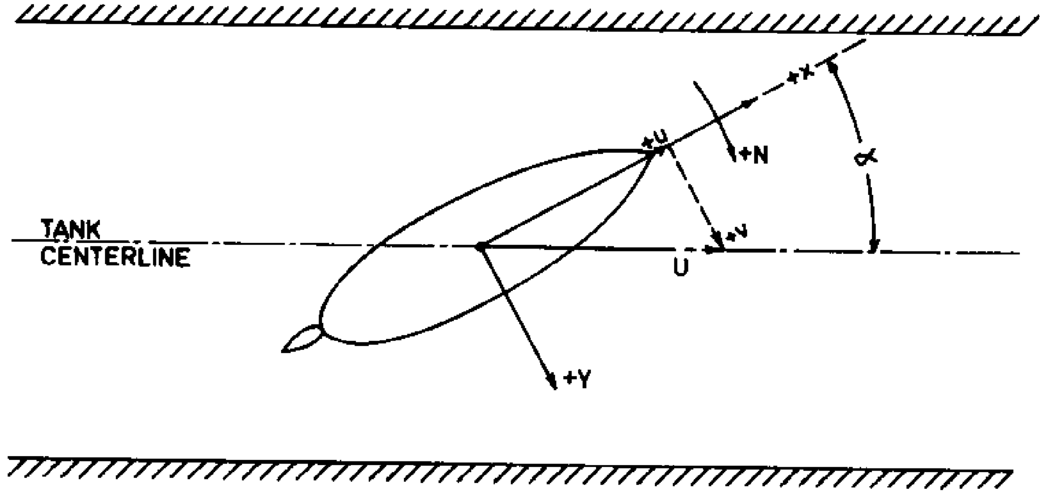
CHAPTER V

Testing Techniques Used for the Measurement of Hydrodynamic Derivatives

The various hydrodynamic derivatives, which appear in the equations of motion and in the criteria for motion stability have numerical values which depend on the geometry (i.e. design) of the ship. In the case of stability criteria in roll, the metacentric height can be calculated from the ship lines by rather simple hydrostatic theory. In the case of dynamical stability in straight line motion, the various derivatives involve calculating forces and moments acting on a given ship design not only while it is moving ahead with a velocity u_o but also while it is experiencing sidewise velocity and angular velocities as well. In the case of the relatively simple motion case of straight ahead motion at constant speed, no adequate theory or calculation exists to predict the resistance of the ship, and resort is made to testing ship models to obtain the necessary information. The more complicated motion involving velocity components in addition to straight ahead motion has not been rendered solvable by present theories or calculations, hence resort to ship model tests of a special nature must be made in order to determine certain hydrodynamic derivatives for a given design.

Some of the model testing techniques used for measuring the derivatives, especially the four derivatives which appear in the stability criterion, will be discussed.

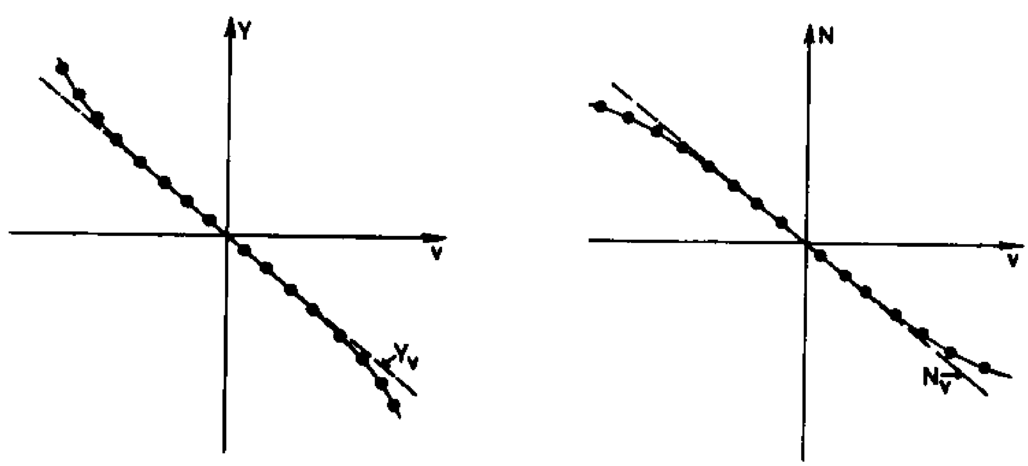
The derivatives Y_v and N_v can be measured on a model by towing a model of the ship at the proper speed (proper Froude number) at various angles of attack to the model path. The sketch below indicates the nature of the model test.



From the sketch, it can readily be observed when a model is towed down the centerline of the tank with a velocity U and at an angle of attack α from starboard, that a velocity component v along the positive y axis is produced such that

$$v = U \sin\alpha$$

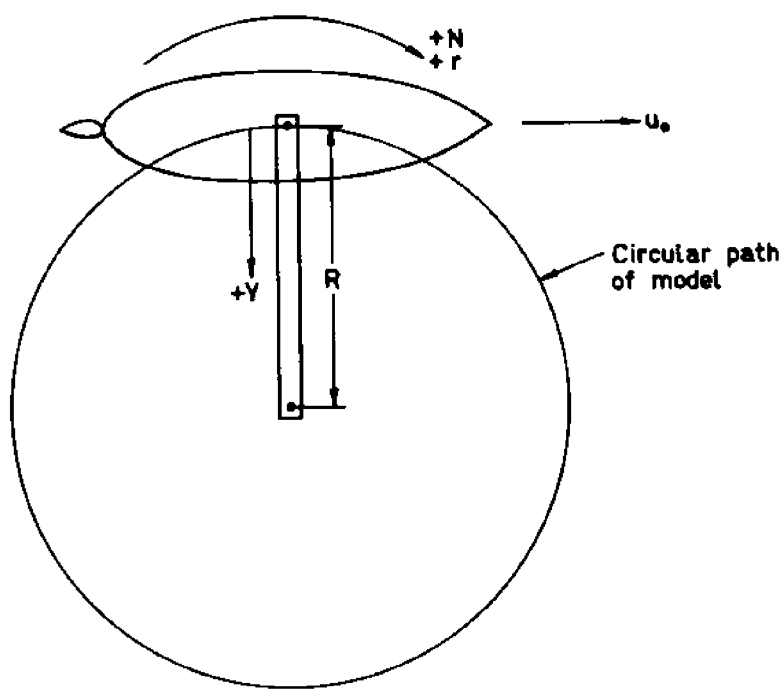
A dynamometer measures the force Y and moment N experienced by the model at each of a series of angles of attack. These measurements are then plotted versus v, producing the typical plots shown below.



The slopes of these plots, taken at v = 0, give numerical values for the derivatives Y_v and N_v for the model. These derivatives can be reduced to non-dimensional form or converted to ship dimensions using the dimensions of length L, speed U, and density ρ as will be indicated later.

Since a rotating propeller acts as a lifting surface, the various model tests should be conducted with propellers operating, preferably at the ship propulsion points. Since the undeflected rudder acts as a lifting surface, model tests should include the rudder in the undeflected position.

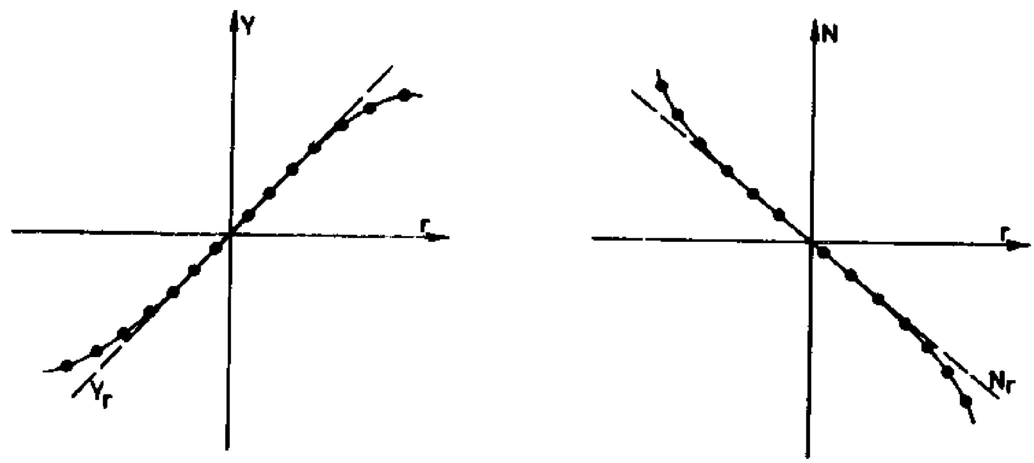
In order to measure the derivatives Y_r and N_r on a model, it is necessary to tow the model at regular forward speed and impose various values of angular velocity r on the model and measure the Y force and N moment for each of these different angular velocities. To do this directly requires a special type of towing tank and apparatus called the "rotating arm" tank. In this facility, an angular velocity is imposed on the model by rotating it in a circle at the end of an arm rotating about an axis, as can be seen from the following sketch.



The model is towed at speed u_o (or Froude no., $\frac{u_o}{\sqrt{gL}}$) at various radii R, and a dynamometer measures the force Y and moment N during each of these tests. Since, for a given model speed u_o (or Froude no. $\frac{u_o}{\sqrt{gL}}$), the angular velocity r is given by

$$r = \frac{u_o}{R}$$

the only way to vary r (at constant $\frac{u_o}{\sqrt{gL}}$) is to vary R. Hence, tests are carried out at various R values. Typical plots of the resulting measurements (after model inertial effects are deducted) are shown below, and the derivatives Y_r and N_r are obtained by evaluating the slope at $r = 0$.



If the model is properly ballasted so that its weight equals its buoyancy and the center of gravity of the model is at the same geometrical position as that of the ship (i.e. $\frac{x_G}{L}$ of model is the same as that of the ship), then the dynamometer measurements will provide values for $(N_r - mx_G u_o)$ and $(Y_r - mu_o)$ directly for use in the stability criteria and the equations of motion. The reduction of these derivative values to non-dimensional form or expansion to ship dimensions will be indicated later. For the same reasons as indicated in the model tests for N_v and Y_v, the model used in the rotating arm tests should have the rudder in the undeflected position and with propellers operating.

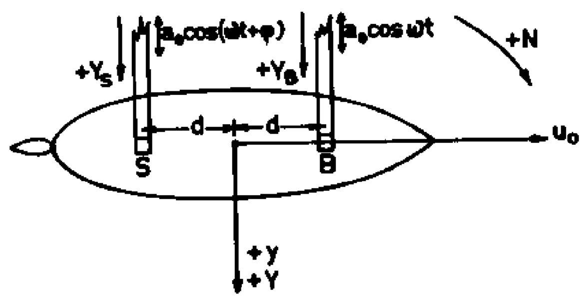
A few problems associated with rotating arm tests and techniques are

1. They require a special towing tank and cannot use the usual long and narrow tank used for resistance testing.

2. The model must be accelerated and data obtained before one revolution. Otherwise, the model will be running in its own wake and the actual u_o will not be known, or the speed will need to be corrected for this wake.

3. In order to obtain the derivatives (i.e. the slope at $r = 0$) sufficient data at small values of r are necessary. This means the radius of turn R, or more correctly, the ratio of R to model length L must be large. For large models, a large facility is required. Smaller models require a smaller tank, but too small models will lead to scale effects in the ship predictions.

In order to avoid the large expense of an additional facility such as the rotating arm tank, a device known as a planar motions mechanism was devised for use in the regular long and narrow towing tank to measure the derivatives Y_r and N_r and some of the other derivatives, such as the acceleration derivatives $N_{\dot v}$, $Y_{\dot r}$, $Y_{\dot v}$, and $N_{\dot r}$. The apparatus consists of two oscillators, one produces a transverse oscillation at the bow and the other a transverse oscillation at the stern of the model while the model moves down the towing tank at the speed u_o.



The bow at a point B located a distance d forward of the origin (usually ⊗) is oscillated transversely with a small amplitude a_o and with a circular frequency ω. Point S on the stern at a distance d aft of the origin is oscillated transversely with the same amplitude a_o and frequency ω but the phasing of the oscillation of the stern relative to the bow can be adjusted and is indicated by the phase angle φ. If $\varphi = 0$, then bow and stern have the same transverse displacement and the model experiences a pure transverse oscillation of the form

$$y = a_o \cos \omega t$$

$$\frac{dy}{dt} = v = -a_o \omega \sin \omega t$$

$$\dot{v} = -a_o \omega^2 \cos \omega t$$

Dynamometers at the bow and stern measure the oscillatory Y forces experienced by the model at the bow and stern, i.e. Y_B and Y_S. Since the velocity v (sine function) is out of phase with the displacement y, (cosine function), then the out of phase measurements of Y_B and Y_S are forces arising from the effects of v. Since the acceleration $\dot{v}$ (cosine function) is in phase with y (cosine function), the in phase measurements of Y_B and Y_S denote forces originating form $\dot{v}$. The derivatives Y_v and N_v are obtained by the following relationship

$$Y_v = \frac{\partial Y}{\partial v} = \pm \frac{\text{Out of phase amplitude of } (Y_B+Y_S)}{-a_o \omega}$$

$$N_v = \frac{\partial N}{\partial v} = \pm \frac{\text{Out of phase amplitude of } (Y_B-Y_S)d}{-a_o \omega}$$

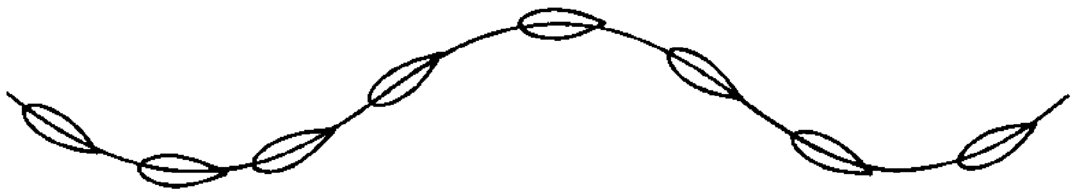
By testing at various frequencies, ω , the frequency dependence of these derivatives can be determined. The derivatives N_v and Y_v can be obtained without oscillating by towing at different angles of attack, i.e. zero frequency as indicated previously. The derivatives $Y_{\dot{v}}$ and $N_{\dot{v}}$ can be obtained by measuring the in phase components of Y_B and Y_S (model inertial forces must be removed).

$$Y_{\dot{v}} = \pm \frac{\text{In phase amplitude of } (Y_B+Y_S)}{-\omega^2 a_o}$$

$$N_{\dot{v}} = \pm \frac{\text{In phase amplitude of } (Y_B-Y_S)d}{-\omega^2 a_o}$$

If the model is properly ballasted ($\frac{x_G}{L}$ same as ship and weight equals buoyancy) then the in phase components will furnish the correct value for the terms $(Y_{\dot{v}}-m)$ and $(N_{\dot{v}}-mx_G)$. Frequency dependence can be determined by testing at different ω values. The time the out of phase Y_B and Y_S are measured,are at the peak values of v, i.e. when $\dot{v} = 0$, and when the in phase components are measured $\dot{v}$ is at its maximum and $v = 0$. Since there is no angular velocity or acceleration, the measurements at the proper phasing are made when $u = u_o$ and only the one variable involved has a value other than zero.

In order to obtain the derivatives Y_r and N_r, the measurements must be made at the time or phasing when $\dot{r} = 0$ and $v = \dot{v} = 0$. Similarly for $Y_{\dot{r}}$ and $N_{\dot{r}}$, the measurements must be taken when $r = 0$, and $v = \dot{v} = 0$. In order to impose an angular velocity and angular acceleration on the body with v and $\dot{v}$ equal to zero, the model must travel down the tank with the centerline of the model always tangent to the path - (this means there is no v component since u = U). The path is oscillatory as shown below.



This type of path will be followed by the model if the phase angle φ , between the bow and stern oscillators, satisfies the condition

$$\frac{\varphi}{2} = \tan^{-1} \frac{\omega d}{U}.$$

With the phase angle set at this value and thereby $v = \dot{v} = 0$ assured, the out of phase components of Y_B and Y_S will provide the forces and moment due to r and the in phase components will provide the forces and moment resulting from $\dot{r}$. If ψ is the orientation angle, then

$$\psi = \psi_o \cos \omega t$$

$$r = - \psi_o \omega \sin \omega t$$

$$\dot{r} = - \psi_o \omega^2 \cos \omega t$$

Hence, r is out of phase with ψ and $\dot{r}$ is in phase with ψ. ψ_o is determined from the amplitude a_o, the distance d, and the phase φ . The derivative values are then, including the inertial effects of the model,

$$(Y_r - mu_o) = \frac{\pm \text{ Out of phase amplitude of } (Y_B + Y_S)}{- \psi_o \omega}$$

$$(N_r - mx_G u_o) = \frac{\pm \text{ Out of phase amplitude of } (Y_B - Y_S)d}{- \psi_o \omega}$$

$$(Y_{\dot{r}} - mx_G) = \frac{\pm \text{ In phase amplitude of } (Y_B + Y_S)}{- \psi_o \omega^2}$$

$$(N_{\dot{r}} - I_z) = \frac{\pm \text{ In phase amplitude of } (Y_B - Y_S)d}{- \psi_o \omega^2}$$

If the model is properly ballasted so that $\frac{x_G}{L}$ of the model and ship are the same and the weight equals the buoyancy, then the above measured values can be directly non-dimensionalized or scaled up to the ship. As in the other tests, the model should be propelled preferably at the ship propulsion point and the rudder included in the undeflected position. The use of $\pm$ in the above terms are associated with the term "amplitude" of oscillation which is always positive. The direction of the forces Y_B and Y_S at the maximum values determine whether + or - should be used.

Some precaution is necessary in applying planar motions tests. Since the ship model is at the water surface, oscillatory motions can create waves whose properties depend on the frequency of generation, hence the derivatives may be frequency dependent. The actual maneuver of a ship going into a turn is at 0 frequency, hence low frequencies are of interest. Since $r = -\omega \psi_o \sin \omega t$, then small ω gives small r which is desirable, since the tests should be carried out at small values of r. The rotating arm test will give data free of frequency effects.

In the case of a deeply submerged submarine model, where surface frequency effects disappear, other frequency effects called "unsteady effects" caused by circulation and lift considerations come into play. The parameter $\frac{\omega L}{U}$ called "reduced frequency" is important for these effects. However, unsteady effects are felt only at high values $\frac{\omega L}{U}$ which are well out of the range of those frequencies used for ship models.

The use of model test data immediately brings to mind the possibility of scale effects. The Froude number is to be satisfied, hence the Reynolds' number will not be satisfied. Since in determining the Y force and N moments, the lift and circulation effects are involved, low aspect ratio airfoil theory indicates very little scale effect on the slope of lift coefficient vs. angle of attack. However, separation or breakdown of lift occurs at a lower angle of attack at the lower Reynolds' number. Fortunately, the various derivatives are determined at the small values of v and r and hence at the small angles of attack before any separation effects come into play. Consideration should be given to possible scale effects if measurements are made at larger values of v and r when obtaining information to be used in any non-linear equations.

It has been indicated that the velocity derivatives N_v, Y_v, N_r, and Y_r for a ship could not be readily calculated and that resort was made to measuring these derivatives by means of special model tests using special dynamometers. Also, the velocity derivatives play the dominant role in the criteria for dynamic stability. On the other hand, the acceleration derivatives appearing in the equations of motion, i.e. $Y_{\dot{v}}$, $N_{\dot{r}}$, $Y_{\dot{r}}$, and $N_{\dot{v}}$, are not involved directly in the criteria and are either small in magnitude as the case of $N_{\dot{v}}$ and $Y_{\dot{r}}$, or are combined with terms of about equal magnitude as $Y_{\dot{v}}$ and $N_{\dot{r}}$ are. The acceleration derivatives, or more specifically the forces arising from acceleration in the fluid, are the result of the inertial properties of the fluid with little, if any, dependence on the viscous properties. Hence, potential theory may be readily employed to estimate these acceleration derivatives - provided such a theory is valid for ship - like bodies and that the free surface is taken into account for bodies operating at the water surface. Calculation for the acceleration derivatives for various submerged bodies of revolution (submarine hulls) have been made by machine calculation, but the theory for arbitrary surface ships remains limited to thin ship theory. Of course, as was the result of the other derivatives for surface ships, they may be frequency dependent.

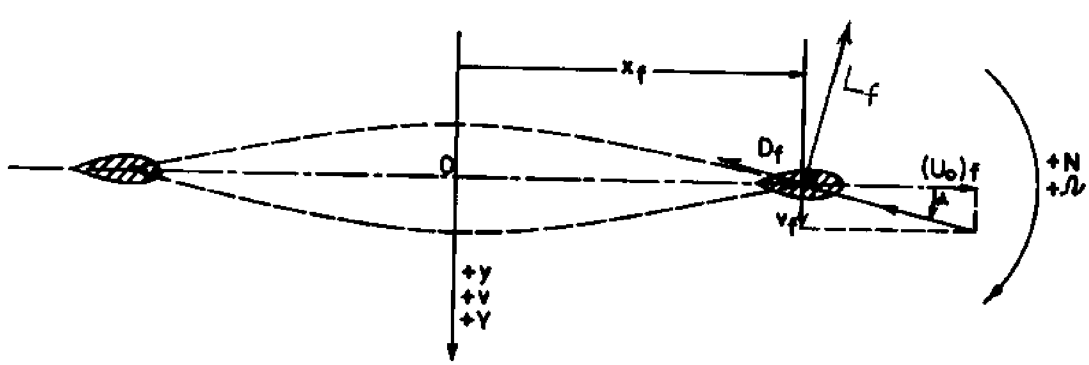
However, a) since we are looking for values of the acceleration derivatives at essentially zero frequency, b) since the significant derivatives $N_{\dot{r}}$ and $Y_{\dot{v}}$ occur in the combinations $(N_{\dot{r}}-I_z)$ and $(Y_{\dot{v}}-m)$ so that a given error in the derivative results in only half that error in the term, and c) since the derivatives are not directly a part of the criteria for stability but are more involved in the magnitude of the roots and the resulting trajectories, the theoretical values for these derivatives may be used. Typical values for use are those of ellipsoids of different length-beam (or length-draft) ratios as calculated and published in Lamb's Hydrodynamics.

The acceleration derivatives can also be readily measured from the record of the in phase components of the planar motion tests on ship models as was mentioned earlier.

CHAPTER VI

Isolated Lifting Surfaces

A special type of body for which the various derivatives can be calculated is the isolated lifting surface, or hydrofoil, located some distance from the axis. The approach is essentially to calculate the angle of attack developed at the surface as the result of a transverse velocity v and an angular velocity r. From the lift characteristics at the resulting angle of attack, the force and moment produced on the body is attached as an appendage to a larger hull, (say a rudder or stabilizing fin), then the effect of the interference of the hull on the water approaching the appendage must be taken into account. Items such as wake and propeller race are examples of such interference. The example is for a lifting surface or fin located either well forward or aft of the origin, a distance at least several foil chord lengths away. The sketch below indicates such a fin arrangement.



The development is for a general location and therefore the example deals with a fin forward a distance $+x_f$ from the origin. Any resulting relationship or formulation from the analysis would give the effect of a fin aft of the origin provided a negative value of x_f is used. The subscript f is used to indicate local condition at the fin. If the body is given a transverse velocity disturbance +v, then the fin also experiences a transverse velocity of v, i.e.

$$v_f = v$$

If at the time of this transverse disturbance the forward velocity component on the fin is $(U_o)_f$, then the change in angle of attack at the fin caused by the transverse velocity v is

$$\alpha = \tan^{-1} \frac{v_f}{(U_o)_f}$$

and the Y force and N moment produced at the fin by this angle of attack is given by

$$Y_f = -(L_f \cos\alpha + D_f \sin\alpha)$$

$$N_f = Y_f x_f$$

where L_f and D_f are the lift and drag forces on the fin. The lift and drag of a foil are usually expressed in terms of the lift coefficient C_L and drag coefficient C_D as defined below

$$L_f = (C_L)_f \, 1/2 \rho A_f U_f^2 = (C_L)_f 1/2 \rho A_f \left[(U_o)_f^2 + v_f^2\right]$$

$$D_f = (C_D)_f 1/2 \rho A_f \left[(U_o)_f^2 + v_f^2\right]$$

where ρ is the fluid density and A_f is the fin (projected) area. Hence

$$Y_f = -1/2 \rho A_f (U_o)_f^2 \left[(C_L)_f \sec\alpha + (C_D)_f \tan\alpha \sec\alpha\right]$$

since

$$\left[(U_o)_f^2 + v_f^2\right] \cos\alpha = ((U_o)_f^2 \quad \frac{\left[(U_o)_f^2 + v_f^2\right]}{(U_o)_f^2} \quad \frac{(U_o)_f}{\left[(U_o)_f^2 + v_f^2\right]^{1/2}}$$

$$= (U_o)_f^2 \frac{\left[(U_o)_f^2 + v_f^2\right]^{1/2}}{(U_o)_f} = (U_o)_f^2 \sec\alpha$$

and by similar substitution

$$\left[(U_o)_f^{\,2}+v_f^{\,2}\right]\sin\alpha = (U_o)_f^{\,2}\operatorname{cosec}^2\alpha\sin\alpha = (U_o)_f^{\,2}\tan\alpha\ \sec\alpha$$

From the relationship $v_f = (U_o)_f \tan\alpha$

$$\frac{dv_f}{d\alpha} = (U_o)_f \sec^2\alpha \qquad \text{or} \quad \frac{d\alpha}{dv_f} = \frac{\cos^2\alpha}{(U_o)_f}$$

and

$$(Y_v)_f = \frac{\partial Y_f}{\partial v_f} = \left(\frac{\partial Y_f}{\partial \alpha}\right)\frac{d\alpha}{dv_f} = -\frac{\rho}{2} A_f (U_o)_f \cos^2\alpha \frac{d}{d\alpha}\left[(C_L)_f \sec\alpha + (C_D)_f \tan\alpha \sec\alpha\right]$$

evaluated at $\alpha = 0$. The derivative of the expression contained within the brackets becomes

$$\frac{d(C_L)_f}{d\alpha}\sec\alpha + (C_L)_f \tan\alpha\sec\alpha + \frac{d(C_D)_f}{d\alpha}\tan\alpha\sec\alpha + (C_D)_f(\sec^3\alpha + \tan^2\alpha\sec\alpha)$$

and at $\alpha = 0$, one obtains

$$(Y_v)_f = -\frac{\rho}{2} A_f (U_o)_f \left[\frac{d(C_L)_f}{d\alpha} + (C_D)_f\right]$$

and obviously

$$(N_v)_f = x_f (Y_v)_f \ .$$

Since $v = v_f$, the above represent contributions of the fin to the overall derivative.

From airfoil and hydrofoil theory the slope of the lift coefficient curve versus angle of attack is given by

$$\frac{\partial C_L}{\partial \alpha} = \frac{2\pi}{1+\frac{2}{AR}}$$

where AR is the aspect ratio, which is the ratio of foil span to chord*). The theoretical value of lift can be used for obtaining numerical values or else the slope as obtained from wind tunnel or towing test on foils (readily obtainable in published literature) can be used. The drag coefficient, D_f, can be estimated as essentially the skin friction drag of the foil at the local Reynolds' number on the foil corresponding to a velocity $(U_o)_f$ or else can be obtained from published data.

If there are no wake effects, or propeller race effects, or no other flow interference effects with the hull, then $(U_o)_f = U_o$ (i.e. inflow to fin same as forward velocity of ship).

If for example, a fin is in the wake of the hull and it is estimated that there is a wake factor of 20% on the fin, then

$$(U_o)_f = (1-0.20)U_o = 0.80\ U_o.$$

From the expression for the contribution of the fin to the Y_v derivative (i.e. $(Y_v)_f$) it is clear that $(Y_v)_f$ is always negative, whether at bow or stern, but that a fin forward decreases the value of N_v and a fin aft increases (makes more positive or less negative) the value of N_v. This confirms an earlier presentation of this point.

The contribution of a fin to the derivatives Y_r and N_r are readily calculable from the expressions already developed. It is clear from the sketch used above, that a small positive angular velocity r produces a linear transverse velocity at the fin, given by

$$v_f = x_f r$$

The force and moment produced at the fin as the result of r is

$$Y_f \text{ due to } r = (Y_v)_f v_f = (Y_v)_f x_f r$$

$$N_f \text{ due to } r = (N_v)_f v_f = x_f(Y_v)_f x_f r = Y_v x_f^2 r$$

On taking the derivative with respect to r, one obtains

$$(Y_r)_f \doteq x_f(Y_v)_f$$

$$(N_r)_f = x_f^2(Y_v)_f$$

*) In determining the proper aspect ratio, consideration must be given to the actual tip losses that occur.

Since $(Y_v)_f$ is negative, lifting surfaces at the bow tend to decrease Y_r (make more negative or less positive) with an opposite effect for a fin located at the stern. A fin at the bow and at the stern will tend to make N_r more negative. This confirms the results indicated previously.

The method developed gives a way of estimating quantitatively, the improvement to be expected from lifting surface addition to a hull. Since a rotating propeller acts as a lifting surface, if the lift characteristics are known, the contribution of propeller to the overall ship derivatives can be calculated.

A similar method of analysis can be used to develop the contribution of the fin to the acceleration derivatives of the body. The analysis results in the following formulations.

$$(N_{\dot{v}})_f = x_f(Y_{\dot{v}})_f$$

$$(Y_{\dot{r}})_f = x_f(Y_{\dot{v}})_f$$

$$(N_{\dot{r}})_f = x_f^2(Y_{\dot{v}})_f$$

$$(Y_{\dot{v}})_f = -\frac{\pi\rho\, s^2c^2}{\sqrt{s^2+c^2}}$$

where s is the span and c is the chord of the fin (i.e. dimensions of the fin)*). The formulation for $(Y_{\dot{v}})_f$ is taken from the calculation of the "added mass" of a flat plate of dimensions s and c for acceleration perpendicular to the plate. It should be noted and stressed that the contribution of a fin like appendage to the acceleration derivatives of a body are small and of minor significance, whereas the contribution of such fins to the velocity derivatives are major, very significant, and often times decisive in their contribution to dynamical stability.

*) For a fin in which one edge does not allow water to flow over it, one must consider using twice this length and taking one half the result.

CHAPTER VII

Solution of Motion Equations for Control Surface Deflections

Let us now turn to the area of control of ship motion. Control forces and moments can be produced on the ship by a deflection of the control surface such as a rudder. If δ designates the deflection of the control surface, then the forces X, Y, and moment N produced on the ship by this deflection, as indicated by the linear terms in the Taylor expansion, are

$$X_\delta \delta$$

$$Y_\delta \delta$$

$$N_\delta \delta$$

It is assumed that the forces and moments produced on the ship as the result of $\dot{\delta}$ and $\ddot{\delta}$ are negligible, although these variables are not necessarily negligible in determining the torque on the rudder stock during a maneuver. It can readily be shown that $X_\delta = 0$ since the rudder is symmetric port and starboard. The equations of motion, including the rudder effect now become

$$a_{11} \Delta u + 0v + 0r = -X_\delta \delta = 0$$

$$a_{21} \Delta u + a_{22}v + a_{23}r = -Y_\delta \delta$$

$$a_{31} \Delta u + a_{32}v + a_{33}r = -N_\delta \delta$$

(Since a_{12} and a_{13} are zero by symmetry, it can be shown that a_{21} and a_{31} must also be zero).

The linear solution for the angular velocity r is readily expressed as

$$r = \frac{\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & -Y_\delta\delta \\ a_{31} & a_{32} & -N_\delta\delta \end{vmatrix}}{\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} = -\frac{a_{11}(a_{22}N_\delta - a_{32}Y_\delta)\delta}{a_{11}(a_{22}a_{33}-a_{23}a_{32})} = -\frac{\left[a_{22}N_\delta - a_{32}Y_\delta\right]\delta}{A(\mathcal{D}-\sigma_1)(\mathcal{D}-\sigma_2)}$$

where A, the coefficient of $\mathcal{D}^2$ has been expressed in terms of derivatives previously, also

$$v = \frac{\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & -Y_\delta\delta & a_{23} \\ a_{31} & -N_\delta\delta & a_{33} \end{vmatrix}}{\text{Det.}} = \frac{\left[a_{23}N_\delta - a_{33}Y_\delta\right]\delta}{A(\mathcal{D}-\sigma_1)(\mathcal{D}-\sigma_2)}$$

$$\Delta u = \frac{\begin{vmatrix} 0 & 0 & 0 \\ -Y_\delta\delta & a_{22} & a_{23} \\ -N_\delta\delta & a_{32} & a_{33} \end{vmatrix}}{\text{Det.}} = \frac{0}{(X_{\dot{u}}-m)A(\mathcal{D}-\sigma_1)(\mathcal{D}-\sigma_2)(\mathcal{D}-\sigma_3)}.$$

The solution for Δu is the homogeneous solution obtained previously, i.e.

$$\Delta u = u_1 e^{\sigma_1 t} + u_2 e^{\sigma_2 t} + u_3 e^{\sigma_3 t}.$$

However, from the nature of the solution for v and r above (and from the fact that $a_{12} = a_{13} = 0$), the X equation can be decoupled from the Y and N equations, so that it can be shown that the constants of integration u_1 and u_2 are zero. This gives

$$\Delta u = u_3 e^{\sigma_3 t}$$

as the solution of the linear equations with or without rudder deflection. This decoupling, a result of the linear theory, indicates no speed loss in a maneuver and is a limitation of the linear theory. Non-linear equations can make the situation more realistic as will be indicated later.

In the solution for v, the numerator is developed by substituting for a_{23} and a_{33} the appropriate expressions.

$$\left[a_{23}N_\delta - a_{33}Y_\delta\right]\delta = \left[(Y_{\dot r}-mx_G)\mathcal{D}+(Y_r-mu_o)\right]N_\delta\delta - \left[(N_{\dot r}-I_z)\mathcal{D}+(N_r-mx_Gu_o)\right]Y_\delta\delta$$

$$= \Big[\overbrace{N_\delta\,(Y_{\dot r}-mx_G)-Y_\delta\,(N_{\dot r}-I_z)}^{b_1}\Big]\mathcal{D}\delta + \Big[\overbrace{N_\delta\,(Y_r-mu_o)-Y_\delta\,(N_r-mx_Gu_o)}^{b_2}\Big]\delta$$

$$= b_1\mathcal{D}\delta \quad +b_2\delta$$

where b_1 and b_2 are defined as the terms within the brackets.

Similarly, the numerator in the solution for r, becomes

$$-\left[a_{22}N_\delta - a_{32}Y_\delta\right]\delta = -\left[(Y_{\dot v}-m)\mathcal{D}+Y_v\right]N_\delta\delta \quad + \left[(N_{\dot v}-mx_G)\mathcal{D}+N_v\right]Y_\delta\delta$$

$$= -\Big[\overbrace{(Y_{\dot v}-m)N_\delta \quad -(N_{\dot v}-mx_G)Y_\delta}^{b_3}\Big]\mathcal{D}\delta \quad - \Big[\overbrace{Y_vN_\delta \quad -N_vY_\delta}^{b_4}\Big]\delta$$

$$= -(b_3\mathcal{D}\delta \; + \; b_4\delta.)$$

For the case where the rudder deflection does not vary with time, as in the case of the rudder being deflected to a position $\delta = \delta_o$ at time $t = 0$, and held at δ_o, then $\mathcal{D}\delta = \frac{d\delta}{dt} = 0$ and the solution for r develops as follows

$$r = \frac{-b_4\,\delta_o}{A(\mathcal{D}-\sigma_1)(\mathcal{D}-\sigma_2)} = \frac{-1}{A}\left(\frac{1}{\mathcal{D}-\sigma_1}\right)e^{\sigma_2 t}\int e^{-\sigma_2 t}(b_4\,\delta_o)dt$$

$$= \frac{-b_4\,\delta_o}{A}\left(\frac{1}{\mathcal{D}-\sigma_1}\right)e^{\sigma_2 t}\left[\int e^{-\sigma_2 t}dt\right] = \frac{-b_4\,\delta_o}{A}\left(\frac{1}{\mathcal{D}-\sigma_1}\right)\left[\frac{e^{-\sigma_2 t}}{-\sigma_2}+C_2\right]e^{\sigma_2 t}$$

$$= \frac{-b_4\,\delta_o}{A}\,e^{\sigma_1 t}\int e^{-\sigma_1 t}\left(\frac{1}{-\sigma_2}+C_2e^{\sigma_2 t}\right)dt = \frac{-b_4\,\delta_o}{A}\left[\frac{1}{\sigma_1\sigma_2}+\frac{C_2e^{\sigma_2 t}}{\sigma_2-\sigma_1}+C_1e^{\sigma_1 t}\right]$$

$$r = \frac{-b_4\,\delta_o}{A\sigma_1\sigma_2} + r_1e^{\sigma_1 t} + r_2e^{\sigma_2 t}$$

where r_1 and r_2 are constants of integration.

The result gives the homogeneous solution (stability equation solution) plus a particular solution $\frac{-b_4\,\delta_o}{A\sigma_1\sigma_2}$.

If the ship is dynamically unstable, the angular velocity will increase in time reaching no steady state angular velocity - as would be the case without rudder deflection. However, if the ship is stable (σ_1 and σ_2 are negative), then the angular velocity changes in time according to the equation, reaching the steady state angular velocity of $\frac{-b_4\delta_o}{A\sigma_1\sigma_2}$ as the transient terms $C_2^1 e^{\sigma_2 t} + C_1^1 e^{\sigma_1 t}$ go to zero as time goes on, for the stable ship.

The solution for v is readily written down (since a similar process of integration is involved) as

$$v = \frac{b_2\delta_o}{A\sigma_1\sigma_2} + v_1 e^{\sigma_1 t} + v_2 e^{\sigma_2 t}.$$

Recalling that $A\sigma_1\sigma_2 = C$, the coefficient independent of $\mathcal{D}$ in the quadratic equation $A\mathcal{D}^2+B\mathcal{D}+C$, and that $C = Y_v(N_r-mx_Gu_o)-N_v(Y_r-mu_o)$, the steady state angular velocity and transverse velocity, for a rudder deflection δ_o is expressed by

$$r = -\left[\frac{Y_vN_\delta - N_vY_\delta}{Y_v(N_r-mx_Gu_o)-N_v(Y_r-mu_o)}\right]\delta_o$$

$$v = \left[\frac{N_\delta(Y_r-mu_o)-Y_\delta(N_r-mx_Gu_o)}{Y_v(N_r-mx_Gu_o)-N_v(Y_r-mu_o)}\right]\delta_o$$

Within the linear theory, the angular velocity and transverse velocity in the steady turn are proportional to the rudder deflection. Since, within the linear theory there is no speed loss, the radius of the turn is given by

$$\text{Radius} = \frac{u_o}{r}$$

It must be recalled here that the deflection of a rudder is positive in the same sense of rotation as r and N, i.e. if one looks from above a clockwise rotation of r, N, and δ is positive. The above expressions for r and v have little meaning if the ship is unstable - no more than knowing the magnitude of a negative metacentric height can give the angle of heel for a given heeling moment. If the σ values are negative, i.e. stable, the above expressions give rather good estimates for small rudder deflections. However, for large rudder deflections and tight turns it becomes necessary to solve the non-linear equations - either by computational technique or by free model turning tests in a maneuvering basin.

In the case where the rudder deflection varies with time such as an exponential rudder deflection

$$\delta = \delta_o(1-e^{-at})$$

or sinusoidal oscillation

$$\delta = \delta_o \cos \; wt = \text{Real part of } \; \delta_o e^{iwt}$$

one must carry through the differentation in the numerator of the algebraic solution before operating with $(\frac{1}{\mathcal{D}-\sigma})$. For example, in the solution for r for an exponential deflection, one has

$$-r = \frac{b_3 \mathcal{D}\delta + b_4 \delta}{A(\mathcal{D}-\sigma_1)(\mathcal{D}-\sigma_2)} = \frac{b_3 \mathcal{D}\left[\delta_o(1-e^{-at})\right] + b_4 \; \delta_o(1-e^{-at})}{A(\mathcal{D}-\sigma_1)(\mathcal{D}-\sigma_2)}$$

$$= \frac{b_3 \; \delta_o a e^{-at} + b_4 \; \delta_o(1-e^{-at})}{A(\mathcal{D}-\sigma_1)(\mathcal{D}-\sigma_2)} = \frac{\delta_o\left[b_4 + (b_3 a - b_4)e^{-at}\right]}{A(\mathcal{D}-\sigma_1)(\mathcal{D}-\sigma_2)} .$$

The integrations resulting from operating with $(\frac{1}{\mathcal{D}-\sigma_1})$ and $(\frac{1}{\mathcal{D}-\sigma_2})$ can be readily accomplished since the integrand is an exponential function of time. In a similar manner, for the case of the sinusoidally oscillating rudder, the complex exponential e^{iwt} can be readily integrated during the process of operating with $\frac{1}{\mathcal{D}-\sigma_2}$ and $\frac{1}{\mathcal{D}-\sigma_1}$.

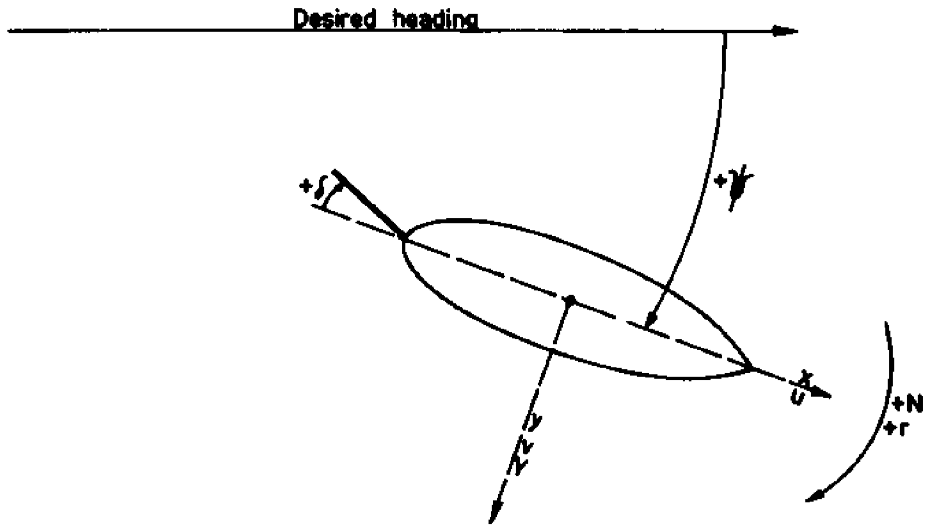
CHAPTER VIII

Automatic Steering Control - Rudder Deflection as a Function of Ship Motion

The linearized equations of motion with rudder deflection were solved previously for the case where the rudder was deflected as some prescribed function of time, such as a constant, exponential, or sinusoidal deflection. The solution of the resulting linearized equations were indicated by using the operational technique previously developed. The LaPlace Transform technique could also have readily been used to obtain the solutions for these equations.

The deflection of the rudder in this sort of prescribed manner is essentially associated with the maneuvering aspect of ships. However, if a rudder or a control surface is deflected as a function of some parameter associated with the ship motion or trajectory, such as ship heading, velocity, etc., the rudder deflection varies with time only as that parameter varies with time. If the control surface is thus automatically deflected according to the value of a parameter or several parameters, then one is dealing with the domain of automatic control, and for motion in the horizontal plane with the rudder being the control surface, one is involved in the area of automatic steering control.

The usual desire is to keep the ship on a desired heading - i.e. a given angular orientation (such as north). Let us designate the angle ψ as the error or deviation from the desired heading. It is obvious that $\dot{\psi} = \frac{d\psi}{dt} = r$, the angular velocity. The rudder deflection, denoted by δ , is positive in the same sense of rotation as ψ , r, and N, as indicated in the following sketch.



In order to deflect the rudder as a function of ψ , one must be able to measure ψ continuously and use the signal from this measurement to activate the rudder mechanism. Since the heading angle is readily measurable by a gyro compass, a directional control signal is readily available. A good helmsman, in attempting to maintain course, will not only deflect the rudder in accordance with his sensing of the deviation from course, but will also ease off on the rudder and perhaps apply a little opposite rudder to meet the "swing of the ship" in order to prevent the angular velocity of the ship from overshooting (swinging the ship beyond) the desired heading. Hence, in addition to a sensitivity to heading, a good control system should have a sensitivity to angular velocity. Let us, therefore, deflect the rudder proportional to the heading error and the angular velocity. (The rudder can be deflected according to any measurable parameter - within the limits of the rudder system - which results in a signal capable of activating the rudder). The equation for the rudder deflection under the conditions mentioned above becomes

$$\delta = k_1 \psi + k_2 r$$

where k_1 and k_2 are the constants of proportionality of the control system. On substitution of this expression for δ into the linearized equations, (after recalling that $X_\delta = 0$, and that the X equation and Δu can be decoupled from the Y and N equations), one obtains

$$\left[(Y_{\dot{v}}-m)\mathscr{D}+Y_v\right] v+\left[(Y_{\dot{r}}-mx_G)\mathscr{D}+(Y_r-mu_o)\right] r+Y_\delta\ (k_1\psi+k_2 r) = 0$$

$$\left[(N_{\dot{v}}-mx_G)\mathscr{D}+N_v\right]v+\left[(N_{\dot{r}}-I_z)\mathscr{D}+(N_r-mx_Gu_o)\right]r+N_\delta\ (k_1\psi+k_2r) = 0$$

Since $r = \dot{\psi} = \mathscr{D}\psi$, the above equations take the form

$$\overset{a_{22}}{\left[(Y_{\dot{v}}-m)\mathscr{D}-Y_v\right]}v+\overset{a_{23}}{\left[(Y_{\dot{r}}-mx_G)\mathscr{D}^2+(Y_r-mu_o+k_2Y_\delta)\mathscr{D}+k_1Y_\delta\right]}\psi = 0$$

$$\overset{a_{32}}{\left[(N_{\dot{v}}-mx_G)\mathscr{D}+N_v\right]}v+\overset{a_{33}}{\left[(N_{\dot{r}}-I_z)\mathscr{D}^2+(N_r-mx_Gu_o+k_2N_\delta)\mathscr{D}+k_1N_\delta\right]}\psi = 0.$$

The above equations can readily be solved for ψ as a function of time in a manner similar to the previous solution for r.

$$\psi = \frac{0}{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}} = \frac{0}{A_1\mathscr{D}^3+B_1\mathscr{D}^2+C_1\mathscr{D}+D_1}$$

$$= \frac{0}{A_1(\mathscr{D}-\sigma_1^1)(\mathscr{D}-\sigma_2^1)(\mathscr{D}-\sigma_3^1)}$$

where σ_1^1, σ_2^1, and σ_3^1 are the roots of the cubic equation in $\mathscr{D}$ obtained by multiplying out the determinant. The fundamental differences between the solution for ψ above and the solution for r in the dynamical stability equations, solved previously, are that

a) the stability roots resulting from the Y and N equations (X equation decoupled) are three in number (σ_1^1, σ_2^1, and σ_3^1) as compared to the two roots (σ_1 and σ_2).

b) the automatic controls have introduced a sensitivity to the heading angle ψ which is not inherent in the hull dynamics. Hence ψ is a basic variable where formerly r was the basic variable.

c) obtaining the three roots of the cubic equation is more complicated and less explicit than the simple quadratic solution for getting the two roots of a quadratic equation.

From our experience with the operational technique, the solution for the linearized equations can readily be written in the form

$$\psi = \psi_1 e^{\sigma_1^1 t} + \psi_2 e^{\sigma_2^1 t} + \psi_3 e^{\sigma_3^1 t}$$

$$v = v_1 e^{\sigma_1^1 t} + v_2 e^{\sigma_2^1 t} + v_3 e^{\sigma_3^1 t}$$

$$\Delta u = u_1 e^{\sigma_4^1 t}$$

where, because the x equation is decoupled, $e^{\sigma_4^1 t}$ does not appear in the ψ and v equations and where σ_4^1 equals the σ_3 value previously obtained from the x equation during the analysis of stability of straight ahead motion. ψ_1, ψ_2, ψ_3, v_1, v_2, v_3, and u_1 are constants of integration depending on initial conditions. The condition for directional stability - i.e. that the ship returns to the original direction of motion as well as straight line motion, after an arbitrary disturbance - is that all three roots σ_1^1, σ_2^1, and σ_3^1 are negative if they are real or have negative real parts if they are complex. (σ_4^1 has already been shown to be real and negative in previous discussions involving σ_3).

By using the quadratic equation solution, it was relatively easy to show in the case of stability of straight line motion, that the terms $\frac{B}{A}$ and $\frac{C}{A}$ had to be positive in order for the σ values to be negative, stable roots. In the case of the cubic equation involving the coefficients A_1, B_1, C_1, and D_1, the condition for all three roots (σ_1^1, σ_2^1, and σ_3^1), to be negative (or real parts negative) is that

$$\frac{B_1}{A_1} > 0, \quad \frac{C_1}{A_1} > 0, \quad \frac{D_1}{A_1} > 0 \quad \text{and} \quad \frac{B_1C_1 - A_1D_1}{A_1^2} > 0.$$

The last condition is often referred to as Routh's discriminant for the cubic equation. (these conditions result from an analysis of stability in dynamical systems as presented by Routh). The discriminant can be written in the form

$$\frac{B_1}{A_1}\frac{C_1}{A_1} - \frac{D_1}{A_1} > 0$$

so that if $\frac{B_1}{A_1}$ is positive and $\frac{D_1}{A_1}$ is positive, the discriminant has the possibility of becoming positive only if $\frac{C_1}{A_1}$ is positive. Hence, the condition $\frac{C_1}{A_1} > 0$ is a redundant condition.

By designing the proper values of the control constants k_1 and k_2 in the control mechanisms (k_1 and k_2 can be negative as well as positive quantities), the ship will be suitably automatically controlled to maintain a given heading when subjected to disturbances (or reasonable magnitude) from the desired heading. The solution of the equations of motion can be carried out (by digital or analogue computer) for various values of k_1 and k_2 and for various expected disturbances and the resulting trajectories can be analyzed to provide the proper choice of these control constants. Of course, the design and construction of the control hardware should allow for some range of adjustment of the constants to be properly adjusted after installation.

By the use of automatic controls, the equations of motion, as compared to the equations without control, have changed in two major respects. There is a sensitivity to the orientation of the ship, ψ , which is not inherent in the hull hydromechanics and secondly certain terms which readily appear in the criteria for straight line motion have been altered in value by the controls. For example, the former term (Y_r-mu_o) now appears as $(Y_r-mu_o+k_2Y_\delta)$ and what was formerly $(N_r-mx_Gu_o)$ now appears as $(N_r-mx_Gu_o+k_2N_\delta)$. If the transverse velocity v could be readily measured and the control made sensitive to v then, also the terms N_v and Y_v would have additional terms added to them. Hence, the second effect of automatic controls is to make the ship behave as if the ship possessed different values of the hydrodynamic derivatives - i.e. as if the hull had different inherent properties. It can be surmised that a dynamically unstable ship can be made dynamically stable (straight line motion) and directionally stable by the use of automatic control (within certain limits). However, the controls need be less sensitive and less worked and in addition the ship can be handled readily without controls, if the ship is dynamically stable in straight line motion to begin with.

The rudder and steering mechanism represent a reasonable amount of inertia and a certain amount of time is necessary to deflect the rudder, once the signal is given. Therefore, although ψ is measured at time t, it requires some time, say Δt, for the rudder to actually reach the deflection $k_1\,\psi(t_1)$. Hence, the deflection of the rudder at time t, i.e. $\delta(t)$ is proportional to ψ at time t_1 (where $t_1 = t-\Delta t$). In functional form, our control system appears as

$$\delta(t) = k_1\,\psi(t-\Delta t)+k_2 r(t-\Delta t)$$

where Δt is referred to as the time lag of the control system.

Since, in the linearized equations of motion the variables ψ and r appear as functions of time t, i.e. $\psi(t)$ and $r(t)$, it is necessary to express $\delta(t)$ in terms of $\psi(t)$ and $r(t)$ before introducing the control forces into the equations of motion. A convenient way to handle this is to use the Taylor expansion of the function in order to express $\psi(t-\Delta t)$ and $r(t-\Delta t)$ in terms of $\psi(t)$ and $r(t)$ respectively.

When the Taylor expansion was used previously to develop the nature of the hydrodynamic function, it was shown that a convenient way to write the expansion was in exponential form using a differential operator. Hence, $\psi(t-\Delta t)$ can be written as follows

$$\psi(t-\Delta t) = e^{-\Delta t \mathcal{D}} \cdot \psi(t) \qquad \text{where } \mathcal{D} = \frac{d}{dt}$$

and on expanding the exponential in a series, one obtains

$$\psi(t-\Delta t) = 1+(-\Delta t)\mathcal{D} + \frac{(-\Delta t)^2}{2\,!}\mathcal{D}^2 + \frac{(-\Delta t)^3}{3\,!}\mathcal{D}^3 + \ldots\ldots \quad (t)$$

$$= \psi(t) - \Delta t\,\mathcal{D}\psi(t) + \frac{(\Delta t)^2}{2\,!}\mathcal{D}^2\,\psi(t) + \ldots\ldots$$

If it is assumed that the time lag Δt is small, the terms greater than the linear term contribute little and the expression up to and including the linear term is sufficient to describe the lag effect. Since $\mathcal{D}\psi(t) = r(t)$, and $\mathcal{D}r(t) = \dot{r}(t) = \mathcal{D}^2\,\psi(t)$, the equation for the rudder deflection, including the linear term in the time lag expression, becomes

$$\delta(t) = k_1\psi(t-\Delta t)+k_2 r(t-\Delta t) = k_1\left[\psi(t)-\Delta t r(t)\right]+k_2\left[r(t)-\Delta t\dot{r}(t)\right]$$

When this formulation for δ is substituted back into the two equations of motion, the following form results

$$\left[(Y_{\dot{v}}-m)\mathcal{D}+Y_v\right]v+\left[(Y_{\dot{r}}-mx_G-\Delta t k_2 Y_\delta)\mathcal{D}^2+(Y_r-mu_o+k_2Y_\delta-\Delta t k_1 Y_\delta)\mathcal{D}+k_1Y_\delta\right]\psi = 0$$

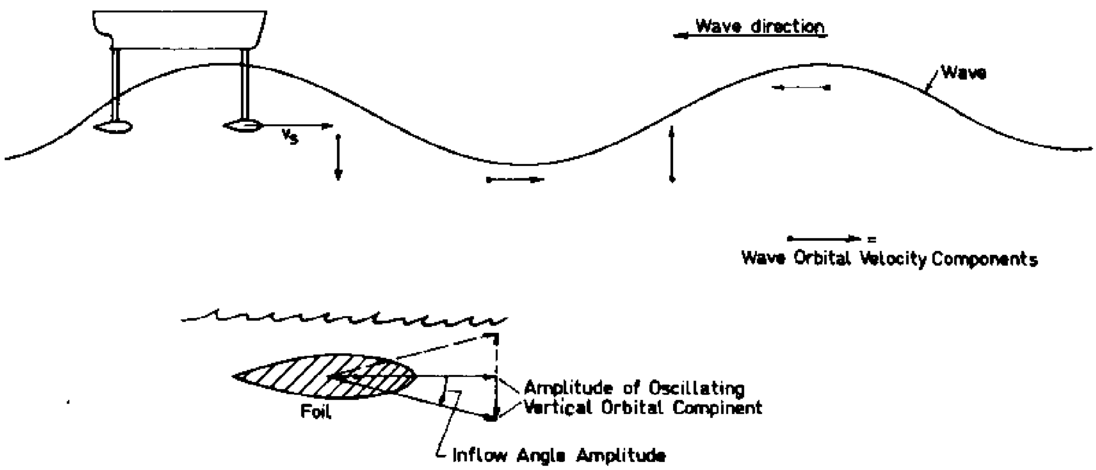
$$\left[(N_{\dot{v}}-mx_G)\mathcal{D}+N_v\right]v+\left[(N_{\dot{r}}-I_z-\Delta t k_2 N_\delta)\mathcal{D}^2+(N_r-mx_Gu_o+k_2N_\delta-\Delta t k_1 N_\delta)\mathcal{D}+k_1N_\delta\right]\psi = 0$$

The above equations can be solved in the usual manner and the three roots of the cubic in $\mathcal{D}$ can be obtained or the criteria for stability can be tested on the values of A_1, B_1, C_1, and D_1. Since the rudder is usually located at the stern, a positive rudder deflection produces a negative N moment (and vice versa), hence N_δ is negative and for proper automatic control k_1 is a positive quantity. It can be seen that the term $-\Delta t k_1 N_\delta$ is positive in quantity and detracts from the negative value of N_r, the damping coefficient. Recalling that a large negative value of N_r encourages stability, it can be concluded that a system with relatively large time lag, at the time it introduces a sensitivity to direction, can also tend to degrade the dynamical stability. In the case above, designing the controls sensitive to r can introduce the term $k_2 N_\delta$ to compensate for the lag effect. The additional term $-\Delta t k_2 N_\delta$ is rather insignificant compared to the large and relatively insensitive quantity $N_{\dot{r}} - I_z$.

Sloppy control systems with unnecessarily large time lags are undesirable. The time lag Δt in any system is rather difficult to ascertain, but certainly depends on the rudder rate. It may depend on the variable ψ (or its derivatives) and in that case non-linear equations result. The purpose of discussing time lags is to indicate qualitatively, rather than quantitatively, the effect of such lags on the performance of the automatic control and the motion stability of the ship.

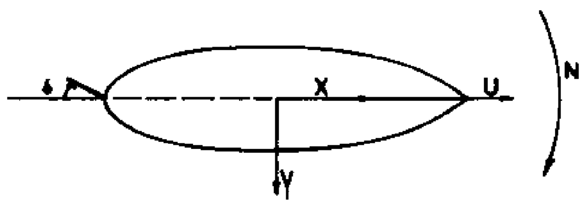
A more accurate and realistic, but much more complicated, analysis of the lags in control mechanisms can be accomplished by writing the equations which describe the actual operation of the mechanism. For example, the electrical equation of the build up of voltage (or amperage) as the result of the gyro measurement of the deviation, the equations describing the actual method of amplification of the signal to produce the power to activate the rudder motor, the equations to describe the electro-mechanical response of the electric motor activating the rudder system, and the equations of motion of the rudder system. These equations are then coupled with the ship motion equations and the overall response analyzed. The results will give a complete test of the stability of the overall system, ship and controls. The controls themselves, if not proper, can introduce instability into the system. The complete control analysis is rather complicated with several more unknowns and equations, resulting in a greater number of roots.

Time lags in control systems become rather important in high speed craft subject to relatively large disturbing forces at sea. Such a vehicle is the hydrofoil boat running into good size waves. The excitation of a hydrofoil boat, with completely submerged foils, is brought on by the varying angle of attack on the foils resulting from the combination of the forward motion of the foil and the vertical component of orbital motion of the water below the wave surface, as demonstrated in the sketch below.

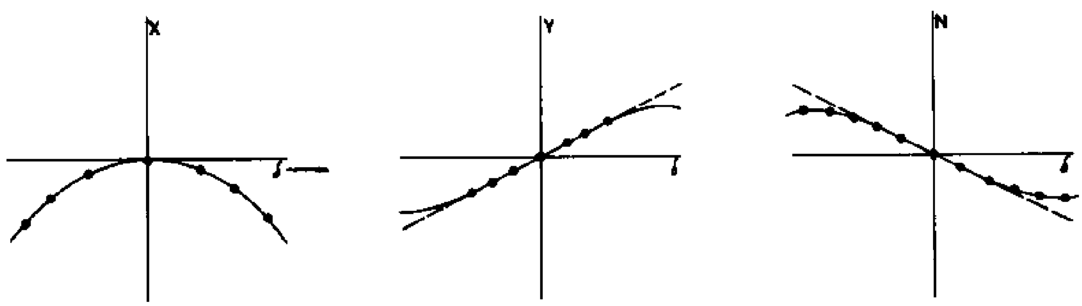


The inflow angle varies approximately sinusoidally in the period of encounter. The period of encounter is rather short in time since, in head seas, the relative speed between the wave and ship is the sum of the ship speed plus the wave speed. Since the hydrofoil, for efficient lift-drag considerations is designed for lifting the weight of the hull with a small angle of attack, small changes in angle of attack brought about by the orbital velocity produce significant exciting forces tending to oscillate the boat in pitch and heave. With such a high speed of encounter, excitations occur so fast that even small time lags still are sufficient to compromise the automatic control based on sensing ship motion parameters. By using a device (acoustic type - or radar principle) that measures the oncoming wave before it excites the ship and coupling this measurement with a control system that deflects the foil by the time the wave excitation reaches the foil, the control system can be made effective. In this case, the measuring or sensing system has enough lead time designed into it to cancel the effect of lag time.

In discussing the effect of controls, the control derivatives X_δ , Y_δ , and N_δ were used to represent the linearization of the force and moment functions resulting from the control surface deflection. The values for these derivatives, for any given design, can be estimated from ship model tests, in a similar fashion as was used to evaluate the derivatives Y_v and N_v. The model (usually with propellers operating) is towed in the towing tank, in straight ahead motion, and a dynamometer capable of measuring resistance, sideforce, and moment, measures these components for various settings of the rudder angle δ . The following is a sketch of such a test.



The measured forces and moments are then plotted versus δ and the slope, taken at $\delta = 0$, will furnish the values of the derivatives. If the rudder (as is usual) is located at the stern, the plots will look something like these shown below.



From symmetry considerations, X_δ is zero. The nature of the forces indicate a positive slope, Y_δ , and negative slope, N_δ , for a rudder located aft. Since rudder forces depend on the inflow velocity, and therefore on any race effects of the propellers, it is more realistic to conduct such tests with model propellers rotating at the ship's corresponding point. The values obtained for the control derivatives may suffer somewhat from scale effect since the rudder operates in the wake region behind the hull. Certain corrections can be estimated to account for the expected difference in wake between the model and the full scale ship. On the other hand, it may be possible to estimate the value of the derivatives based on a lifting surface sufficiently removed from the origin by the use of hydrofoil theory as was demonstrated earlier. An estimate of wake and propeller race effects is necessary with this approach.

Although the low Reynolds' number of the model may not significantly affect the value of the control derivative, the range of linearity of the Y and N versus δ curves (i.e. value of δ when curve departs from the tangent) may be somewhat lower at model Reynolds' number. This results from the fact that separation of flow on a foil will occur at a lower angle of attack at the lower Reynolds' number. In linear theory, there is little concern about this difference. However, in handling the non-linear equations, some consideration should be given to the possible existence of this form of scale effect. Also in dealing with non-linear effects, consideration should be given to the fact that in a maneuver the angle of attack on the rudder is a combination

of the rudder deflection, the drift angle $\frac{v}{U}$, and the inflow angle caused by the angular velocity, $\frac{rd}{U}$, (d is distance the rudder is from the origin).

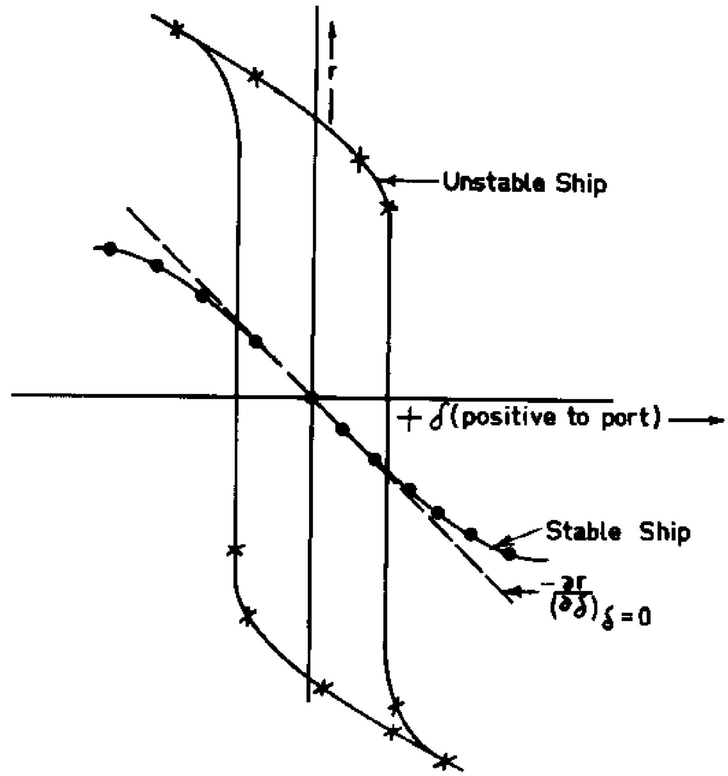
CHAPTER IX

Full Scale Trials for Evaluation of Steering and Maneuvring Characteristics

The theory of dynamic stability and control has been discussed and methods of evaluating these items for a given ship design have been described, using model tests and/or theoretical procedures. The determination of the stability and maneuvering qualities of the ship, as built, requires certain full scale trials and the nature of these trials will be briefly discussed.

Just as in the domain of resistance and powering of ships, where towing the full size ship to find its resistance is out of the question, it is virtually impossible to conduct tests on the ship which duplicate the model tests in order to determine the various hydrodynamic derivatives for the full size ship. Such restrained body tests as planar motions, rotating arm, etc., on the full size ship is inconceivable. The nature of the tests for the ship must necessarily involve an unrestrained, free running ship. Just as the first ship trial is an inclining experiment to determine the value of the metacentric height and the determination of the ship's stability in roll, an analogous type of trial referred to as the spiral maneuver is designed to determine the dynamical stability of the ship.

The spiral maneuver consists of the following operation of the ship. The ship executes a large rudder deflection to one side, say 25 degrees rudder to starboard. The rudder is held in this position until a constant angular velocity is obtained and this angular velocity is recorded. The rudder deflection is then reduced to say 20 degrees starboard and held until a steady angular velocity is reached and recorded. This procedure is continued down through 0 deflection and continues up through say 25 degrees port rudder. Although 5 degree intervals are desired at the large rudder deflections, deflection increments of 1 or 2 degrees are desired in the range between 5 degrees starboard and 5 degrees port rudder. The process is then repeated starting with say 25 degrees port rudder and ending up at 25 degrees starboard rudder. A plot of the measured angular velocity versus the rudder deflection is made as shown below.



If the ship is stable, then for each rudder deflection there is only one steady state angular velocity (i.e. one turning radius) and the curve appears as that indicated by the "stable ship" label in the above figure. Since, for small rudder deflections linear theory holds, the solution of the linearized equations for a rudder deflection should give us a prediction of the slope of the r versus δ curve of the trial (i.e. $(\frac{\partial r}{\partial \delta})_{\delta=0}$. Since the solution for r was

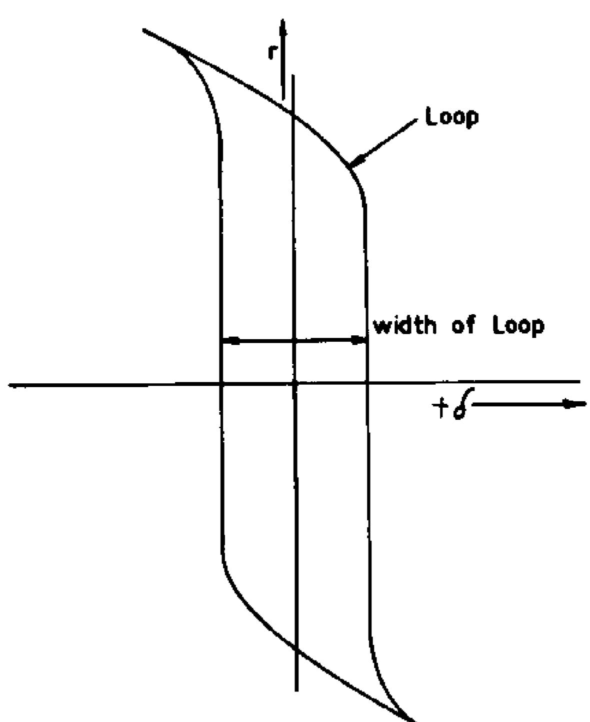
$$r = \left[\frac{N_v Y_\delta - Y_v N_\delta}{Y_v(N_r - mx_G u_o) - N_v(Y_r - mu_o)}\right] \delta_o$$

then

$$(\frac{\partial r}{\partial \delta})_{\delta=0} = \frac{N_v Y_\delta - Y_v N_\delta}{Y_v(N_r - mx_G u_o) - N_v(Y_r - mu_o)} .$$

The solution of the linear equations gives r as a linear function of δ. Hence, the range of validity of the linear theory is determined by observing just where the curve of r vs. δ departs from the tangent as drawn to this curve at $\delta = 0$. The use of non-linear equations and their solution is necessary to predict the curve of r versus δ at the larger rudder deflections.

On the other hand, if the ship is dynamically unstable in straight line motion, then the typical result of the spiral tests will be as that indicated by the label "unstable ship" in the figure above. It should be noted that the ship is unable to go straight ahead ($r = o$) for an undeflected rudder ($\delta = o$) but has two values of angular velocity for $\delta = o$, one positive and one negative. This indicates that the unstable ship, in the absence of any rudder deflection, may go into a left turn or a right turn, on a purely arbitrary basis depending on the arbitrary nature of any infinitesimal disturbance. The process of going through the sequence of rudder increments twice, including a reverse sequence (i.e. 25 degrees starboard to 25 degrees port and then from 25 degrees port to 25 degrees starboard) is needed in order to set up opposite initial disturbances for the condition of $\delta = o$ in order to clearly establish the two points on the curve. This reverse procedure establishes a loop in the curve for the unstable ship. The width of this loop determines the range of δ within which the ship may turn against its rudder. Hence the width of the loop, as shown below, is an indication of the magnitude of the instability of the ship.



When the linearized equations of motion, with rudder deflection, were solved, it was indicated that the solution for the case of the unstable ship was rather useless in that no steady state angular velocity was obtained since the transient terms involving $e^{\sigma_1 t}$ and $e^{\sigma_2 t}$ did not go to zero in time but increased. Hence, for the unstable ship, linear theory is unable to predict the nature of the loop but it can indicate whether a loop will exist (because of instability) and give some information as to the relative width of the loop. Non-linear theory and equations and their solutions are required in order to predict the size and shape of the loop for the unstable ship.

Some understanding of the significance of the spiral tests and the loop can be obtained by making an analogy between the spiral test plot and the curve of statical stability for a ship. The curves below are analogous - a righting or heeling moment (or arm) is the ordinate in the curve of statical stability and a minus rudder deflection*) (or positive rudder turning moment since with rudder at stern N_δ is negative) is the ordinate in the curve resulting from the spiral tests (have rotated the axis from the previous figure).

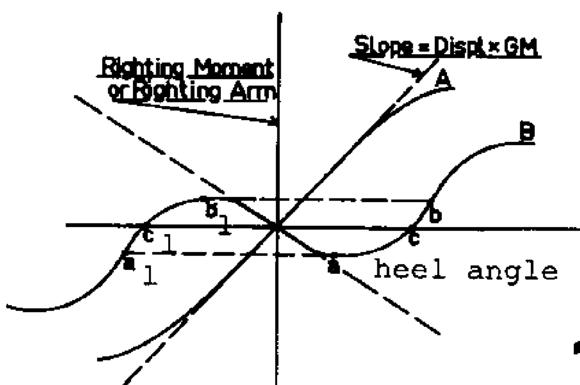


FIG.1

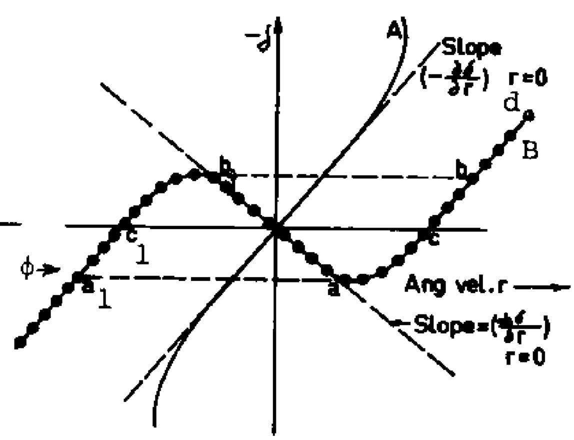


FIG. 2

In the case of the curve of statical stability the abscissa is the angle of heel φ , produced by the heeling moment, whereas in the spi-

*) Since for a rudder at the stern, a deflection to starboard is negative δ and yet an angular velocity of the ship to starboard is a positive r, then labelling the positive ordinate "rudder deflection to starboard" and the positive abscissa "angular velocity to starboard" will produce the proper graph.

ral tests the abscissa is the angular velocity resulting from the turning force and moment produced by the rudder deflection.

In the case of the ship stable in heel, curve A in Figure 1, the slope of the righting moment versus φ (at $\varphi = 0$) curve is positive, indicating stability, and, since this slope equals the ship's displacement multiplied by the metacentric height, it represents a positive metacentric height which is the criteria for stability in heel. Similarly, for the ship which is dynamically stable in straight line motion, curve A in Fig. 2, the slope of the $-\delta$ versus r (at r = 0) is positive - i.e. $\left(\frac{-\partial\delta}{\partial r}\right)_{r=0} > 0$. From the solution of r versus δ from the linearized equations of motion, we have for the equilibrium, or steady turning condition, for a given rudder deflection,

$$-\delta = \left[\frac{Y_v(N_r - mx_G u_o) - N_v(Y_r - mu_o)}{Y_v N_\delta - N_v Y_\delta}\right] r$$

$$\left(\frac{-\partial\delta}{\partial r}\right)_{r=0} = \left[\frac{Y_v(N_r - mx_G u_o) - N_v(Y_r - mu_o)}{Y_v N_\delta - N_v Y_\delta}\right].$$

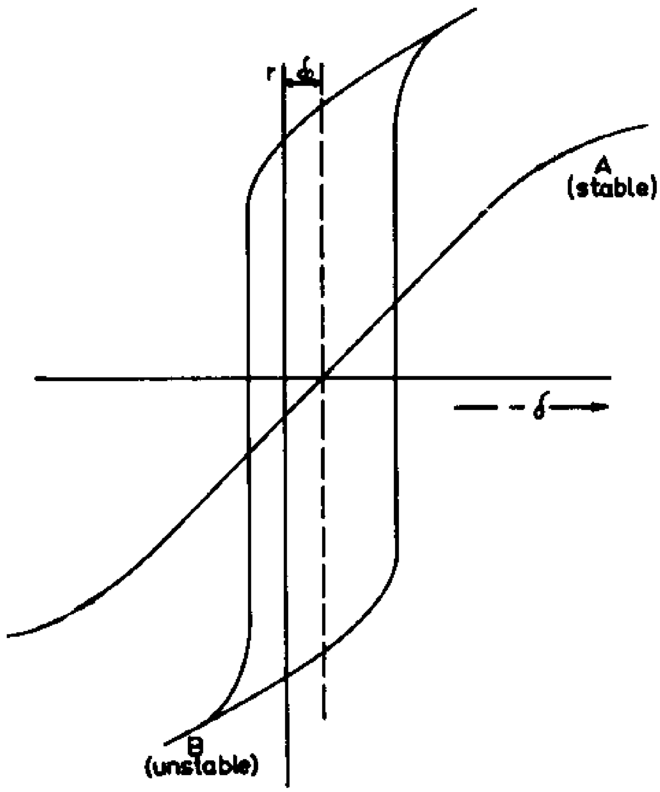
Since Y_v is negative and since N_δ is negative for a rudder at the stern (and this analysis has been developed for a rudder at the stern), the product $Y_v N_\delta$ is positive. Also, since for practically all ships N_v is slightly negative and since Y_δ is positive, the product $N_v Y_\delta$ is negative. Hence, the two terms in the denominator add to assure that the denominator, (in the above equation) is a positive quantity. (Very seldom, for ships, if ever, is N_v a positive quantity. If in some extreme case it is, it is only slightly so and the first term in the denominator will be predominant, thereby causing the denominator to be positive). With the denominator a positive quantity, whether the slope $\left(\frac{-\partial\delta}{\partial r}\right)_{r=0}$ is positive or negative depends on whether the numerator is positive or negative. But the numerator is exactly the criteria for dynamical stability in straight line motion as was derived from the solution of the homogeneous linear equations of motion. The criteria for stability, as previously developed, is that $Y_v(N_r - mx_G u_o) - N_v(Y_r - mu_o)$ be positive. Hence, if this term is positive, then the slope $\left(\frac{-\partial\delta}{\partial r}\right)_{r=0}$ is positive, and a positive slope of the curve of $-\delta$ versus r (for rudder aft) indicates stability of straight line motion. It has been demonstrated that a rather direct analogy exists between the curve of statical stability and the curve of $-\delta$ versus r (results of spiral trials) in that the criteria for stability, that GM be positive, determines the nature of the initial slope of the curve of statical stability and the criteria for dynamical stability determines the nature of the slope of the curve of $-\delta$ versus r. Similarly, if the ship is unstable, curve B in Figures 1 and 2, the slope at the origin of each of the B curves is negative, resulting in the expression in the criteria becoming negative, a negative GM in the case of the curve of statical stability and a negative value for $Y_v(N_r-mx_G\ u_o)-N_v(Y_r-mu_o)$ in the curve of $-\delta$ versus r. The larger the negative values of these slopes, the more unstable the ship is.

In the case of the stable ship (A in Figs. 1 and 2) there is only one resulting angle of heel for any given heeling disturbance and there is only one angular velocity (or turning radius) for any given rudder deflection. In the case of the unstable ship (B in Figures 1 and 2), there are regions, (between the lines aa_1 and bb_1) where there is more than one equilibrium angle of heel for a given heeling moment in the case of Fig. 1 and more than one turning angular velocity for a given rudder deflection in the case of Fig. 2. For the unstable ship in roll, there is a region in which a ship can heel against the direction of the heeling moment. Similarly, for the dynamically unstable ship, there is a region in which the ship can turn against its rudder.

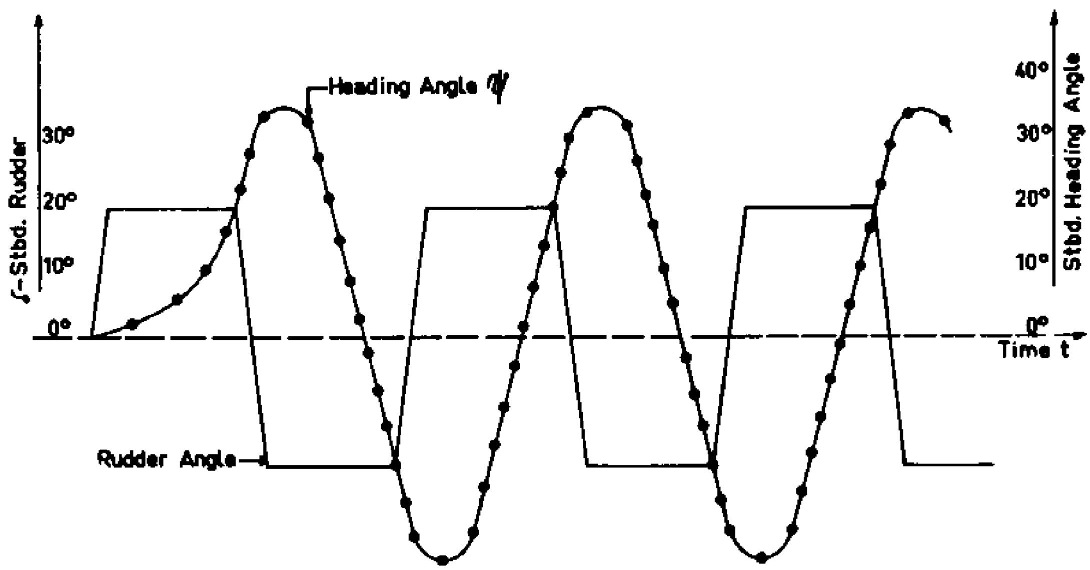
During the spiral tests, no data could be obtained for the unstable ship in the region between a and b_1 on curve B in Figure 2 (marked by points, O) even though these are points of equilibrium for the given rudder deflections. The reason for this is that these are points where the equilibrium is unstable and the ship cannot remain at the unstable equilibrium position but moves to a position of stable equilibrium for that rudder deflection. Hence in the absence of any turning disturbance at $\delta = 0$, the ship cannot go straight ahead ($r = 0$) since this equilibrium condition is unstable, and therefore it goes into a turn to either port or starboard ending up at a turning angular velocity (turning radius) denoted by points c or c_1 (on curve B Figure 2), which are <u>stable</u> equilibrium conditions for zero rudder deflection. (Note that the slope of the $-\delta$ versus r curve at these points are positive i.e. stable equilibrium). For a dynamically unstable ship with a large rudder deflection $-\delta$, the ship has an angular velocity denoted by point d on curve B of Fig. 2. As the rudder angle is reduced, the angular velocity is reduced following curve B, until at zero rudder deflection an angular velocity indicated by point c is obtained. On further reducing the rudder angle (i.e. deflecting the rudder to the other side), we find that the ship still continues to turn in the original direction against the direction of the rudder deflection until a rudder deflection and angular velocity indicated by 'a' on curve B. Any further change of the rudder angle will cause the ship to shoot over to a large angular velocity in the opposite direction as indicated by point a_1 (and perhaps temporarily overshoot a_1). Hence the unstable ship can turn against its rudder up to a point and then suddenly it will swing to the other direction to a stable condition for the rudder deflection. Similarly, as a spiral test progresses from point c_1 to b_1 further increase in rudder angle causes the ship to quickly swing as fast as its inertia will let it to an angular velocity indicated by b.

This type of behavior of the dynamically unstable ship is analogous to the ship which is unstable in roll. The unstable ship cannot remain upright even in the absence of a heeling disturbance. It will heel either to port or starboard reaching an angle of heel (for no disturbing moment) indicated by either c or c_1 on curve B Figure 1, which are positions of equilibrium (slope of the curve is positive at these points but negative at $\varphi = 0$). For a ship, heeled at the angle to starboard denoted by point c, if a port heeling moment is applied the angle of heel is reduced, but the ship is now heeled against the direction of the heeling moment. When this port heeling moment is increased the heel angle moves along curve B until point a is reached. Any further increase in the heel moment to port will cause the ship to swing quickly from point a to point a_1, which is a stable position of large heel to port. (Of course, the heel angle will overshoot a_1 but will finally settle down at a_1). Hence, the ship behaves in its inclining experiment in a fashion quite analogous to that of the ship in its spiral tests. No points in the unstable region between a and b, on curve B Figure 1, can be obtained during the inclining experiment.

For the case of a single screw ship wherein the direction of rotation of a single propeller is an asymmetry which produces a turning moment (and slight side force) on the ship, a certain amount of rudder deflection is required to overcome this moment and produce an equilibrium condition. The results of any spiral tests on this type of ship would cause a shift of the curve of r versus δ in the δ direction by the amount of rudder angle necessary to counteract the single screw effect (δ_e). A sample curve for this case is shown below.



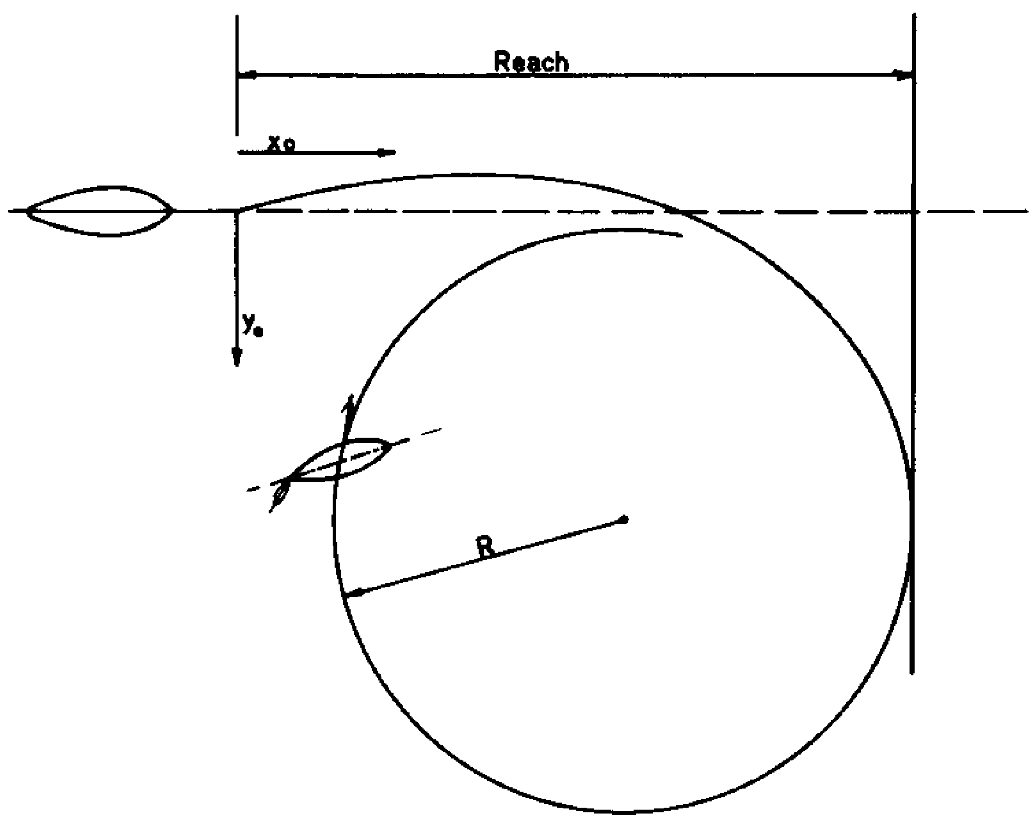
Another type of ship maneuvering trial is the Z maneuver consisting of the following steering sequence. With the ship proceeding at speed with zero rudder, the rudder is deflected to say 20 degrees to starboard (numbers are nominal and can be chosen to best suit the individual tests) and is held in this position until the ship has changed its heading to say 20° starboard. At this point the rudder is then suddenly deflected 20 degrees to port and held in that position until the ship reaches a heading of 20° to port at which time the rudder is suddenly deflected to 20° starboard. This sequence is repeated through several cycles, and the rudder angle and heading angle are recorded as a function of time during these maneuvers. The following sketch indicates typical measured response for the trials described above.



The Z maneuver measures or indicates the ability of a ship to rectify the motion, brought about by a maneuver, by a countermaneuver. Hence, the degree of overshoot of the heading angle curve (ratio of amplitude of ψ to amplitude of δ) and the phasing between the two curves are indicative of the dynamical stability and maneuverability of the ship. One notices the almost sinusoidal response in ψ for this maneuver. The response of the ship to such a rudder program may be predicted from the sinusoidal rudder oscillation equations and solution which was indicated previously. Or a more accurate calculation can be made by solving the equations for a square wave input. The linearized equations should give a fairly good estimate of the ship response provided moderate rudder angles and changes of heading are used in the trials.

A ship which is slightly unstable can turn rather quickly to starboard for application of starboard rudder (although it has an ability to turn against the rudder for certain small rudder deflections). Since it turns in the absence of any rudder, a first maneuver can be accomplished somewhat more quickly by the less stable ship. However, in a second countering maneuver it takes more time for the less stable ship to pull out of its original turn and go into the turn called for by the second maneuver. Hence the less stable ship tends to overshoot more during a Z maneuver. The ability to react properly during countering maneuvers is important when sailing in restricted waters (harbor or canal) in that in taking action to avoid one situation, the ship may find itself in a second situation from which it cannot pull out in sufficient time. Hence, besides desiring dynamical stability for the purposes of wanting to go straight when no rudder angle is applied, one also desires to have such stability to improve the ship's ability in countermaneuvers.

Of course the normal steering trial of ships is well-known, wherein the ship proceeds at speed in straight line and then suddenly deflects the rudder and the resulting turning trajectory is measured. A typical ship's path for such a maneuver is shown below.

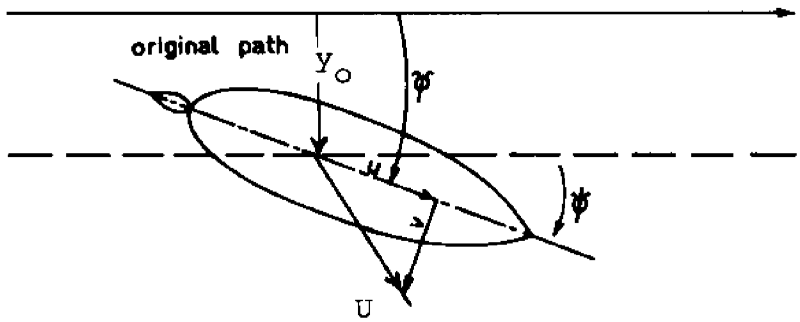


The trajectory of the ship y_o versus x_o is plotted and from this the radius of turn, R, and the reach is determined. For small rudder angles (not too tight a turn) the linear theory may give a pretty good prediction of the turning circle radius. Linear theory does not give the speed reduction in the turn nor can it readily give the trajectory.

The rate of motion sidewise to the original path is given by $\frac{dy_o}{dt} = \dot{y}_o$ and along the direction of the original path by $\frac{dx_o}{dt} = \dot{x}_o$. It can be seen from the sketch below that

$$\dot{y}_o = u \sin\psi + v \cos\psi$$
$$\dot{x}_o = u \cos\psi - v \sin\psi$$

where ψ is the heading angle of the ship relative to the original path.



To get the trajectory plot of y_o vs. x_o, one needs to integrate y_o and x_o with respect to time. Hence

$$y_o = \int (u \sin\psi + v \cos\psi)dt$$
$$x_o = \int (u \cos\psi - v \sin\psi)dt .$$

If one considers small angles of ψ and linearizes, then $\sin\psi \approx \psi$, $\cos\psi \approx 1$, and the path calculated by the linear theory would only hold until ψ reached about 15 degrees and then would be quite invalid beyond this point. In order to determine the "reach", the integrals must be carried out until $\psi = 90^\circ$ which is well beyond the range of linearity. Hence, non-linear equations are necessary for estimating the trajectory of the steering maneuver as well as predicting the speed loss in the maneuver. However, if Δu and v remain within the linear range, the above integration for obtaining the trajectory can be easily calculated.

CHAPTER X

Non-Linear Equation of Motion

The usefulness and the limitations of the linear equations for steering, stability, and maneuvering and their solutions are mentioned previously. It was indicated that certain important information in the prediction of certain ship maneuvers would require the use and solution of non-linear equations of motion. The nature or form of these non-linear equations will be developed and discussed from the point of view of certain hydrodynamic aspects. First let us consider the equation for the X force. The linear equation was

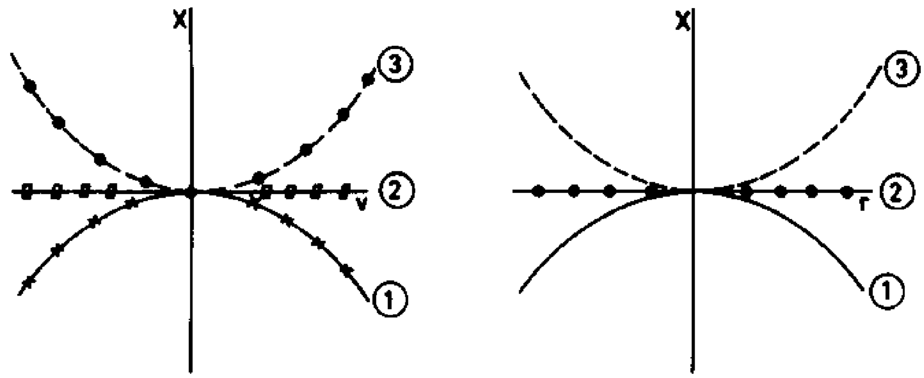
$$X_{\dot{u}}\dot{u}+X_u\,\Delta u+X_{\dot{v}}\dot{v}+X_v v+X_{\dot{r}}\dot{r}+X_r r+X_\delta\delta = m\dot{u}.$$

On the right hand side of the equation, the non-linear terms, as previously developed in the general equations of motion, become

$$m\left[\dot{u}-rv-x_G r^2\right]$$

for the condition of motion in the horizontal plane, without roll, with $y_G = 0$ (G on centerline plane).

It was shown, previously, that the derivatives X_v, X_r, X_δ, $X_{\dot{v}}$, and $X_{\dot{r}}$, were all zero because of the symmetry of port and starboard (present the same shape to the flow). Because of these symmetry properties, it was shown that the functions, from which each of the above derivatives were obtained, had to be even functions of the variables, that is, the graph was symmetrical relative to the ordinate (X) axis, as indicated by the sketches below, (general types of curves 1, 2, or 3.



Similar symmetrical graphs result for the X force as functions of δ, $\dot{r}$, and $\dot{v}$. Let us take as a typical case, the curve of X versus v. If X is to be expressed by an expansion in powers of v beyond the first power - i.e. in non-linear polynomial form - then, because X is an even function of v, only the even powers of v can appear in the expansion and the coefficients of the odd powers must be zero. Hence X as a function of v takes the form

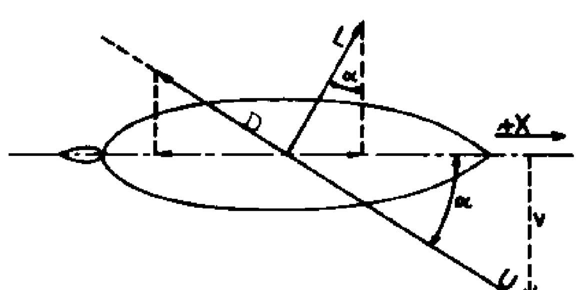
$$X(v) = a_2 v^2 + a_4 v^4 + a_6 v^6 + \dots\dots\dots$$

where the a's are coefficients depending on body shape.

For a given u_o, as v is changed, the angle of attack of the flow to the hull changes with v. From the hydrodynamics of low aspect ratio foils, when a body is placed at an angle of attack to the flow, in addition to a lift force, there is also created an increase in drag resulting from an increase in profile drag and the production of induced drag due to flow over the foil edge (in this case - the keel). The X force in terms of lift (L) and drag (D) can be expressed as

$$X = L \sin\alpha - D \cos\alpha$$

and $\alpha = \sin^{-1} \frac{v}{U}$.



Since the lift is linear in α for small α and since the drag varies approximately as the square of α, the X versus v should initially take the form of a quadratic (and perhaps higher even powers), probably of the shape indicated by ①, since it is expected that the drag force term is the larger of the two terms. Since an angular velocity, r, produces a local flow approach angle $\frac{rd}{u_o}$ at a distance d from the origin, there will be "localized L and D" at each point with a net force on the body from the integral effect. Hence, the nature of the X versus r curve will be similar to those of the X versus v curve because similar hydrodynamic effects are present. The curve of X versus r is an even function*) because of symmetry, and, because of the reasons just mentioned, the curve will probably have the form indicated by ① in the sketch above. Similar reasons hold for the X versus δ curve. Hence we have

$$X(r) = b_2r^2+b_4r^4+b_6r^6+\ldots\ldots$$

$$X(\delta) = c_2\delta^2+c_4\delta^4+c_6\delta^6+\ldots\ldots$$

where the b and c are various coefficients depending on the body shape.

Actually, in the general situation, δ, v, and r vary at the same time. The localized flow approach angle along the hull is $\frac{v}{U} + \frac{rx_s}{U}$, and this angle at the rudder is $-\delta + \frac{v}{U} + \frac{rx_s}{U}$ **). Considering the symmetry of port and starboard (calling for even power expansion, and the hydrodynamic aspects discussed above), the form of X as a function of δ, v, and r will be the even power expansion of the effective sum of these parameters as shown below.

$$X(r,v,\delta)=(k_1r+k_2v+k_3\delta)^2+(k_1r+k_2v+k_3\delta)^4+\ldots\ldots$$

*) Even functions can be produced using absolute values of odd powers, such as where the b and c are various coefficients depending on the body shape.

$$a_1|v|+ a_2v^2+a_3|v^3|+a_4v^4+\ldots\ldots$$

but the term $|v|$ makes the function discontinuous at v = 0 and the use of absolute values seems unnatural and further complicates computation.

**) Where x_s is positive forward of the origin.

The expansion gives terms in

$$r^2,\ v^2, \delta^2,\ rv,\ r\delta\ ,\ v\delta\ ,\ r^4, v^4, \delta^4,\ r^3 v,\ + \ldots\ldots$$

For the purpose of calculating or predicting ship maneuvers, it is sufficient to carry the non-linearity through the 3^{rd} order, or cubic term. Hence, the 4^{th} order terms are to be dropped and the non-linear terms of interest are

$$r^2,\ v^2,\ \delta^2,\ rv,\ r\delta\ ,\ v\delta\ .$$

Since the hydrodynamic forces contributing to X are, in part, attributable to angle of attack situations, and since the angle of attack and hydrodynamic forces depend on the forward velocity u, the forces arising from r, v, and δ may vary as u varies. If for a constant velocity u_o, X varies, say, as kr^2 (where k is a constant), then for a different velocity $u_o + \Delta u$ (and therefore a different Froude number, $\frac{u}{\sqrt{gL}}$, the force at a given r will change as u changes. If this change in force brought about by a change in forward velocity component, Δu, is expressed as a power series expansion in Δu, one obtains

$$X = kr^2 \left[1^{*)} + k_1\, \Delta u + k_2 (\Delta u)^2 + \ldots\ldots \right]$$

Since only terms through the cubic are to be retained, the above expression reduces to

$$kr^2(1+k_1\, \Delta u) = kr^2 + kk_1 r^2\, \Delta u = kr^2 + k_2' r^2\, \Delta u$$

where the k's are constants depending on ship geometry and size. The term r^2 was taken as an example. Similar terms will be derived from considering v^2, δ^2, rv, $r\delta$, and $v\delta$. (For example, one obtains terms $k_3\, \delta v + k_4\, \delta v\, \Delta u$). The non-linear equation in X (through the cubic term) will include terms in

$$v^2,\ r^2, \delta^2,\ rv,\ r\delta\ , \delta v,\ v^2 \Delta u,\ r^2 \Delta u,\ \delta^2 \Delta u,\ rv\, \Delta u,\ r\, \delta\, \Delta u, \text{ and } \delta v\, \Delta u.$$

In the linear theory, $X_u \Delta u$ gives the variation of X with Δu. The variation of the X force with Δu can be expressed as a power expres-

*) This 1 takes care of the value when $u = u_o$, i.e. when $\Delta u = 0$.

sion in Δu; hence, one expects terms in Δu, $(\Delta u)^2$, $(\Delta u)^3$, etc. Such a form can be shown to exist for the actual ship. The X force on a ship moving straight ahead is the difference between the propeller thrust (working with the thrust deduction) and the resistance (drag) of the ship. In equation form this becomes

$$X = K_t \rho n^2 d^4 (1-t) - 1/2\ C_R \rho S u^2$$

where

- K_t is the propeller thrust coefficient
- d is the propeller diameter
- ρ is the water density
- n is the revolutions per second of the propeller
- t is the thrust deduction factor
- C_R is the resistance coefficient
- S is the wetted surface of the ship
- u is the forward speed component of the ship

The coefficient K_t is a function of the speed coefficient, $J = \frac{(1-w)u}{nd}$, where w is the wake fraction. The resistance coefficient, C_R, is a function of u (or the Froude number, $\frac{u}{\sqrt{gL}}$). The K_t vs. J curve is quite linear in the region of the design speed. Hence, the change in K_t with u, in the region of a speed change caused by a maneuver, can be expressed as $\frac{\partial K_t}{\partial \Delta u} \Delta u = \frac{\partial K_t}{\partial J} \frac{\partial J}{\partial \Delta u} \Delta u$.

Similarly, let us assume, and it is sufficiently valid for the Δu occurring in a maneuver, that the variation in C_R with u (i.e. with Froude number) can be expressed by $\frac{\partial C_R}{\partial \Delta u} \Delta u$, (this means the slope of the curve multiplied by the change in u). Assuming that the revolutions of the propeller do not change during a maneuver, the expression for the X force for a disturbance, Δu, becomes (assuming t does not vary with Δu)

$$u_o^2 + 2u_o \Delta u + (\Delta u)^2$$

$$X(\Delta u) = \left[(K_t)_o + \left(\frac{\partial K_t}{\partial \Delta u}\right)_{u=o} \Delta u\right] \rho n^2 d^4 (1-t) - \left[(C_R)_o + \left(\frac{\partial C_R}{\partial \Delta u}\right)_{u=o} \Delta u\right] 1/2\, \rho S (u_o + \Delta u)^2$$

where $(K_t)_o$ is the thrust coefficient at $u = u_o$ and $(C_R)_o$ is the resistance coefficient at $u = u_o$. Since at the speed u_o, the ship is in an equilibrium condition, i.e. constant speed, then

$$(K_t)_o \rho n^2 d^4 (1-t) - (C_R)_o 1/2 \rho S u_o^2 = 0.$$

Also, $\frac{\partial J}{\partial \Delta u} = \frac{\partial}{\partial \Delta u} \frac{(u_o + \Delta u)}{nd} (1-w) = \frac{(1-w)}{nd}$ where w is the wake fraction.

The expression for $X(\Delta u)$ becomes

$$X(\Delta u) = \left[\left[\left(\frac{\partial K_t}{\partial J}\right)_{u=u_o} \left(\frac{1-w}{nd}\right)(1-t) \rho n^2 d^4\right] - 1/2 \rho S \left[2u_o (C_R)_o + \left(\frac{\partial C_R}{\partial \Delta u}\right)_{u=u_o} u_o^2\right]\right] \Delta u$$

$$-1/2 \rho S \left[(C_R)_o + 2u_o \left(\frac{\partial C_R}{\partial \Delta u}\right)_{u=u_o}\right] (\Delta u)^2 - 1/2 \rho S \left[\left(\frac{\partial C_R}{\partial \Delta u}\right)_{u=u_o}\right] (\Delta u)^3.$$

Hence, this analysis indicates that a power function in Δu is required and the series is carried through the cubic term. In any practical example, the K_t vs. J curve for the ship's propeller along with the C_R versus u for the ship, can be used to estimate the coefficients of Δu, $(\Delta u)^2$, and $(\Delta u)^3$. The coefficients can also be estimated by running self propelled model tests, but consideration should be given to scale effects and the equivalent propulsion point of the ship. The X equation will therefore include terms in Δu, $(\Delta u)^2$, and $(\Delta u)^3$.

The X force also depends on the acceleration parameters $\dot{u}$, $\dot{v}$, and $\dot{r}$. In the linear equations, one obtained for the linear form

$$X(\dot{u}) \approx X_{\dot{u}} \dot{u}$$

$$X(\dot{v}) \approx X_{\dot{v}} \dot{v} = 0$$

$$X(\dot{r}) \approx X_{\dot{r}} \dot{r} = 0$$

since, as was shown previously, $X_{\dot{v}}$ and $X_{\dot{r}}$ were zero because of the symmetry of port and starboard. In considering the non-linear form for these parameters one would form a power series in these accelerations, i.e. $\dot{u}^2$, $\dot{u}^3$, $\dot{v}^2$, $\dot{v}^3$, etc. It is expected that the coefficients of the higher power acceleration terms will be zero (or negligibly small) from the following considerations.

The acceleration forces are essentially the result of the inertia property, i.e. density, of the fluid. Since there is no significant interaction between the inertial and viscous forces, it was indicated previously that calculation of the hydrodynamic acceleration forces by potential theory gave adequate values. Non-linear equations of potential theory when applied to submerged bodies give forces resulting from accelerations which are linear in the acceleration. Hence, one expects no second or higher order terms in the acceleration parameters. Also, on the right hand side of the general equations which represent the inertial properties of the body, acceleration terms appear only in the linear term.

In a general Taylor expansion of the variables to the higher order terms, terms of the form $\dot{r}v^2$, $\dot{u}rv$, etc. would appear as cubic terms. These terms represent interaction between acceleration and velocity parameters. Since these terms a) do not appear in the non-linear potential theory solution, b) are essentially interference between viscous and inertial aspects and c) do not appear on the right side of the equation (body inertial forces), then the coefficients of these type of terms are considered zero or at least negligibly small.

An additional argument is that, because of the magnitude of the inertia of the ship, during even tight maneuvers the accelerations remain small.

The expected form of the non-linear equation for X becomes

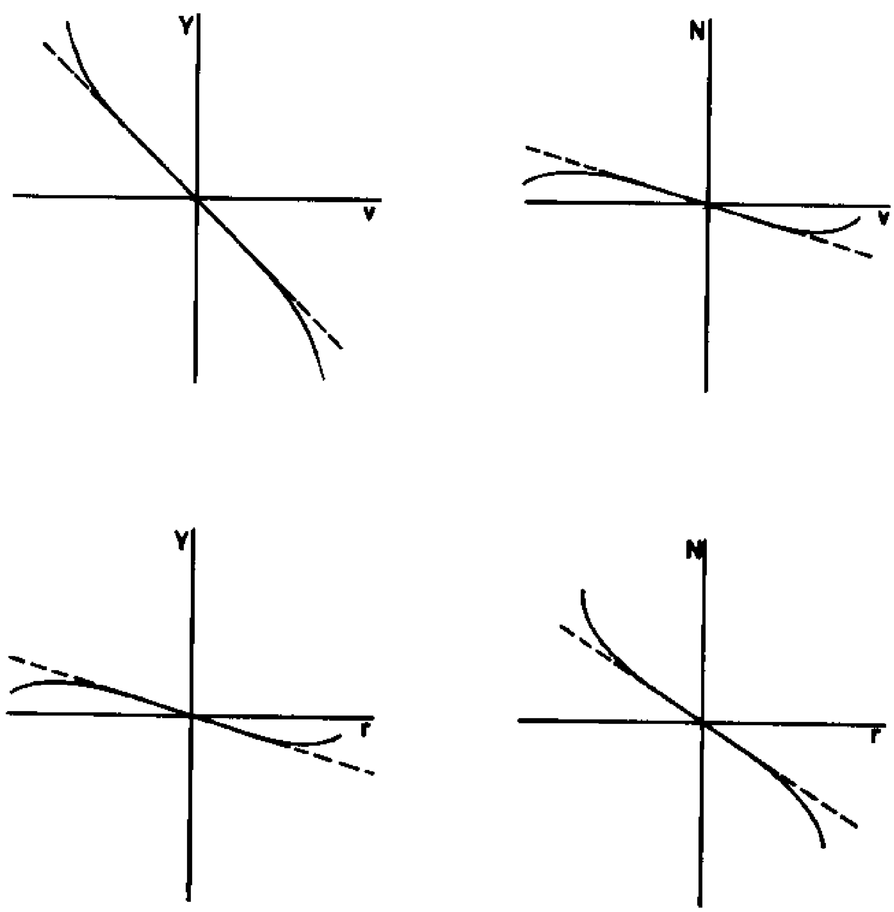
$$a\dot{u}+b\Delta u+c(\Delta u)^2+d(\Delta u)^3+ev^2+fr^2+g\delta^2+hrv+jr\delta+kv\delta+e_1v^2\Delta u$$

$$+f_1r^2\Delta u+g_1\delta^2\Delta u+h_1rv\Delta u+j_1r\delta\Delta u+k_1v\delta\Delta u = 0.$$

The coefficients a, f, and h include those terms brought over from the right hand side of the X equation.

Some of the coefficients in the above equation are small and difficult to measure in a model experiment or to calculate and little is lost from the accuracy of prediction if these coefficients are set equal to zero. Certain of the other coefficients can be readily measured on models or calculated, as indicated previously for the case of a, b, c, and d. The terms e, f, and g can be obtained from model tests by measuring the X force at various values of the parameter (v, r, and δ), and determining the value of the coefficient from the resulting curve (essentially parabolic). Similarly, by running these model tests at different speeds, the coefficients e_1, f_1 and g_1 can be determined. Considerations of inflow angle at the rudder may lead to an estimate of the coefficients j, k, and possibly j_1 and k_1 (although these terms may be negligibly small). (The counterparts of these terms in the Y and N equation can similarly be estimated).

The arguments for the form of the non-linear equations for Y and N are similar to those used in the X equation. With regard to the acceleration parameters only the linear terms will appear in the non-linear equations for the same reasons as were discussed for the acceleration terms in the X equation. The terms involving r, v, and δ , in the Y and N equations will be odd functions of these parameters, that is the graph of the function (say Y versus v) will be reflected about the origin. This results again from the symmetry of port and starboard of the ship. The curves of Y vs. v, N vs. v, Y vs. r, and N vs. r will appear something like the sketches shown below.



The curve of Y versus v will be taken as an example for discussion, realizing that similar conclusions will be obtained from consideration of the other curves.

Any expansion of the Y(v) in a power series in v will give 0 for the coefficients of the even powers of v, since the function is an odd function of v. Hence, for Y(v) one obtains

$$Y(v) = s_1 v + s_2 v^3 + s_3 v^5 + \ldots\ldots *$$

It is obvious that the coefficient s_1 represents the slope of the curve at v = 0 and therefore $s_1 = Y_v$. In the discussion of the X equation, it was indicated that the hydrodynamic forces due to (v, r, and δ) would be power expansions of the combination of these terms. Therefore, one expects in the Y and N functions an odd power series in the sum of these variables of the following form

$$(k_5 v + k_6 r + k_7 \delta) + (k_8 v + k_9 r + k_{10} \delta)^3 + (k_{11} v + k_{12} r + k_{13} \delta)^5 + \ldots\ldots$$

If the series is stopped after the cubic term, then the following terms appear in the expansion of Y and N in r, v, and δ .

$$r, v, \delta, r^3, v^3, \delta^3, r^2\delta, \delta^2 r, r^2 v, v^2 r, \delta^2 v, v^2\delta, \text{ and } \delta rv.$$

As indicated in the discussion of the X equation, the change in velocity, Δu, will affect the forces involved and a power series in Δu was indicated. Hence, in the Y and N equations, a power expansion in (Δu) is used to describe the effect of a change in velocity Δu from the original velocity u_o. If terms up through the cubic are retained, then terms like $r\Delta u$, $r(\Delta u)^2$, etc. will be included in the function.

If for the case of zero rudder deflection, the ship has a turning moment N_o and a force Y_o due to the asymmetry of a single screw propeller (rotating in one direction), this initial moment and force must be taken into account. Since, the forces due to single screw action may be altered by a change in forward speed, other terms in Y_o and N_o in combination with (Δu) may be present. The non-linear equation for Y takes the following form on the basis of the previous discussion.

$$A\dot{v} + B\dot{r} + Cv + Dr + E\delta + Fv^3 + Gr^3 + H\delta^3 + Ir^2\delta + J_1\delta^2 r + Kv^2 r + Mr^2 v + N_1\delta^2 v + Pv^2\delta + P_1 rv\delta$$

$$+ Qv\Delta u + Rv(\Delta u)^2 + Sr\Delta u + Tr(\Delta u)^2 + V\delta\Delta u + W\delta(\Delta u)^2 + Y_o\left[1 + A_1\Delta u + B_1(\Delta u)^2\right] = 0$$

The coefficients A, B, D, and S incorporate terms brought over from the right hand side of the equation of motion (i.e. body inertial forces).

*One can consider a term of the form $v|v|$, which is also an odd function, if the basic hydrodynamic phenomenon indicates this type of relationship.

The form of the N equation is identical, with different coefficients of course, and the coefficients corresponding to A, B, D, and S will also contain terms brought over from the right hand side.

The coefficients C and F can be determined from the plot of model test results on the planar motions mechanism (or three component dynamometer). The coefficients D and G can be obtained by the plot of model test results from the rotating arm tests or planar motions test. The coefficients E and H can be determined by the model tests described previously where the measurement of Y_δ and N_δ was discussed. By testing at different speeds, the coefficients Q, R, S, T, V, and W, A_1 and B_1 can be estimated. A and B are derivable from planar motions model tests. The order of magnitude of other coefficients may be estimated from hydrodynamic considerations and thereby it can be determined if they are small enough to neglect. The more important non-linear terms in the Y and N equations are the terms in v^3, r^3, and δ^3 followed in importance by perhaps some terms in Δu. In the X equation the important non-linear terms are the terms in $(\Delta u)^2$, v^2, r^2, and perhaps rv followed by some of the terms involving Δu.

The solution of the non-linear equations for ship maneuvering has been programmed for the computer. Using the various coefficients determined by model tests or calculated, the path, velocity, angular velocity, etc. are calculated as a function of time for a ship in the several types of maneuver (or trials) such as the spiral, turning circle, and oscillating rudder maneuvers.

The equations of motion have been developed and solved in dimensional form. For example, Y has the dimensions of a force, v has the dimensions of a velocity, $\dot{r}$ has the dimensions of angular acceleration. Since the values of many of the coefficients for a given ship are derived from model tests, it may be advantageous in certain cases to express these coefficients, the equations of motion, and their solution in non-dimensional form. As indicated earlier, the physical quantities of length (L), mass density (ρ), and velocity (U) are the most convenient quantities to use to produce dimensionless coefficients in hydrodynamics. ρ has the dimension of mass per unit volume $(\frac{M}{L^3})$, L has the dimension of length, and U has the dimension of length per unit time $(\frac{L}{T})$. Hence, time, T, has the dimension of $\frac{L}{U}$, $(\frac{L}{\frac{L}{T}} = T)$.

The non-dimensional form of the various derivatives can be obtained by writing both the numerator and denominator of the derivative in non-dimensional form.

For example, if the derivative Y_v in non-dimensional form is expressed by $(Y_v)^1$, then

$$(Y_v)^1 = \frac{\partial Y^1}{\partial v^1} = \frac{\partial\left[\frac{Y}{\frac{1}{2}\rho L^2U^2}\right]}{\partial\left[\frac{v}{U}\right]} = \frac{\partial Y}{\partial v}\left[\frac{1}{\frac{1}{2}\rho L^2U}\right] = \frac{Y_v}{\frac{1}{2}\rho L^2U}$$

since $\frac{1}{2}\rho L^2U^2$ has the dimension of a force and v has the dimension of a velocity. Hence, dividing Y_v by $\frac{1}{2}\rho L^2U$ will reduce it to non-dimensional form. The coefficient $(Y_{\dot{v}}-m)$ is reduced to non-dimensional form by dividing $\dot{v}$ by $\frac{U^2}{L}$ which has the dimensions of an acceleration, since $\frac{U^2}{L} = \frac{L^2}{T^2} \cdot \frac{1}{L} = \frac{L}{T^2}$ = dimensions of acceleration. Therefore

$$(Y_{\dot{v}})^1 = \frac{\partial Y^1}{\partial \dot{v}^1} = \frac{\partial\left[\frac{Y}{\frac{1}{2}\rho L^2U^2}\right]}{\partial\left(\frac{\dot{v}L}{U^2}\right)} = \frac{\partial Y}{\partial \dot{v}} \; \frac{1}{\frac{1}{2}\rho L^3} = \frac{Y_{\dot{v}}}{\frac{1}{2}\rho L^3}$$

and the mass m is non-dimensionalized by dividing by the dimension of mass - i.e. $m^1 = \frac{m}{\frac{1}{2}\rho L^3}$. It is clear that $Y_{\dot{v}}$ has the dimensions of mass, as it should, and the total term $Y_{\dot{v}}-m$ reduces to the non-dimensional form as follows.

$$(Y_{\dot{v}})^1 - m^1 = (Y_{\dot{v}}-m)^1 = \frac{(Y_{\dot{v}}-m)}{\frac{1}{2}\rho L^3} .$$

When a similar procedure is used to reduce $N_r - mx_Gu_o$ to non-dimensional form, one obtains (since r has the dimension of $\frac{1}{T}$ and N has the dimensions of a moment = force · length)

$$(N_r)^1 = \frac{\partial\left(\frac{N}{\frac{1}{2}\rho L^3U^2}\right)}{\partial\left(\frac{rL}{U}\right)} = \frac{N_r}{\frac{1}{2}\rho L^4U}$$

$$(mx_Gu_o)^1 = \frac{m}{\frac{1}{2}\rho L^3} \cdot \frac{x_G}{L} \cdot \frac{u_o}{U} = \frac{mx_Gu_o}{\frac{1}{2}\rho L^4U}$$

$$(N_r - mx_G u_o)^1 = \frac{N_r - mx_G u_o}{\frac{1}{2}\rho L^4 U}.$$

The non-dimensional forms of the other derivatives can be derived in the same manner. In using the non-dimensional form of the equations, the Froude number, $\frac{U}{\sqrt{gL}}$, (or $\frac{u_o}{\sqrt{gL}}$) is to be used as the parameter for speed. For the maneuvering of a submerged submarine in the horizontal plane,* where there are no free surface effects, and therefore no dependence on Froude number, the non-dimensional equations become independent of speed.

*Without roll.

CHAPTER XI

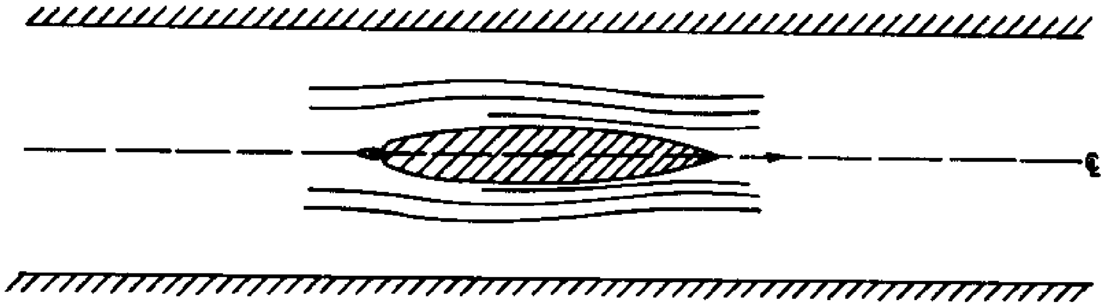
Maneuvring in Restricted Waters and Stability of Towed Bodies

Up to the present, the discussions have dealt with the ship in unrestricted water both in water surface size and water depth. Also, we have considered the ship as a free body with no extraneous forces acting on it, only hydrodynamic forces arising from its motion and control surface deflection.

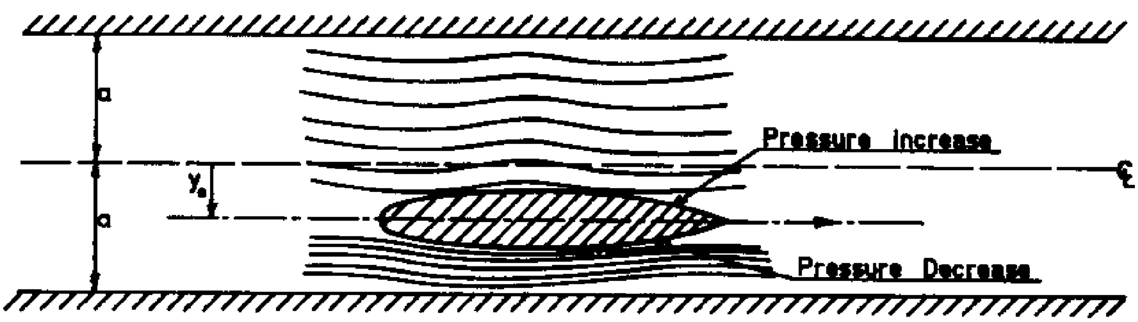
If a ship (barge) is being towed, forces are applied to the ship, through the tension in the tow line, at the point of attachment of the line to the ship. Hence, one must add the force and moment caused by the tow rope to the usual hydrodynamic force and moment acting on the ship.

If a ship moves in water, restricted either in depth or width or both, the water flow lines about the ship are altered from the situation of deep depth and width. The change in flow lines cause a change in the hydrodynamic forces acting on the hull. For example, when a ship moves ahead through water, the lines of flow not only go around the sides of the ship but also go down the bow and along the bottom of the ship. If the water is shallow, the water flow under the hull is somewhat restricted causing more water to flow along the sides. This in turn changes the side force and moment acting on a ship and therefore can change the hydrodynamic derivatives (such as Y_v, N_v, etc.). Hence, shallow water may change the value of the derivatives for a given hull and, if maneuvering in very shallow water is being considered, model tests to measure these derivatives, should be carried out in shallow water.

If a ship moves in water where close boundaries restrict the motion, such as in a canal, then the flow around the sides of the ship are altered from those existing in unrestricted water. This in turn causes certain hydrodynamic forces on the hull. Consider a ship moving in a canal as indicated in the sketch below.



If the ship is on the centerline of the canal, there is symmetrical flow on port and starboard side, hence, no moment or side force. If the ship is moving along the canal off of the centerline and closer to one wall, the symmetry of flow is disturbed, as indicated in the following sketch.

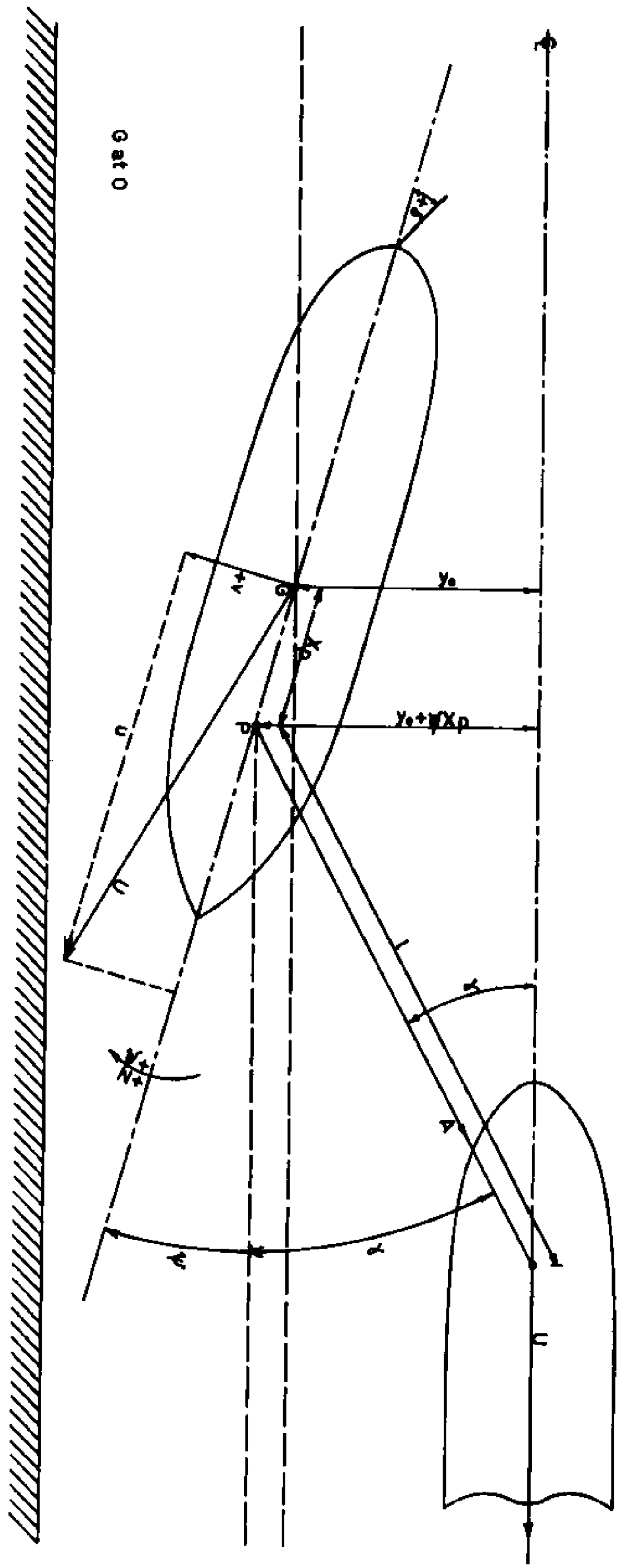


The velocity of the water between the hull and the near side of the canal is increased whereas the velocity of the water between the hull and the far side is decreased. This causes a decrease in pressure on the near side of the hull and an increase in pressure on the far side (Venturi effect). This pressure creates a force drawing the ship towards the near wall of the canal and a moment tending to swing the bow away from the near canal wall. If the distance from the centerline of the canal to the origin on the centerline of the ship is designated by y_o, then in a narrow canal there is a force and moment produced on the ship as a function of y_o (i.e. orientation in space) and the linear approximations (linear term in the Taylor expansion of the function) are $Y_{y_o} y_o$ and $N_{y_o} y_o$. The derivatives Y_{y_o} and N_{y_o} are zero for the case of a ship in open ocean but become very significant for motion of a ship in a canal since they represent essentially destabilizing (and hazardous) effects - tending to draw the ship away from the center of the canal (original equilibrium condition) towards the near bank and tending to swing the ship away from its heading straight down the canal and head it toward the far bank. This hazardous situation calls for very experienced piloting since a maneuver tending to avoid hitting the near side (or even without any rudder application) may cause the ship to head for the other side requiring an appropriate counter-maneuver to avoid hitting the canal side. Here we have a case where not only the stability of the ship has been reduced because of its location in a canal but the ship is faced with counter-maneuvers and a greater tendency to overshoot on these maneuvers because of this reduced stability. A ship with insufficient or marginal dynamical stability in the open ocean, may have its stability degraded in a canal, where stability and control are essential, to a point where the pilot has extreme difficulty in avoiding collision with the canal wall.

In order to handle the case of stability of towed bodies and the case of stability in a canal, a general treatment of the case of stability of a towed body in a canal is carried out. The equations and solutions can be used to cover any specialized case by setting equal to zero those terms which are zero for the special case. Hence, the linear equations and development hold for

1. Towed or partially towed bodies in a canal.
2. Towed bodies in open sea.
3. Ship moving in a canal (under own power).
4. Ship moving in open sea.
5. Any of the above, with or without automatic control.

In order to make the treatment of the problem more realistic, the rudder is deflected under automatic controls sensitive to the heading angle ψ , the distance off the centerline y_o, and the angular velocity r. This is to simulate the action of an experienced pilot who would take into account these items in ordering the various rudder applications. A time lag t^1 is used for the control system. The following sketch shows the ship being towed in a canal with the ship oriented in some general position in the canal. It is assumed that the towing vessel moves at all times down the center of the tank,


G at O

wherein the towed vessel is free to move over the water surface. (A more complicated analysis can be made for the case where the towing vessel has the freedom to move off the centerline).

The following is a definition of certain symbols indicated in the sketch.

ℓ is the length of the towline.

x_p is the location of the tow point (point of line attachment) relative to the origin, G. x_p is positive if forward of G and negative if aft of G.

G is the location of the center of gravity which has been chosen as the origin in this development.

P is the point of attachment of the tow line to the hull.

ψ is the heading angle of the ship relative to canal centerline.

γ is the angle between the canal direction and the towline direction.

T is the tension in the towline.

With the origin located at the center of gravity, the linearized equations for Y and N become

$$Y = m(\dot{v}+ru_o) = Y_{\dot{v}}\dot{v}+Y_v v+Y_{\dot{r}}\dot{r}+Y_r r+Y_\psi \psi+Y_{y_o} y_o+Y_\delta \delta - T \sin(\gamma+\psi) \quad (1)$$

$$N = I_z\dot{r} = N_{\dot{v}}\dot{v}+N_v v+N_r r+N_{\dot{r}}\dot{r}+N_\psi \psi+N_{y_o} y_o+N_\delta \delta - T \sin(\gamma+\psi)x_p. \quad (2)$$

The X equation has been decoupled since, as shown in a previous development, certain derivatives in the X equation are zero because of port and starboard symmetry.

For small ψ and γ:

$$\sin(\gamma + \psi) = \psi + \left(\frac{y_o+x_p\psi}{\ell}\right) = \left(1+\frac{x_p}{\ell}\right)\psi + \frac{y_o}{\ell}.$$

If we have controls proportional to ψ, r, and y_o with an equivalent time lag t^1, then the control function becomes:

$$\delta = k_1(1-t^1D)\psi+k_2(1-t^1D)r + k_3(1-t^1D)y_o \quad (3)$$

where D is the operator $\frac{d}{dt}$.

$$\dot{y}_o = \frac{dy_o}{dt} = v \cos\psi + u \sin\psi.$$

For small ψ, within the linear theory $\cos\psi \approx 1$, $\sin\psi \approx \psi$, and $u \approx u_o$.
Then

$$\dot{y}_o \approx v + u_o \psi$$

$$v = \dot{y}_o - u_o \psi = Dy_o - u_o \psi \tag{4}$$

$$\dot{v} = \ddot{y}_o - u_o \dot{\psi} = D^2 y_o - u_o D\psi \tag{5}$$

$$r = \dot{\psi} = D\psi, \quad \dot{r} = D^2\psi.$$

Substituting equations (3), (4), and (5) into equations (1) and (2) results in:

$$(Y_{\dot{v}}-m)(D^2y_o-u_oD\psi)+Y_v(Dy_o-u_o\psi)+\left[Y_{\dot{r}}(D^2)+(Y_r-mu_o)D+Y_\psi\right]\psi+Y_{y_o}y_o$$

$$+Y_\delta\left[k_1(1-t^1D)\psi+k_2(1-t^1D)D\psi+k_3(1-t^1D)y_o\right]-T(1+\frac{x_p}{l})\psi-\frac{Ty_o}{l}=0 \tag{6}$$

$$N_{\dot{v}}(D^2y_o-u_oD\psi)+N_v(Dy_o-u_o\psi)+\left[(N_{\dot{r}}-I_z)D^2+N_rD+N_\psi\right]\psi+N_{y_o}y_o$$

$$+N_\delta\left[k_1(1-t^1D)\psi+k_2(1-t^1D)D\psi+k_3(1-t^1D)y_o\right]-Tx_p(1+\frac{x_p}{l})\psi-\frac{Tx_py_o}{l}=0 \tag{7}$$

When terms are regrouped, there results

$$\left[D^2(Y_{\dot{v}}-m)+D(Y_v-k_3Y_\delta t^1)+(Y_{y_o}-\frac{T}{l}+k_3Y_\delta)\right]y_o+\left[D^2(Y_{\dot{r}}-k_2Y_\delta t^1)\right.$$

$$\left.+D\left\{Y_r-u_oY_{\dot{v}}-k_1Y_\delta t^1+k_2Y_\delta\right\}+\left\{-Y_vu_o+Y_\psi+k_1Y_\delta-T(1+\frac{x_p}{l})\right\}\right]\psi=0 \tag{6a}$$

$$\left[D^2(N_{\dot{v}})+D(N_v-N_\delta k_3t^1)+(N_{y_o}+k_3N_\delta-\frac{Tx_p}{l})\right]y_o$$

$$+\left[D^2(N_{\dot{r}}-I_z-N_\delta k_2t^1)+D(-N_{\dot{v}}u_o+N_r-k_1N_\delta t^1+k_2N_\delta)\right.$$

$$\left.+\left\{-N_vu_o+N_\psi+N_\delta k_1-Tx_p(1+\frac{x_p}{l})\right\}\right]\psi=0. \tag{7a}$$

The determinant of the coefficients of equations (6a) and (7a) is:

$$aD^4 + bD^3 + cD^2 + dD + e = 0$$

where:

$$a = (Y_{\dot{v}}-m)(N_{\dot{r}}-I_z-N_\delta k_2 t^1)-(N_{\dot{v}})(Y_{\dot{r}}-k_2 Y_\delta t^1)$$

$$b = (Y_{\dot{v}}-m)(-N_{\dot{v}}u_o+N_r-k_1 N_\delta t^1+k_2 N_\delta)+(Y_v-k_3 Y_\delta t^1)(N_{\dot{r}}-I_z-N_\delta k_2 t^1)$$
$$-(Y_r - u_o Y_{\dot{v}} - k_1 Y_\delta t^1+k_2 Y_\delta)(N_{\dot{v}})-(Y_{\dot{r}}-k_2 Y_\delta t^1)(N_v-N_\delta k_3 t^1)$$

$$c = (Y_v-k_3 Y_\delta t^1)(-N_{\dot{v}}u_o+N_r-k_1 N_\delta t^1+k_2 N_\delta)+(N_{\dot{r}}-I_z-N_\delta k_2 t^1)(Y_{y_o}-T/\ell+k_3 Y_\delta)$$
$$+(Y_{\dot{v}}-m)(-N_v u_o+N_\psi+N_\delta k_1-Tx_p\left\{1+\frac{x_p}{\ell}\right\})-(Y_r - u_o Y_{\dot{v}}$$
$$-k_1 Y_\delta t^1+k_2 Y_\delta)(N_v-N_\delta k_3 t^1)-(Y_{\dot{r}}-k_2 Y_\delta t^1)(N_{y_o}+k_3 N_\delta-\frac{Tx_p}{\ell})$$
$$-(N_{\dot{v}})(-Y_v u_o+Y_\psi+k_1 Y_\delta-T(1+\frac{x_p}{\ell}))$$

$$d = (Y_v-k_3 Y_\delta t^1)(-N_v u_o+N_\psi+N_\delta k_1-Tx_p(1+\frac{x_p}{\ell}))+(Y_{y_o}-\frac{T}{\ell}+k_3 Y_\delta)$$
$$(-N_{\dot{v}}u_o+N_r-k_1 N_\delta t^1+k_2 N_\delta)-(Y_r - u_o Y_{\dot{v}} - k_1 Y_\delta t^1+k_2 Y_\delta)$$
$$(N_{y_o}+k_3 N_\delta-\frac{Tx_p}{\ell})-(-Y_v u_o+Y_\psi+k_1 Y_\delta-T(1+\frac{x_p}{\ell}))(N_v-N_\delta k_3 t^1)$$

$$e = (Y_{y_o}-\frac{T}{\ell}+k_3 Y_\delta)(-N_v u_o+N_\psi+N_\delta k_1-Tx_p(1+\frac{x_p}{\ell}))\cdot$$
$$-(-Y_v u_o+Y_\psi+k_1 Y_\delta-T(1+\frac{x_p}{\ell}))(N_{y_o}+k_3 N_\delta-T\frac{x_p}{\ell}).$$

The fourth order equation in the differential operator D, when the determinant of the coefficients in the Y and N equations are set equal to zero, will give four roots.

$$a(D-\sigma_1)(D-\sigma_2)(D-\sigma_3)(D-\sigma_4) = 0.$$

The condition for stability, as developed previously, is that the all four roots (σ_1, σ_2, σ_3, and σ_4) must be negative if real or the real part must be negative if the root is complex. The condition of equilibrium in this general case is straight ahead motion at constant speed, in the direction of the canal centerline and on the canal centerline. A stable condition would result in the ship returning to the centerline of the canal, and heading in the canal direction, after a disturbance from this condition. For a fourth degree equation in D, the following necessary and sufficient conditions for stability (negative roots) have been established (by Routh).

$$\frac{b}{a} > 0, \ \frac{d}{a} > 0, \ \frac{e}{a} > 0 \text{ and } \frac{bcd-ad^2-b^2e}{a^3} > 0$$

($\frac{c}{a} > 0$ is implied in the last condition).

The equations of motion can be solved, to give the trajectory resulting from any given disturbance. This more general case can be readily reduced to give the solution for any of the more specialized cases.

1. To obtain the case of a ship freely moving in a canal under its own power (i.e. not being towed), the term T is set equal to zero.

2. To obtain the case of a ship (or barge) being towed in the open sea (unrestricted waters), the derivatives, Y_{y_o}, N_{y_o}, Y_{ψ}, and N_{ψ} are also set equal to zero.

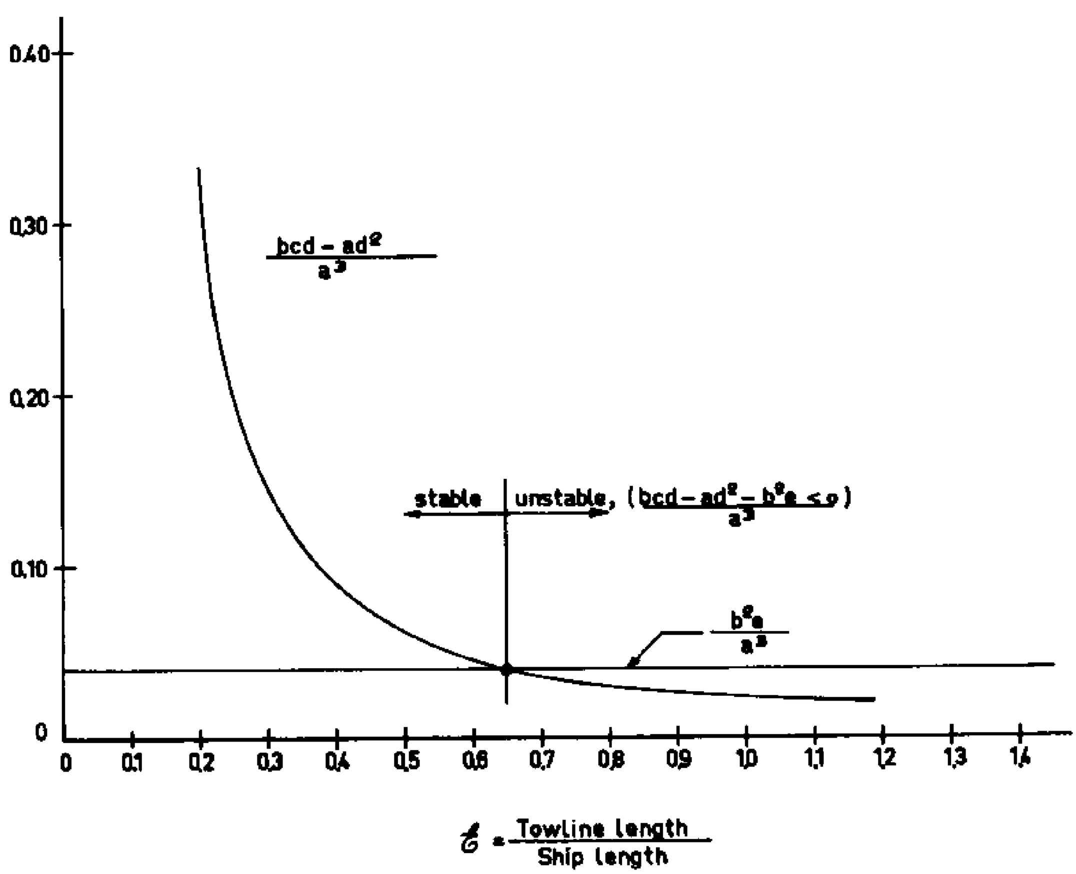
3. To obtain the case of a ship moving on its own in unrestricted water, the terms Y_{y_o}, N_{y_o}, Y_{ψ}, N_{ψ}, and T are set equal to zero.

4. In each of the cases above, to obtain the case of a ship without automatic controls, the values of k_1, k_2, and k_3 are set equal to zero, or if the controls are not sensitive to certain of the parameters indicated, then the appropriate k is set equal to zero.

In cases 1 and 2 above, there will be four roots as in the more general case developed. In case 3, if the controls are sensitive to heading ψ, and not y_o, there will be three roots. The fourth root will be D = 0, since the ship is not sensitive to y_o (location in space). If there are no controls in case 3, then one obtains the two roots and the criterion developed previously for this special case of stability in straight line motion. In this latter case, a factor $D^2 = 0$ can be taken from the determinant equation in D. (This indicates insensitivity to both ψ and y_o).

One should remember that the hydrodynamic derivatives used must be evaluated under the conditions of the ship, i.e. shallow water, restricted water, etc.

An analysis of the equations and solution, for a body being towed, is that the stability, as it depends on the tow line length ℓ and the towpoint P, is improved as ℓ decreases and as x_p increases. The graph below is indicative of the trend in stability brought about by the parameters of towline length for the practical case of a ship being towed with the tow point at the extreme bow.



APPENDIX II

NOTES ON AXIS TRANSFORMATIONS

1) A transformation of an axes system takes a quantity described in one frame of reference and transforms it into another frame of reference such that if we measured the same quantity in the second frame of reference the transformed quantity and the measured quantity would be identical.

2) Transforms between frames are needed in the study of the motions of ocean vehicles because the equations of motion for such a vehicle are most easily derived in the inertial frame attached to the earth (x_o , y_o , z_o) frame, while the forces acting on the vehicle are most easily evaluated in the frame attached to the vehicle (x,y,z). Hence, we ultimately desire to transform the equations of motion from the inertial frame into the non-inertial frame fixed in the vehicle.

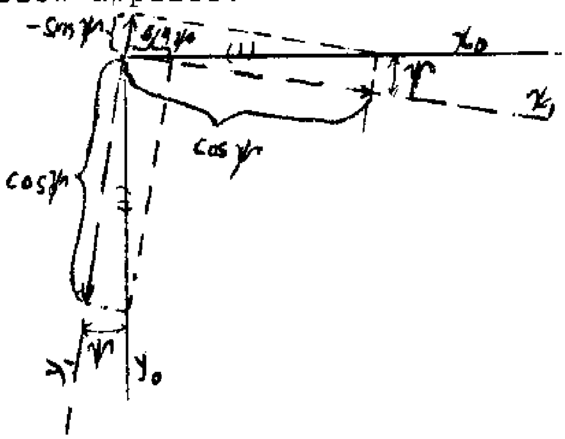
3) The angular or orientational relationship between the two frames is determined by the three Euler angles: yaw (ψ), rotation about z axis: trim (θ), rotation about y axis: and roll (ϕ), rotation about z axis.

4) If $\vec{V}_o$ is some vector measure in the x_o , y_o , z_o frame and $\vec{V}$ is same vector measured in the x,y,z, frame which is only changed in orientation then:

$\vec{V} = T(\psi, \theta, \phi)\, \vec{V}_o$ where $T(\psi, \theta, \phi)$ = the transform

5) Suppose we have rotation about z axis from position where x,y,z axis are coincident with $x_0 y_0 z_0$ axis. to another position.

The diagram below applies:



From the above we see that

$$x_1 = x_0 \cos\psi + y_0 \sin\psi + 0\, z_0$$

$$y_1 = -x_0 \sin\psi + y_0 \cos\psi + 0\, z_0$$

$$z_1 = 0\, x_0 + 0\, y_0 + z_0$$

or in matrix form

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = T(\psi)\ \vec{x}_0$$

in the same manner we can show

$$T(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

and

$$T(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}$$

6) Any change of orientation can be broken up into 3 individual rotations. To do this in the standard manner we first yaw, then trim, and finally roll. Thus:

$$\vec{x}_1 = T(\psi)\,\vec{x}_0 \qquad \text{yaw}$$

$$\vec{x}_2 = T(\theta)\,\vec{x}_1 \qquad \text{trim}$$

$$\vec{x} = T(\phi)\,\vec{x}_2 \qquad \text{roll}$$

or $\vec{x} = T(\phi)\ T(\theta)\ T(\psi)\ \vec{x}_0 = T(\phi,\theta,\psi)\ \vec{x}_0$

thus we have our desired result

$$T(\phi,\theta,\psi) = T(\phi)\ T(\theta)\ T(\psi)$$

7) Now evaluate $T(\phi,\theta,\psi)$

$$T(\phi,\theta,\psi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Preforming the first multiplication

$$T(\phi,\theta,\psi)=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\phi & \sin\phi\\ 0 & -\sin\phi & \cos\phi\end{bmatrix}\begin{bmatrix}\cos\psi\cos\theta & \sin\psi\cos\theta & -\sin\theta\\ -\sin\psi & \cos\psi & 0\\ \cos\psi\sin\theta & \sin\psi\cos\theta & \cos\theta\end{bmatrix}$$

Finally

$$T(\phi,\theta,\psi)=\begin{bmatrix}\cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta\\ -\sin\psi\cos\phi+\sin\phi\sin\theta\cos\psi & \cos\phi\cos\psi+\sin\phi\sin\theta\sin\psi & \sin\phi\cos\theta\\ \sin\phi\sin\psi+\cos\phi\cos\psi\sin\theta & -\sin\phi\cos\psi+\cos\phi\sin\theta\sin\psi & \cos\phi\cos\theta\end{bmatrix}$$

8) If this had been done in a different order say ,

$T(\theta,\phi,\psi)$,then

$$T(\theta,\phi,\psi)=\begin{bmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\phi & \sin\phi\\ 0 & -\sin\phi & \cos\phi\end{bmatrix}\begin{bmatrix}\cos\psi & \sin\psi & 0\\ -\sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{bmatrix}$$

Preforming the multiplication we find

$$T(\theta,\phi,\psi) = \begin{bmatrix} \cos\theta\cos\psi + \sin\theta\sin\phi\sin\psi & \cos\theta\sin\psi + \sin\theta\sin\phi\cos\phi & -\sin\theta\cos\phi \\ -\cos\phi\sin\psi & \cos\psi\cos\phi & \sin\phi \\ \cos\psi\sin\theta - \cos\theta\sin\phi\sin\psi & \sin\theta\sin\psi - \cos\theta\sin\phi\cos\psi & \cos\theta\cos\phi \end{bmatrix}$$

9) It is obvious by inspection that $T(\theta,\phi,\psi) \neq T(\phi,\theta,\psi)$. Therefore, in general, the order in which the rotations are made is very important, hence the process is not commutative.

10) But to first order

$$T(\theta,\phi,\psi) = \begin{bmatrix} 1 & \psi & -\theta \\ -\psi & 1 & \phi \\ \theta & -\phi & 1 \end{bmatrix}$$

$$T(\phi,\theta,\psi) = \begin{bmatrix} 1 & \psi & -\theta \\ -\psi & 1 & \phi \\ \theta & -\phi & 1 \end{bmatrix}$$

or to first order;

$T(\theta,\phi,\psi) = T(\phi,\theta,\psi)$ or the result is independent of the order that the rotations are made in.

11) To derive the transform for angular velocity we first start with the x,y,z axis coincident with the x_0, y_0, z_0 axis. An angular velocity $\dot{\psi}$ is then imparted to the vehicle. To transform this into the frame attached to the vehicle at the end of the

process of rotation we must apply the three transformations $T(\phi)\ T(\theta)\ T(\psi)$ or the angular velocities due to yaw are given by:

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix}_{yaw} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}$$

$$\begin{bmatrix} p \\ g \\ r \end{bmatrix}_{yaw} = T(\phi)\quad T(\theta)\quad T(\psi) \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}$$

Then an angular velocity $\dot{\theta}$ is imparted in trim. However, since we have already yawed the vehicle and we are involved in the angular velocity about the axis of yaw, we need only apply $T(\theta)\quad T(\phi)$ to transform the result or

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix}_{trim} = T(\phi)\quad T(\theta) \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix}_{trim} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix}_{trim} = \begin{bmatrix} 0 \\ \dot{\theta} \cos \phi \\ -\dot{\theta} \sin \phi \end{bmatrix}$$

Next we impart a rolling angular velocity $\dot{\phi}$ Having already yawed and trimed, we need only apply T (ϕ) or:

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix}_{roll} = T(\phi) \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix}$$

The total angular velocity equals the sum of these three vector components

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix}_{total} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}_{yaw} + \begin{bmatrix} p \\ q \\ r \end{bmatrix}_{trim} + \begin{bmatrix} p \\ q \\ r \end{bmatrix}_{roll}$$

or finally:

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} -\sin\theta & 0 & 1 \\ \sin\phi \cos\theta & \cos\phi & 0 \\ \cos\theta \cos\phi & -\sin\phi & 0 \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix}$$

$$p = -\sin\theta\dot{\psi} + \dot{\phi}$$

$$q = \sin\phi \cos\theta\dot{\psi} + \cos\phi\dot{\theta}$$

$$r = \cos\theta \cos\phi\dot{\psi} - \sin\phi\dot{\theta}$$

The above is exact

12) But to first order

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} -\theta & 0 & 1 \\ \phi & 1 & 0 \\ 1 & -\phi & 0 \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix}$$

$$\begin{aligned} p &= -\theta\dot{\psi} + \dot{\phi} = \dot{\phi} \\ q &= \phi\dot{\psi} + \dot{\theta} = \dot{\theta} \qquad \text{to first order} \\ r &= \dot{\psi} - \phi\dot{\theta} = \dot{\psi} \end{aligned}$$

13) Assorted Things of Interest

1) Not only does the result of a rotation vary when it is made but also the actual axis of rotation is different for different sequences of rotation. This is because all rotations are made about axis in x, y, z frame and not x_o, y_o, z_o frame.

14. It is understood from the sequence of rotation that θ is the angle that the x axis (longitudinal axis) of the body makes with the horizontal plane $x_o\ y_o$, and that ψ is the angle that a plane through the x axis perpendicular to the horizontal plane x_o, y_o makes with the original x_o axis.

APPENDIX III

Bibliography from paper "Recent Developments in Seakeeping Research and Its Application to Design" by M. A. Abkowitz, L. Vassilopoulos, F. Sellars IV, Transactions of the Society of Naval Architects and Marine Engineers, New York, N.Y. 1966

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APPENDIX IV

13.98 - STABILITY AND MOTION CONTROL OF OCEAN VEHICLES

Fall Term 1968 - 69
Instructor in Charge - Professor Martin A. Abkowitz
Office Room 5-319 Ext. 4334

- SCHEDULE OF LECTURES -

Lecture No.	Date	Topic
1	9/26/68	Introduction - Philosophy and Organization of Subject.
2	9/30	Nature and Origin of Forces on the Vehicle in the Environment.
3	10/1	Forces from Fluid Dynamic Phenomena.
4 & 5	10/3 10/7	Equations of Motion as Derived from Analysis of Physical Phenomena.
6	10/8	Degeneralization of Math. Model, by Restraints, into Self-Propelled, Towed, Moored, etc. vehicles and into Specific Degrees of Freedom.
7 & 8	10/10 10/14	Concepts and Determination of Stability.
9 & 10	10/15 10/17	Concepts of Control Forces, Effectors, and Automatic Control.
11	10/21	Nature and Operation of Control Effectors Including Sensors, Time Lags and Leads.
12	10/22	Methods of Determining Parameters or Coefficients in Body Force Equations.
13	10/24	Methods of Determining Parameters or Coefficients in Body Force Equations.
14 & 15	10/28 10/29	Methods of Determining Excitation and Control Forces of the Equations.
16	10/31	System Responses Under Regular Excitation.

Lecture No.	Date	Topic
17 & 18	11/4 11/5	Spectral Techniques for Irregular Excitation and Response.
19	11/7	Roll Excitation, Response and Control
-	11/11	HOLIDAY
20	11/12	Dynamics of Anti-Rolling Tanks and Fins.
21	11/14	Case Discussion of Stability and Automatic Steering of Ships. Restricted Water.
-	11/18	WRITTEN EXERCISE
22 & 23	11/19 11/21	Case Discussion of Torpedo and Submarine Motion.
24	11/25	Surface Ship Motion in Waves - Case Study of Commercial Displacement Ship.
25	11/26	Surface Ship Motion in Waves - Case Study of Commercial Displacement Ship
-	11/28	HOLIDAY (Thanksgiving)
26	12/7	Surface Ship Motion - Case Study of Military Displacement Types.
27	12/3	Case Discussion of Flip Ship Motion and Free Falling Body.
28	12/5	Case Discussion of Hydrofoil Boat Motion.
29	12/9	Case Discussion of Hovercraft Motion.
30	12/10	Case Discussion of Drilling Platform Motion.
31	12/12	Case Discussion of Catamaran Motion and Trawler Motion.
-	12/16	Written Exercise
32	12/17	Case Discussion - The Deep Submergence Rescue Vehicle - General Aspects.
33	12/19	DSRV - Control System Sophistication, Development and Design.

Lecture No.	Date	Topic
-	-	CHRISTMAS VACATION
34	1/6/69	Stability of Towed Bodies.
35	1/7	Equilibrium of Towed Submerged Cable Configurations.
36	1/9	Coupled Towed Systems.
37	1/13	Case Discussion of Tethered Sonar System and Instrument Towed System.
38	1/14	Response of Towed System to Towing Ship and Environmental Disturbance.
39	1/16	Case Discussion of Free Floating and Moored Surface Buoys.
-	1/20	Written Exercise.
40	1/21	Case Discussion of Bottom Moored Buoys.

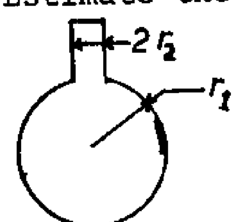
APPENDIX A - SAMPLE WRITTEN EXERCISES

The following are typical of quizes given in this course.

Written Exercise #1

Question I

A glass flask consists of a six inch diameter sphere attached to a one inch diameter cylinder. The flask is weighted so that the sphere and ten inches of the cylinder are below the water when neutrally bouyant and stable. Estimate the natural period in heave.



Solution

The equation of motion in the vertical plane with heave only is:

$$(m-z_{w})w + z_{w} w + z_{z} z = 0$$

Remembering that for this case $z_w = 0$, we solve for ω_n and get:

$$\omega_n = \sqrt{\frac{z_z}{(m-z_{\dot{w}})}}$$

Now z_z equals the area of the waterline times the weight density of water or

$$z_z = \pi r_2^2 \rho g$$

Also

$$m = \rho \text{ (vol)}$$

$$m = \rho \left(\frac{4\pi}{3} r_1^3 + \pi r_2^2 (10)\right)$$

$z_{\dot{w}}$ for the cylinder can be neglected, therefore:

$$z_w = \frac{-1}{2} \text{ (mass sphere) } = \frac{-2}{3} \rho \pi r_1^3$$

$$\omega_n = \sqrt{\frac{\pi r_2^2 \rho g}{\rho (2 \pi r_1^3 + \pi r_2^2 (10))}} = \sqrt{\frac{96}{56.5}} = 1.3 \text{ radian/sec}$$

Finally

$$T_n = \frac{2\pi}{\omega_n}$$

$$T_n = \frac{6.28}{1.3} = 4.84 \text{ sec}$$

Question 2

The linearized equation for the single degree of freedom in roll with an anti-roll control δ is given as:

$$(k_{\dot{p}} - I_x)\dot{p} + k_p p + k_\phi \phi + k_\delta \delta = 0$$

a) Without controls, show that stability in roll is obtained only if both k_p and k_ϕ are negative.

b) Suppose the vehicle is stable and we have proportional control, with linear time lag at such that:

$$\delta(t) = k_1 \phi(t_1)$$

where $t_1 = t - \Delta t$

In terms of the hydrodynamic coefficients, at what value of k_1 and $k_1 \Delta t$ will the vehicle become unstable?

Solution

a) Without controls, the differential equation reduces to

$$\left[(k_{\dot{p}} - I_x)\, D^2 + k_p\, D + k\,\phi\right]\phi = 0$$

Dividing by $k_{\dot{p}} - I_x$ yields

$$\left[D^2 + \frac{k_p}{k_{\dot{p}} - I_x} D + \frac{k\,\phi}{k_{\dot{p}} - I_x}\right]\phi = 0$$

Solving the above by the quadratic formula we get:

$$\left.\begin{matrix}\sigma_1\\ \sigma_2\end{matrix}\right\} = \frac{-k_p}{2(k_{\dot{p}} - I_x)} \pm \sqrt{\left[\left(\frac{k_p}{k_{\dot{p}} - I_x}\right)^2 - \frac{4k\,\phi}{k_{\dot{p}} - I_x}\right]} \div 2$$

For stability both σ_1, and σ_2 must be negative.

For this to happen, the following must occur

$$\frac{-k_p}{k_{\dot{p}} - I_x} < 0$$

and

$$\left|\frac{-k_p}{k_{\dot{p}} - I_x}\right| > \left|\pm\sqrt{\left(\frac{k_p}{k_{\dot{p}} - I_x}\right)^2 - \frac{4k\phi}{k_{\dot{p}} - I_x}}\right|$$

It should be remembered that $k_{\dot{p}}$ is strongly negative and therefore:

$$k_{\dot{p}} - I_x < 0$$

We see then, that for the first condition to be fulfilled

$$k_p < 0$$

We also observe that the second condition can only be fulfilled if

$$\frac{-4k_\phi}{k_{\dot{p}} - I_x} < 0$$

This, in turn implies that

$$k_\phi < 0$$

b) From the problem statement

$$\delta (t) = k_1 \phi (t_1)$$

and

$$t_1 = t - \Delta t$$

thus

$$\delta(t) = k_1 \phi (t- \Delta t)$$
$$\delta(t) = k_1 \phi(t) - \Delta t \dot{\phi} (t) k_1$$

or $$\delta(t) = k_1 \phi(t) - \Delta t k [p(t)]$$

Substituting the expression for $\delta(t)$ in the original equation yields

$$(k_{\dot{p}} - I_x) \dot{p} + \underbrace{(k_p - \Delta t k_1 k_\delta)}_{A} p + \underbrace{(k_\phi + k_1 k_\delta)}_{B} \phi = 0$$

The system will become unstable when either A or B become positive. Therefore the system is just on the verge of instability when A=0 or B=0. Thus for just marginal stability

$$k_p - \Delta t k_1 k_\delta = 0$$

or $$\Delta t k_1 = \frac{k_p}{k_\delta}$$

and

$$k\phi + k_1 k_\delta = 0$$

or $$k_1 = -k_\phi / k_\delta$$

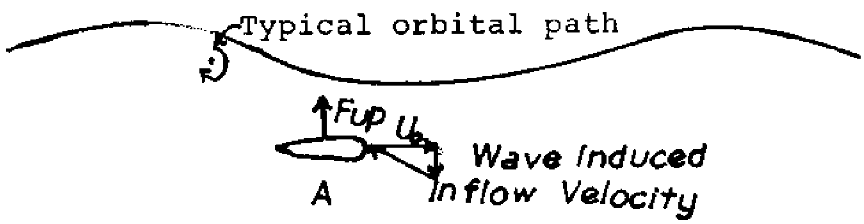
Question 3

A hydrofoil just below the surface moves with a velocity V_o in the same direction as the propagation of a surface wave moving with a speed v_ω .

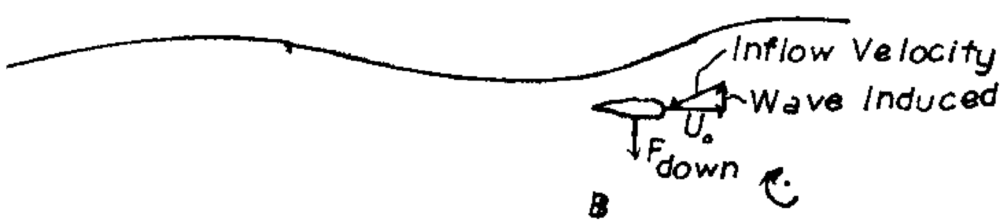
Show that if $v_o > v_\omega$, the excitation force tends to move the foil upward at the same time that the water surface above the foil is dropping (or the force tends to move the body downward at the same time the water surface is rising).

Solution

For wave height going down



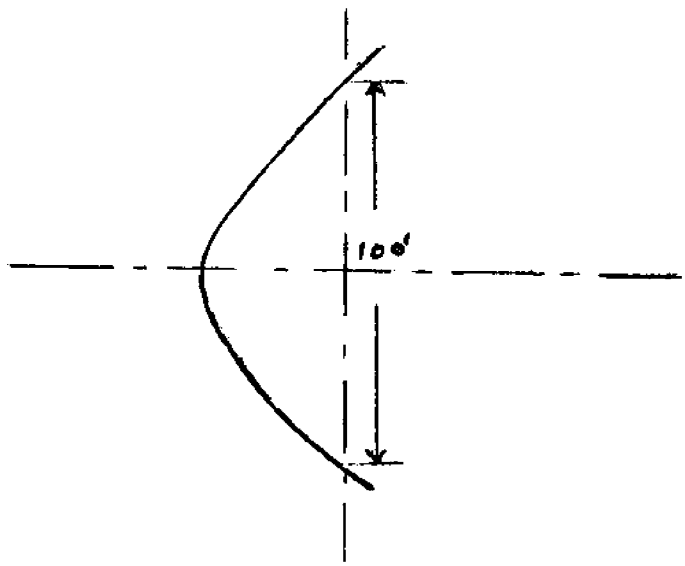
For wave height going up



In understanding the above, it should be noted that when the foil is on the back of a wave (position A) the particle velocity of the water is down. When the foil is in position B, then the velocity of the water particles is up.

Written Exercise #2

1) A one inch diameter plastic towline has a weight equal to its buoyancy, so that its weight under water is zero. The drag of the cable oriented transverse to the flow is 5 pounds per foot of cable length. The cable is towed in the water with its ends 100 feet apart, as shown in the Figure below.



SOLUTION

$\phi_c = 0$ and $f = 0.01$

if $\phi = 10^{\circ}$ then

$T = 1.0584$, $\sigma = 5.8352$, $\varepsilon = 4.9129$ and $\eta = 2.4846$

if $\phi = 90^{\circ}$ then

$T = 1.0000$, $\sigma = 0$, $\varepsilon = 0$ and $\eta = 0$

we know that

$$T = \frac{\tilde{T}}{T_o}$$

therefore

$$\frac{R_x}{T_o} = 4.9129 \qquad \frac{R_y}{T_o} = 2.4846 \quad \text{and} \quad \frac{R_s}{T_o} = 5.8352$$

and $$\frac{R_s}{T_o} = \frac{5.8352}{2.4856} = 2.35$$

from $\frac{s}{y} = 2.35$ we see that

or $1/2$ cable length $= 2.35\ (50)$

The total length of Cable therefore equals

$$\ell = 235 \text{ feet}$$

2) On the Deep Submergence Rescue Vehicle

a. How does incorporating the mathematical model for the vehicle motion into the control system improve the overall vehicle control?

b. What does the term "critical velocity" refer to?

c. The vehicle is directed into a current of 5 feet per second. The vehicle is then pitched through an angle of $\theta = 38^{\circ}$ and then rolled through an angle of $\phi = 30^{\circ}$. What are the u,v, and w components of the velocity?

SOLUTION

a) The feed forward system allows simultaneous controls to effect unidegree of motion control. Also the feedback loop is able to handle an error in the prediciton of the vehicles course.

b) The "critical velocity" is the velocity below which an equlibrium in horizontal flight cannot be achieved with the crafts diving planes.

c) From the notes, we see that the transformations for pitch and roll are

$$T_p = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

and

$$T_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

First pitching the model yields

$$\begin{bmatrix} U_1 \\ V_1 \\ W_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ 0 & 1 \\ -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \sin\theta & U \\ 0 & 0 \\ \cos\theta & 0 \end{bmatrix} = \begin{bmatrix} U\cos\theta \\ 0 \\ -U\sin\theta \end{bmatrix}$$

Next rolling the model yields

$$\begin{bmatrix} U_f \\ V_f \\ W_f \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Phi & -\sin\Phi \\ 0 & \sin\Phi & \cos\Phi \end{bmatrix} \begin{bmatrix} U_1 \\ V_1 \\ W_1 \end{bmatrix} = \begin{bmatrix} V\cos\theta \\ +U\sin\Phi \quad \sin\theta \\ -U\sin\theta \quad \cos\Phi \end{bmatrix}$$

Therefore at completion of the roll and pitch maneuvers are completed

$U = U\cos\theta$

$V = U\sin\phi \quad \sin\theta$

$W = -U\cos\phi \quad \sin\theta$

since $\sin 30 = .5$ and $\cos 30 = .867$,

$U = 5\ (.865) = 4.33$ knots

$V = 5\ (.5)(.5) = 1.25$ knots

$W = -5\ (.5)\ (.865) = 2.17$ knots

APPENDIX B - SAMPLE HOME EXERCISES

Question 1

Find the steady state turning response [r (t)] of a vehicle to a rudder deflection given by $\delta = \delta_0 \cos \omega t$.

Solution

When the equations of motion of a vehicle are written down only in Y and N, they become:

$$[(Y_{\dot{v}} - m) D + Y_v] v + [(Y_{\dot{r}} - m x_G) D + (Y_r - m u_o)] r$$

$$+ Y_\delta \delta e^{i\omega t} = 0$$

$$[N_{\dot{v}} - m x_G) D + N_v] v + [(N_{\dot{r}} - I_z) D + (N_r - m x_G u_o)] r$$

$$+ N_\delta \delta e^{i\omega t} = 0$$

where $\delta e^{i\omega t}$ represents the rudder deflection given.

If
$$a_{11} = [(Y_{\dot{v}} - m) D + Y_v]$$
$$a_{12} = [(Y_{\dot{r}} - m x_G) D + (Y_r - m u_o)]$$
$$a_{21} = [(N_{\dot{v}} - m x_G) D + N_v] v$$
$$a_{22} = [(N_{\dot{r}} - I_z) D + (N_r - m x_G u_o)]$$

then
$$a_{11}\, v + a_{12}\, r + Y_\delta\, \delta\, e^{i\omega t} = 0$$
$$a_{12}\, v + a_{22}\, r + N_\delta\, \delta\, e^{i\omega t} = 0$$

The solution of this, in determinant form, is given by:

$$r\,(t) = \frac{\begin{vmatrix} a_{11} & -Y_\delta \\ a_{21} & -N_\delta \end{vmatrix} \delta_o\, e^{i\omega t}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

or:

$$r(t) = \frac{\delta\,[\,-N\delta a_{11} + Y_\delta\, \delta\, a_{21}\,]\, e^{i\omega\, t}}{(Y_{\dot{v}} - m)\,(N_{\dot{r}} - I_z)\,(D - \sigma_1)(D - \sigma_2)}$$

where σ_1 and σ_2 are the roots of the homogeneous equation. The above then becomes

$$r(t) = \left\{\frac{\delta_o}{(Y_{\dot{v}} - m)(N_{\dot{r}} - I_z)}\right\}$$
$$\left\{\frac{\{-N_\delta\, (i\omega)\, (Y_{\dot{v}} - m) - N_\delta Y_v + Y_\delta\, (i\omega)(N_{\dot{v}} - m\, x_o) + Y_\delta\, N_v\}\; e^{i\omega\, t}}{(D - \sigma_1)(D - \sigma_2)}\right\}$$

if
$$P_o = \frac{\delta_o}{(Y_{\dot{v}} - m)(N_{\dot{r}} - I_z)}$$

$$A = Y_\delta\, N_v - N_\delta\, Y_v$$

$$B = \omega\,[Y_\delta\, (N_{\dot{v}} - mX_G) - N_\delta (Y_{\dot{v}} - m)]$$

Then the above expression for r(t) becomes

$$r(t) = \frac{P_o (A + i B) e^{i\omega t}}{(D - \sigma_1)(D - \sigma_2)}$$

Using the operation $\frac{1}{D-\sigma}$, we obtain

$$r(t) = P_o (A + iB) \frac{1}{D - \sigma_1} e^{\sigma_2 t} \int e^{(i\omega - \sigma_2) t} dt$$

$$r(t) = \frac{P_o (A + iB)}{} \frac{1}{D - \sigma_1} \frac{e^{i \omega t} + C_1 e^{\sigma_2 t}}{(i\omega - \sigma_2)}$$

or finally

$$r(t) = \frac{P_o (A + iB) e^{i\omega t}}{(i\omega - \sigma_1)(i\omega - \sigma_2)} + c_1 e^{\sigma_1 t} + c_2 e^{\sigma_2 t}$$

If σ_1 and σ_2 are stable roots, then the transient $(c_1' e^{\sigma_1 t} + c_2' e^{\sigma_2 t})$ dies out in time and we are left with the oscillatory term:

$$r(t) = \frac{P_o (A + iB) e^{i\omega t}}{(i\omega - \sigma_1)(i\omega - \sigma_2)} \times \frac{(i\omega + \sigma_1)(i\omega + \sigma_2)}{(i\omega + \sigma_1)(i\omega + \sigma_2)}$$

$$r(t) = \frac{P_o e^{i\omega t}[A + iB][\sigma_1 \sigma_2 - \omega^2) + i\omega (\sigma_1 + \sigma_2)]}{(\sigma_1^2 + \omega^2)(\sigma_2^2 + \omega^2)}$$

or finally

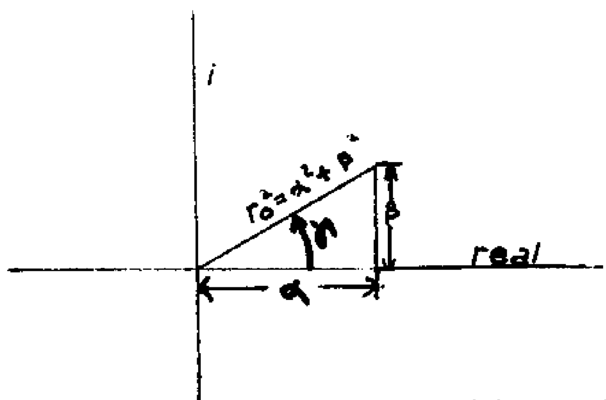
$$r(t) = (\alpha + iB)\, e^{i\omega t}$$

where

$$\alpha = \frac{P_o\,[A(\sigma_1 \sigma_2 - \omega^2) - B_{\omega}(\sigma_1 + \sigma_2)]}{(\sigma_1^2 + \omega^2)(\sigma_2^2 + \omega^2)}$$

$$\beta = \frac{P_o\,[B(\sigma_1 \sigma_2 - \omega^2) + A\omega(\sigma_1 + \sigma_2)]}{(\sigma_1^2 + \omega^2)(\sigma^2_2 + \omega^2)}$$

Now, we shall construct a rotating vector diagram of the previous expression. This diagram is shown below.



$$r(t) = \text{Real Part } [r_o\, e^{i(\omega T + \gamma)}]$$

$$r(t) = r_o \cos(\omega T + \gamma)$$

where

$$r_o = \sqrt{\alpha^2 + B^2}$$

$$\gamma = \tan^{-1} \frac{\beta}{\alpha}$$

2. Derive the steady state roll response of the DSRV to a rudder deflection of one-third of a radian. Most coefficients needed for this problem are given in Table I. Assume that the DSRV has the following characteristics:

$$x_G' = o, \; z_G' = 4.0\mathrm{x}10^{-3} \;\; \& \;\; L = 50 \text{ feet}$$

Also assume that the final solution should be for a velocity of 5 knots. The frequency of rudder oscillation should be taken as the natural frequency of the vehicle in roll only

Solution:

First calculate k ϕ. It should be recalled that

$$k_\phi = \frac{dk}{d\phi} = \frac{-d}{d\phi}(mgz_g \sin \phi)$$

or

$$k_\phi' = -m_g \; z_g \cos \phi$$

non-dimensionalalizing

$$k_\phi{}' = \frac{-1}{\frac{1}{2}\rho \ell^3 \; U_o^2} [m' \; (1/2\rho\ell^3) \; g \; z_G' \; (\ell)]$$

$$k_\phi' = m' \; z_G{}' \; \frac{g\ell}{U^2}$$

Substituting numerical values

$$k_\phi{}' = -.00398$$

The generalized equations of motion for this problem are

X_{EQU}

$$-X_{\delta}\delta = \overbrace{[(X'_{\dot{u}'} - m')D + X'_{u'}]}^{a_{11}} \Delta u' + \overbrace{[X'_{\dot{v}} D + X'_{v}]}^{a_{12}} v' + \overbrace{[X'_{\dot{r}} D + X'_{r}]}^{a_{13}} r'$$

$$+ \overbrace{[X'_{\dot{p}} D^2 + X'_{p} D + X'_{\phi}]}^{a_{14}} \phi$$

Y_{EQU}

$$\overset{a_{21}}{[Y'_{\dot{u}'}D + Y'_{u'}]}\Delta U' + \overset{a_{22}}{[(Y'_{\dot{v}'}-m')D + Y'_{v'}]}v' + \overset{a_{23}}{[(Y'_{\dot{r}'}-m'X_{G'})}$$

$$D + (Y'_{r'}-m'v'_{o})]r' + [Y'_{\dot{p}'} + m'z'_{G})D^2 + Y'_{p'} D + Y'_{\phi'}]$$

$$\phi = -Y'_{\delta}{}'\delta'$$

N_{EQU}

$$\overset{a_{31}}{[N'_{\dot{u}} D + N'_{u'}]}\Delta U' + \overset{a_{32}}{[(N_{\dot{v}'}-m'X_{G'})D + N'_{v}]}v' + [N'_{\dot{r}'}-I'_{z})$$

$$D + (N'_{v'}-m'X'_{G} U'_{o'})]v' + [N'_{\dot{p}'}D^2 + N'_{p} D + N'_{\phi}]$$

$$\phi = -N'_{\delta'}\delta'$$

K_{EQU}

$$[K'_{\dot{u}'} D + K'_{u'}]\Delta U' + [(K'_{\dot{v}'} + m'Z'_{G})D + K'_{v'}]$$

$$v' + [K'_{\dot{r}}D + (K_{r'}+m'Z'_{G}V'_{o'})]r' + [(K'_{p} - I'_{x})D^2 + K'_{p}D + K'\phi']$$

$$\phi = -k'_{\delta'}\delta'$$

When the proper numerical values are substituted in the above it becomes:

TABLE I

Nondimensional Stability and Control Coefficients and Prototype Constants

I_x'	0.000150	M_{vp}'	0.0
I_y'	0.002328	M_{vr}'	0.0
I_z'	0.002328	M_{vv}'	0.0
K_p'	-0.000065	M_w'	0.004181
$K_{p\|p'\|}$	-0.000011	$M_{\|w\|q}'$	-0.005850
$K_{\dot{p}}'$	-0.000075	$M_{w\|w'\|}$	0.017597
K_{qr}'	0.0	$M_{\dot{w}}'$	-0.000481
K_r'	0.0	$M_{\delta s}'$	-0.021772
$K_{\dot{r}}'$	0.0	$M_\theta' \; V_{k^2}$	-0.083004
K_v'	-0.000591	M_*'	-0.000480
$K_{v\|v'\|}$	-0.001363	m'	0.044010
$K_{v\|w'\|}$	0.006230	N_p'	0.0
$K_{\dot{v}}'$	-0.000576	N_{pq}'	-0.002468
K_{wp}'	0.000576	$N_{\dot{p}}$	0.0
$K_{\delta r}$	0.0	N_r'	-0.015463
K_*'	-0.000560	$N_{\|r\|\delta r}'$	0.0
M_q'	-0.015347	$N_{\dot{r}}'$	-0.002543
$M_{\|q\|\delta s}'$	0.0	N_v'	-0.012670
$M_{\dot{q}}'$	-0.002543	$N_{\|v\|r}'$	-0.005850
M_{rp}'	-0.002468	$N_{\|v\|v}'$	0.000407
M_{rr}'	0.0	$N_{\dot{v}}'$	0.000129

TABLE I (Con't)

N_{wv}'	0.0	$Y_{v\|r\|}'$	-0.016000
$N_{\delta r}'$	-0.021772	$Y_{v\|v\|}'$	-0.090738
N_{*}'	0.0	$Y_{\dot{v}}'$	-0.042418
X_{qq}'	0.0	Y_{wp}'	0.036187
X_{rp}'	0.000576	Y_{wv}'	0.0
X_{rr}'	0.0	$Y_{\delta r}'$	0.048128
$X_{\dot{u}}'$	-0.002290	Y_{*}'	0.0
X_{vr}'	0.042420	Z_{q}'	-0.024771
X_{vv}'	0.0	$Z_{\|q\|\delta s}'$	0.0
X_{wq}'	-0.036190	$Z_{\dot{q}}'$	0.0
X_{ww}'	0.0	Z_{rp}'	0.0
$X_{\delta r\delta r}'$	-0.021475	Z_{rr}'	0.0
$X_{\delta s\delta s}'$	-0.021475	Z_{vp}'	-0.042418
Y_{p}'	0.0	Z_{vr}'	0.0
$Y_{p\|p\|}'$	0.0	Z_{vv}'	0.0
Y_{pq}'	0.0	Z_{w}'	-0.050949
$Y_{\dot{p}}'$	-0.000576	$Z_{w\|q\|}'$	-0.016000
Y_{r}'	0.024857	$Z_{w\|w\|}'$	-0.048213
$Y_{\|r\|\delta r}'$	0.0	$Z_{\dot{w}}$	-0.036187
$Y_{\dot{r}}'$	0.0	$Z_{\delta s}'$	-0.048128
Y_{v}'	-0.050852	Z_{*}'	-0.000600

NOTE: The contribution of the main propeller to the longitudinal force as a function of the propulsion coefficient n'(ratio of ordered speed to instantaneous speed of the submarine)is expressed as follows for n' greater than zero:

$$X' = -0.00450 - 0.00325n' + 0.00775n'^2$$

and for n' less than zero:

$$X' = -0.00450 - 0.00130n' - 0.0042n'^2$$

X_{equ}

$$[-.0463\ D + X'_u]\ \Delta U' + 0 + 0 + 0 = 0$$

Y_{equ}

$$0 + [-.0864\ D - .0509]\ V' - .0192\ r' - .0004\ D^2\phi' = -.0481\ \delta'$$

N_{equ}

$$0 + [.0001\ D - .0127]\ v' + [.00487\ D - .0155]\ r' = .0218\ \delta'$$

K_{equ}

$$0 + [-.0004\ D - .0006]\ v' + .00018\ r' + [-.0002\ D^2 - .00007\ D - .0039]\ \phi' = 0$$

It can be seen from the K equation that the frequency of rudder excitation used should be:

$$\omega'_n = \sqrt{\frac{k_{\dot{\phi}}}{k'_{\dot{p}} - I'_x}}$$

or

$$\omega'_n = \sqrt{\frac{-.00398}{-.000225}}$$

$$\omega'_n = 4.2$$

Solving these equations by determinant yields

$$\phi = \frac{\begin{vmatrix} a_{22} & a_{23} & -Y_{\delta}\delta \\ a_{32} & a_{33} & -N_{\delta}\delta \\ a_{42} & a_{43} & -K_{\delta}\delta \end{vmatrix}}{\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}}$$

Expanding this determinant and deleting terms equal to zero yields:

$$\phi = \frac{\{-[a_{23}\ a_{24}\ N_{\delta}\delta\ +\ a_{32}\ a_{43}\ Y_{\delta}\delta\ ]+a_{42}\ a_{33}\ Y_{\delta}\delta\ +\ a_{43}\ a_{22}\ N_{\delta}\delta\}}{\{a_{22}\ a_{33}\ a_{44}\ +\ a_{24}\ a_{32}\ a_{43}\ -[a_{42}\ a_{33}\ a_{24}\ +\ a_{44}\ a_{32}\ a_{23}\ ]\}}$$

The individual terms in this expression are evaluated below:

$$-a_{23}\ a_{42}\ N_{\delta}\ \delta' = [-167D - 247] \times 10^{-9}\delta'$$

$$-a_{32}\ a_{43}\ Y_{\delta}\ \delta' = [D - 906] \times 10^{-9}\delta'$$

$$a_{42}\ a_{33}\ Y_{\delta}\delta = [-94D^2 - 437D - 440] \times 10^{-9}\delta'$$

$$a_{43}\ a_{22}\ N_{\delta}\ \delta' = [-331D - 195] \times 10^{-9}\delta'$$

The sum of the above is:

$$[-94D^2 - 926D - 1788] \times 10^{-9}\delta' \equiv [b_1\ D^2 + b_2\ D + b_3]\delta'$$

and

$$a_{22}\ a_{33}\ a_{44} = [-95\ D^4 - 383D^3 - 1955D^2 - 6354D - 3129] \times 10^{-9}$$

$$a_{32}\ a_{43}\ a_{24} = D^2 \times 10^{-9}$$

$$-a_{42}\ a_{33}\ a_{24} = [D^4 + 4\ D^3 + 4\ D^2] \times 10^{-9}$$

$$-a_{32}\ a_{44}\ a_{23} = [-D^3 + 54D^2 + 6D + 965] \times 10^{-9}$$

The sum of the above is:

$$[-94D^4 - 380D^3 - 1896D^2 - 6348D - 2164] \times 10^{-9} = AD^4 + BD^3 + CD^2 + DD + E$$

Substituting this in our expression for ϕ yields

$$\phi = \frac{\delta_o[-b_1\ \omega^2 + ib_2\ \omega + b_3\ ]\ e^{i\ \omega t}}{A\ (D-\sigma_1)(D-\sigma_2)(D-\sigma_3)(D-\sigma_4)}$$

$$\phi = \frac{[P + iQ]\delta_o\ e^{i\omega\ t}}{[R + is]} + \frac{C_1\ e^{\sigma_1 t}}{(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)(\sigma_1 - \sigma_4)} + \frac{C_2\ e^{\sigma_2 t}}{(\sigma_2 - \sigma_3)(\sigma_2 - \sigma_4)} + \frac{C_3\ e^{\sigma_3 t}}{(\sigma_3 - \sigma_4)} + C_4\ e^{\sigma_4 t}$$

Where:

$$P + i\ Q \equiv [(b_3 - b_1\omega^2) + i(b_2\omega)] = [-126 - i\ 3892] \times 10^{-9}$$

and:

$$R + is \equiv [(A\omega^4 - C\omega^2 + E) + i(D\omega - B\omega^3)] = [1947 + i\ 1570] \times 10^{-9}$$

For the steady state solution we assume that all the roots of the characteristic equation are negative. This means that the transient goes to zero and we have the steady state ϕ defined by:

$$\phi = \frac{[P + iQ]}{[R + iS]} \delta_o e^{i\omega t} = \phi_o \; e^{i(\omega t + \varepsilon)}$$

$$\phi_o = \left(\frac{P^2 + Q^2}{R^2 + S^2}\right)^{1/2} \delta_o = \frac{15.18}{6.23}^{1/2} \delta_o = 1.558 \; \delta_o$$

$$\delta_o = {}^1/3 \text{ radian}$$

$$\varepsilon = \tan^{-1} \left(\frac{PR + QS}{QR - PS} \right) = \tan^{-1} (.863)$$

Therefore:

$$\phi_o = 29.9^\circ$$

$$\varepsilon = 40.8^\circ$$

Note: The values for $\sigma_1, \sigma_2, \sigma_3$, and σ_4 were evaluated by computer program and found to be equal to -0.381, - 1.38 $\pm$ 4.221i, and -3.385 respectively.